Solvable Markov Models With Stochastic Volatility and Jumps

Alexey Kuznetsov
Department of Mathematics
University of Toronto
kuznecov@math.toronto.edu

Claudio Albanese
Department of Mathematics
University of Toronto
albanese@math.toronto.edu

Introduction

There are only few known examples of Markov processes for which the transitional probability density can be computed in analytically closed form. Our method gives a general framework for building a family of analytically tractable models, which include all previously known processes, and enables us to construct new solvable martingale processes having such important properties as state-dependent volatility and/or stochastic volatility and/or jumps. The probability kernel is expressed as expansion in orthogonal polynomials. The method can be summarized as the change of measure, change of phase space for the process and the spectrum deformation for the generator of the Markov semigroup.

Method

1) Orthogonal expansion.

We start from the Markov generator for the process X_t:

$$\mathcal{L}^X f = m(x)f' + \frac{1}{2}\sigma^2(x)f'' \qquad (1)$$

and find the complete set of eigenvectors $\psi_n(x)$ with eigenvalues λ_n. Then the probability kernel for the process X_t can be computed as:

$$p^X(t, x_0, x_1) = e^{t\mathcal{L}^X} = \sum_{n=0}^{\infty} e^{\lambda_n t}\psi_n(x_0)\psi_n(x_1)\mu(x_1). \qquad (2)$$

It is convenient to start with an operator $\mathcal{L}^X$ which admits as its eigenfunctions a set of polynomials $p_n(x)$, which are orthogonal in $L_2(D,\mu)$. Then one can use the three term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x)$$

with "initial conditions" $p_{-1} = 0$, $p_0 = 1$ to compute iteratively the values of the polynomials and the probability kernel.

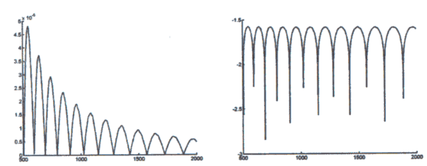

Fig. 1: *Approximation error and the exponent of convergence.*

2) Measure change and change of variables

are defined by the following equations:

$$\frac{dQ_t}{dP_t} = e^{-\rho t}g(X_t), \qquad X_t \Rightarrow Y_t = Y(X_t)$$

where $Y(x) = f(x)/g(x)$ and functions f and g are two linearly independent solutions to equation $\mathcal{L}^X f = \rho f$ (ρ is a positive fixed parameter).

Theorem:
Under the new measure Q the process $Y_t = Y(X_t)$ is a Markov martingale and its probability kernel can be computed explicitly as follows:

$$p^Y(t, y_0, y_1) = \frac{g(x_1)}{g(x_0)}\frac{1}{Y'(x_1)}e^{-\rho t}p^X(t, x_0, x_1) =$$
$$= \frac{g(x_1)}{g(x_0)}\frac{1}{Y'(x_1)}\sum_{n=0}^{\infty} e^{-(\rho-\lambda_n)t}\psi_n(x_0)\psi_n(x_1)\mu(x_1). \qquad (3)$$

where $(y_0, y_1) = (Y(x_0), Y(x_1))$.

3) Spectrum deformation/time change.

A *stochastic time change process* T_t is defined as a right-continuous non-decreasing process started from 0 and with values in $[0,\infty)$. The time changed version of the process Y_t is

$$Y_t \Rightarrow \tilde{Y}_t = Y_{T_t}.$$

We assume that the time change process T_t is independent of the underlying process Y_t. Then the probability kernel of the time changed process $\tilde{Y}_t$ is given by the following expression:

$$p^{\tilde{Y}}(t, y_0, y_1) = \frac{g(x_1)}{g(x_0)}\frac{1}{Y'(x_1)}\mu(x_1)\times$$
$$\times\left[\sum_{n=0}^{\infty} L(t, \rho-\lambda_n)\psi_n(x_0)\psi_n(x_1)\right], \qquad (4)$$

where $L(t,\lambda) = E\left[e^{-\lambda T_t}\right]$ is the Laplace transform of T_t. Another interesting formula can be derived for the value of the following expectation (which represents the price of a call option in Mathematical Finance):

$$E^Q(\tilde{Y}_t - K)^+ = (y_0 - K)^+ + \frac{1}{2}\nu^2(K)\frac{g(k)}{g(x_0)}\frac{1}{Y'(k)}\times$$
$$\times\left[G(\rho, x_0, k) + \mu(k)\sum_{n=0}^{\infty}\frac{L(t,\rho-\lambda_n)}{\rho-\lambda_n}\psi_n(x_0)\psi_n(k)\right] \qquad (5)$$

where $k = Y^{-1}(K)$ and the function $G(\rho, x_0, x_1)$ is the Green function defined as the integral kernel of the operator $(\mathcal{L}^X - \rho)^{-1}$ or by the integral $\int_0^\infty e^{-\rho s}p^X(s, x_0, x_1)ds$. This formula also gives a convenient way of computing the expectation of the local time $L_t(\tilde{Y})$ of the process $\tilde{Y}_t$, which can be shown to be equal to $E^Q(\tilde{Y}_t - K)^+ - (y_0 - K)^+$.

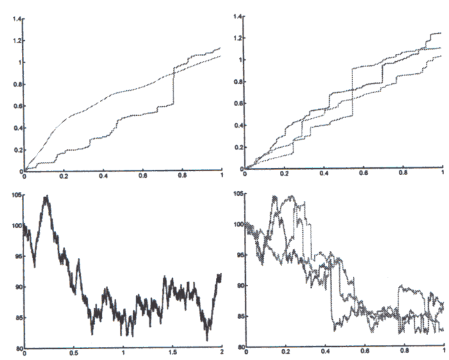

Fig. 2: *Three samples of the time change process, which is combined of continuous part and pure jump Levy process, sample trajectory of Markov martingale and its time changed versions.*

From equations (4) and (5) it can be observed that *analytical solvability of our model is equivalent to being able to compute the Laplace transform* $L(t,\lambda)$. This is why it is convenient to use Levy processes (for example the gamma process Γ_t) and the so-called *affine processes*, for which the Laplace transform of the $T_t^c = \int_0^t u_s ds$ can be computed explicitly (one can use CIR, Ornstein-Uhlenbeck processes and their lattice approximations given by Meixner and Charlier process). We model the time change process as

$$T_t = \Gamma_t + \int_0^t u_s ds, \qquad (6)$$

where Γ_t is Gamma process (or any other increasing Levy process) and u_s is any affine process independent of Γ_t. This construction gives us both jumps and stochastic volatility while preserving the solvability of the model.

Applications of time change to semigroup theory.

If we perform the time change by a Levy process we have the following orthogonal expansion of the semigroup:

$$Ee^{T_t\mathcal{L}^X} = e^{-t\phi(-\mathcal{L}^X)} = \sum_{n=0}^{\infty} e^{-t\phi(-\lambda_n)}\psi_n(x_0)\psi_n(x_1)\mu(x_1) \quad (7)$$

where $\phi(\lambda)$ is a Berstein function defined by $Ee^{-\lambda T_t} = e^{-t\phi(\lambda)}$ (this function can be computed using the formula: $\phi(\lambda) = \int[\exp(\lambda u) - 1]d\nu(u)$, where $\nu(u)$ is the Levy measure). Thus the generator of the time changed process has the same eigenfunctions as $\mathcal{L}^X$ but different eigenvalues.

Theorem:
If the time change process if given by Levy process, then performing a time change is equivalent to deforming the spectrum of the generator of the semigroup:

$$\lambda_n \Rightarrow -\phi(-\lambda_n).$$

Example: Subordination by gamma process Γ_t deforms the spectrum:

$$\lambda_n \Rightarrow -log(1-\lambda_n).$$

It is interesting to note that the converse is also true for Markov semigroups: every spectrum deformation that preserves the Markov property can be modelled by some Levy time change.

Computational results

In this section we sketch a step-by-step construction to illustrate the theory. As for X_t, we choose the CIR process of equation $dX = (a - bX)dt + \sigma\sqrt{x_t}dW$. The Markov generator $\mathcal{L}^X$ in the new variable $\xi = \theta x$ is given by

$$\mathcal{L}^X = \frac{1}{b}\left((\alpha+1-\xi)\frac{d}{d\xi} + \xi\frac{d^2}{d\xi^2}\right). \qquad (8)$$

where $\theta = \frac{2b}{\sigma^2}$ and $\alpha = \frac{2a}{\sigma^2} - 1$. Two linearly independent solutions to the equation $\mathcal{L}^X f = \rho f$ are:

$$f_1(x) = {}_1F_1\left(\frac{\rho}{b}, \alpha+1, \theta x\right), \quad f_2(x) = \frac{{}_1F_1\left(\frac{\rho}{b} - \alpha, 1-\alpha, \theta x\right)}{(\theta x)^\alpha}$$

where ${}_1F_1(A, B, z)$ is the confluent hypergeometric function. The eigenfunctions ψ_n can be expressed in terms of Laguerre polynomials:

$$\psi_n(x) = \sqrt{\frac{n!}{\Gamma(n+\alpha+1)}}L_n^{(\alpha)}(\theta x),$$

where $L_n^{(\alpha)}$ are Laguerre polynomials of order α. The corresponding eigenvalues are $\lambda_n = -bn$ and the orthogonality measure $\mu(x)dx$ is equal to $\theta^{\alpha+1}x^\alpha e^{-\theta x}dx$. Laguerre polynomials can be computed efficiently by means of the recurrence relation:

$$(n+1)L_{n+1}^{(\alpha)}(z) = (2n+\alpha+1-z)L_n^{(\alpha)}(z) - (n+\alpha)L_n^{(\alpha)}(z)$$

with initial conditions $L_{-1}^{(\alpha)} = 0$, $L_0^{(\alpha)} = 1$.

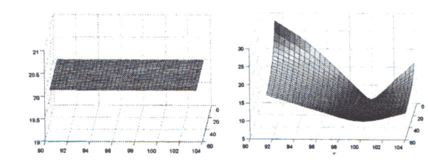

Fig. 3: *We were able to model an implied volatility surface which captures the real life behavior of stock prices (on the left is the implied volatility surface in the classical Black-Scholes model).*