Generalizations of the convexity theorem of Chandler Davis.

Yury Grabovsky
and
Omar Hijab

Department of Mathematics
Temple University
Philadelphia, PA 19122

Motivation

(From the theory of composite materials)

For $U \subset \mathrm{Sym}^+(\mathbb{R}^3)$ find $\mathcal{L}(U)$—the smallest set such that

1. $U \subset \mathcal{L}(U)$

2. $\mathcal{L}(U)$ is $SO(3)$-invariant

3. $W(\mathcal{L}(U))$ is convex

$$W(\mathbf{L}) = [(\mathbf{I} - \mathbf{L})^{-1} - \mathbf{e}_1 \otimes \mathbf{e}_1]^{-1}$$

Reformulation

$V = \mathsf{Sym}(\mathbb{R}^3)$—a vector space
$C = W(\mathsf{Sym}^+(\mathbb{R}^3))$ is a convex subset of V
$G = SO(3)$ acts non-linearly on C

$$\mathbf{R} \cdot \mathbf{K} = W(\mathbf{R}W^{-1}(\mathbf{K})\mathbf{R}^{-1})$$

Problem: For $U \subset C$ find the smallest convex and G-invariant subset of C containing U.

Question: Can we tell convexity of U by looking at $U \cap \{\text{diagonal matrices}\}$?

Answer: Maybe! If we can solve

Two difficult problems

Problem 1: For $\{\mathbf{a}, \mathbf{b}\} \subset (\mathbb{R}^3)^+$ find $L(\mathbf{a}, \mathbf{b})$—the smallest subset of $(\mathbb{R}^3)^+$ such that

1. $\{\mathbf{a}, \mathbf{b}\} \subset L(\mathbf{a}, \mathbf{b})$
2. $L(\mathbf{a}, \mathbf{b})$ is convex
3. $L(\mathbf{a}, \mathbf{b})$ is S_3-invariant

$\sigma \in S_3$ acts nonlinearly:

$$\sigma \cdot \mathbf{x} = (j \circ \sigma \circ j)(\mathbf{x}), \qquad j(\mathbf{x}) = \left(\frac{1}{x_1}, x_2, x_3\right)$$

Two difficult problems

Problem 2(Conjecture): $\forall\{\mathbf{a},\mathbf{b}\} \subset (\mathbb{R}^3)^+$

the set $W(SO(3)J(L(\mathbf{a},\mathbf{b})))$ is convex.

$$J : (\mathbb{R}^3)^+ \to \mathrm{Sym}(\mathbb{R}^3), \quad J(\mathbf{x}) = \begin{bmatrix} \frac{1}{x_1} & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

$$W(\mathbf{L}) = [(\mathbf{I}-\mathbf{L})^{-1} - \mathbf{e}_1 \otimes \mathbf{e}_1]^{-1}$$

Reformulation

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Problem: For $U \subset C$ find the smallest convex and G-invariant subset of C containing U.

Theorem: If the conjecture is true then an $SO(3)$-invariant susbset U is convex if and only if $U \cap \{\text{diagonal matrices}\}$ is convex.

Another approach

Chandler Davis theorem: $SO(3)$-invariant function $f(\mathbf{A})$ is convex if and only if its restriction to diagonal matrices is convex.

Philosophy: CXF(C) $\rightarrow$CXS(C) (level sets)
CXS($C\times\mathbb{R}$) $\rightarrow$CXF(C) (epigraphs)

G acts on $C \times \mathbb{R}$ by $g \cdot (x, \alpha) = (gx, \alpha)$.

Definitions

- A convex subset T of C is called a transversal if
$$\forall_{x \in C} \quad \mathcal{O}_x^T = T \cap \mathcal{O}_x \neq \emptyset.$$

- A function $f : C \to \mathbb{R}$ is called G-invariant if
$$\forall_{x \in C} \, \forall_{g \in G} \; f(gx) = f(x).$$

- $U \subset T$ is called G-invariant if $U = GU \cap T$.

- For $U \subset T$, let $L(U)$ be the smallest convex, G-invariant subset of T containing U.

- For $U \subset C$, let $\mathcal{L}(U)$ be the smallest convex, G-invariant subset of C containing U.

First generalization

Theorem: $\mathcal{L}(U) = GL(U \cap T)$, provided $GU = U$ and $\forall_{\{x_1,x_2\}\subset T}\ GL(\{x_1, x_2\})$ is convex.

Proof: Almost obvious.

Corollary (first generalization):
G acts on $C\times\mathbb{R}$ by $g\cdot(x,\alpha) = (gx,\alpha)$. $\widehat{T} = T\times\mathbb{R}$

Assume $\forall_{\{\widehat{x}_1,\widehat{x}_2\}\subset\widehat{T}}\ G\cdot\widehat{L}(\{\widehat{x}_1, \widehat{x}_2\})$ is convex.

Then a G-invariant function is convex if and only if its restriction to T is convex.

Proof: Let f be the restriction of a G-invariant function F from C to T. Then epi$(F) = G\cdot$epi(f).

More Definitions

- $T^* \subset V'$. $f(x)$ is called (T, T^*)-regular if

$$f(x) = \sup_{y \in T^*} \left(y(x) - \alpha(y) \right)$$

- For $x \in T$, let $L(x) = L(\{x\})$.

- $\psi_y(x) = \sup_{z \in L(x)} y(z)$

- $\widehat{\psi}_y(x) = \psi_y(\mathcal{O}_x^T)$

Second generalization

G acts on $C \subset V$. C—convex.
T—transversal.
$T^* \subset V'$ arbitrary.
$\widehat{\psi}_y(x)$—G-invariant extension of $\psi_y(x)$.

Assume $\widehat{\psi}_y(x)$ is convex on C $\forall_{y \in T^*}$.

Then a G-invariant function $F : C \to \mathbb{R} \cup \{+\infty\}$ is convex on C if its restriction f to T is convex and (T, T^*)-regular.

Proof

$f = F \restriction T, \quad f(x) = \sup_{y \in T^*} \{y(x) - \alpha(y)\}$

Let $\mathcal{F}_{x_0} = \{x \in T : \ f(x) \leq f(x_0)\}$.

Then $L(x_0) \subset \mathcal{F}_{x_0}$. Therefore,

$$f(x_0) \leq \sup_{x \in L(x_0)} f(x) \leq \sup_{x \in \mathcal{F}_{x_0}} f(x) \leq f(x_0).$$

$$f(x_0) = \sup_{x \in L(x_0)} \sup_{y \in T^*} \{y(x) - \alpha(y)\}$$

$$f(x_0) = \sup_{y \in T^*} \{\psi_y(x_0) - \alpha(y)\}.$$

$$F(x) = \sup_{y \in T^*} \{\widehat{\psi}_y(x) - \alpha(y)\}.$$

Linear group action

Theorem: Assume $\sup_{z\in\mathcal{O}_x^T} y(z) = \sup_{z\in\mathcal{O}_x} y(z)$ for all $x \in C$ and all $y \in T^*$. Then

$$\widehat{\psi}_y(x) = \sup_{g\in G} y(gx).$$

Proof: $\text{conv}(\mathcal{O}_x^T) \subset L(x) \subset \text{conv}(\mathcal{O}_x) \cap T$.

$$\sup_{z\in\text{conv}(\mathcal{O}_x^T)} y(z) \le \psi_y(x) \le \sup_{z\in\text{conv}(\mathcal{O}_x)} y(z).$$

So, $\psi_y(x) = \sup_{z\in\mathcal{O}_x} y(z)$.

Remark: Kostant's convexity theorem yields the assumption for the co-adjoint action of a semi-simple Lie group on its Lie algebra dual.

Example: 2D conductivity

$$V = \mathsf{Sym}(\mathbb{R}^2)$$

$$C = \left\{ \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \; : \; k_{11} > -1, \; k_{22} < 1 \right\}.$$

$$T = \{\mathbf{K} \in C : \; (1 + k_{11})(1 - k_{22}) \geq 1, \; k_{12} = 0\}.$$

Theorem: $T^* = \{\mathbf{Y} : y_{11} \geq 0, \; y_{22} \leq 0\}.$

$$\widehat{\psi}_{\mathbf{Y}}(\mathbf{K}) = y_{22} - y_{11} + \psi_0 \left(\Delta + \sqrt{\Delta^2 - 4\lambda} \right) \Big/ \, 2\lambda,$$

$$\begin{aligned} \psi_0 &= y_{11} a_{11} - y_{22} a_{22} \\ \Delta &= \lambda + 1 + k_{12}^2 \\ a_{11} &= 1 + k_{11} \\ a_{22} &= 1 - k_{22} \\ \lambda &= a_{11} a_{22} \end{aligned}$$

Applications