NB No class Monday, 21 March

E.g. Extensions of the form $1 \to \mathbb{Z}/_2 \to G \to \mathbb{Z}/_2 \times \mathbb{Z}/_2 \to 1$

$\quad$ Claim: $\# H^2(V, \mathbb{Z}/_2) = 8 \quad (\cong \mathbb{Z}/_2{}^3)$

Assume $E \equiv$
$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "a"] & G \arrow[r, "b"] & K \arrow[r] & 1 \\ 1 \arrow[r] & H \arrow[r] \arrow[u, equal] & G \times_K K' \arrow[r, "b' = pr_2"'] \arrow[u, "\varphi' := pr_1"] & K' \arrow[r] \arrow[u, "\varphi"'] & 1 \end{tikzcd}$$
exact seq., H, G, K groups

$\varphi^* E \equiv 1 \to H \to G \times_K K' \to K' \to 1$

$h \mapsto (a(h), 1)$

$\varphi : K' \to K$ group hom.

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

Claim: $\varphi^* E$ is an extension of K' by H

$\quad$ b' surjective: $x \in K'$, $\exists\, g \in G : b_g = \varphi x$ as b is surjective

$\qquad$ $b'(a, x) = x \Rightarrow b'$ is surjective

$\quad$ $a' : H \to G \times_K K'$ injective

$\quad$ $h \mapsto (a(h), 1)$

$\quad$ Want $\operatorname{Ker} b' = \operatorname{Im} a'$

$\qquad (g, x) \in G \times_K K'$; $\; b'(g, x) = 1 \iff x = 1 \in K'$

$\qquad \Rightarrow b(g) = \varphi(1) = 1 \Rightarrow g \in \operatorname{Ker} b \iff g = a(h)$

$\qquad \Rightarrow (g, x) = (a(h), 1) = a'(h)$

$\qquad b' a'(h) = b'(a(h), 1) = 1$

$\qquad \Rightarrow \varphi^* E$ is an exact sequence

Dually, $E \equiv$
$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "a"] \arrow[d, "\psi"'] & G \arrow[r, "b"] \arrow[d, "\psi'"', "g \mapsto (g,1)"] & K \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & H' \arrow[r, "a'"'] & G \times H' \arrow[r, "b"'] & K \arrow[r] & 1 \end{tikzcd}$$
Need H, H' abelian

$\psi^* E \equiv 1 \to H' \to G \times H' \to K \to 1$ exact sequence

$(a(h), \psi(h))$

$(g, h') \longmapsto b(g)$

The "push-out" of ψ

Lemma. $\psi_* \varphi^* E \underset{\text{equiv}}{=} \varphi^* \psi_* E$

NB. If E, E' are equivalent, so are $\varphi_* E$ and $\psi^* E'$, etc.

Special case: $\varphi \in \mathrm{Aut}\, K$

$$\begin{tikzcd}
E \equiv 1 \arrow[r] & H \arrow[r, "a"] \arrow[d, equal] & G \arrow[r, "b"] \arrow[d, equal] & K \arrow[r] & 1 \\
\varphi^*E \equiv 1 \arrow[r] & H \arrow[r, "a"] \arrow[d, equal] & G \arrow[r, "\varphi^* b"'] \arrow[d, "\cong"] & K \arrow[r] \arrow[u, "\varphi"'] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r] & G \times_K K \arrow[r, "b'"] & K \arrow[r] & 1
\end{tikzcd}$$

Question: Are E and φ^*E equivalent? In general, no.

$$\begin{tikzcd}
1 \arrow[r] & H \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "1"] & G \arrow[r, "b"] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r, "a"] & G \arrow[r, "\varphi^* b"] \arrow[u, "\psi"'] & K \arrow[r] & 1
\end{tikzcd}$$

1. $\psi a = a$ and
2. $b\psi = \varphi^* b$

Concrete example: $H = \mathbb{Z}/2$, $K = V$; $\mathrm{Aut}\, K = \mathfrak{S}_3$

$|G| = 8$ and as sets $G \cong H \times K$

Groups of order 8:

3 Abelian ones: $\mathbb{Z}/8$, $\mathbb{Z}/4 \times \mathbb{Z}/2$, $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$

2 non-Abelian ones: D_8 (dihedral, symmetries of a square)

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

$\mathbb{Z}/8$ cannot occur in the middle, so

$H^2(V, \mathbb{Z}/2) = [\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2] + 3[\mathbb{Z}/4 \times \mathbb{Z}/2] + 3[D_8] + [Q_8]$

(orbits under $\mathrm{Aut}(V) \cong \mathfrak{S}_3$)

$$1 \to \mathbb{Z}/2 \xrightarrow{a} G \xrightarrow{b} \mathbb{Z}/2 \times \mathbb{Z}/2 \to 1$$

Observation: $\mathrm{Ker}\, b$ has to contain all "squares":

$x \in G$ is a square if $\exists\, y \in G : x = y^2$

$b(x) = b(y^2) = b(y)^2 = 1$

Groups	squares
$\mathbb{Z}/4 \times \mathbb{Z}/2$	$(0,0), (2,0)$
$D_8 = \{r, s \mid s^2 = r^4 = 1, sr = r^{-1}s\}$	e, r^2
Q_8	± 1

$\Big\}$ Ker b is uniquely determined

$$\mathbb{Z}/4 \times \mathbb{Z}/2 \,/\, \ker b = \left\{ \begin{array}{lc} & \text{order} \\ (\text{even}, \text{even}) & 2 \\ (\text{odd}, \text{even}) & 4 \\ (\text{even}, \text{odd}) & 2 \\ (\text{odd}, \text{odd}) & 4 \end{array} \right\} \begin{array}{l} \text{identity} = \ker b \\ \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \end{array}$$

$\therefore$ The only automorphism of $\mathbb{Z}/2 \times \mathbb{Z}/2$ that can be lifted is the permutation of the cosets with elements of order 4

Thus there are three inequivalent extensions with $\mathbb{Z}/4 \times \mathbb{Z}/2$ in the middle

Q_8 gives only one extension class

$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$

squares are $1, r^2$

$$D_8 / \text{squares} = \left\{ \begin{array}{ll} \{1, r^2\} & \text{trivial} = \ker b \\ \{r, r^3\} & \text{order } 4 \\ \{s, sr^2\} & \text{order } 2 \\ \{sr, sr^3\} & \text{order } 2 \end{array} \right.$$

Thus D_8 gives rise to 3 distinct extensions

Finally: Any extension with three copies of $\mathbb{Z}/2$ in the middle splits, and any two split extensions are equivalent

$\Rightarrow \exists$ one extension class with $\mathbb{Z}/2^3$ in the middle

Given H, K, $\{1 \to H \xrightarrow{a} G \xrightarrow{b} K \to 1\} \mapsto G$

$\{1 \to H \to G \to K \to 1\} / \text{equiv} = H^2(K, H) \mapsto [G]$ isom. class of G

$$\begin{tikzcd} H^2(K,H) \arrow[r, "\text{``middle term''}"] \arrow[d] & \text{isom. classes of groups of order } |H||K| \\ \text{Aut } H \backslash H^2(K,H) / \text{Aut } K \arrow[ur, "\pi"'] & \end{tikzcd}$$

In our example π is injective

NB. φ^*, ψ_* of a split homomorphism is split

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[u, equal] & G \arrow[r] \arrow[u, "\text{pullback}"] & \mathbb{Z}/2 \times \mathbb{Z}/2 \arrow[r] \arrow[u, "\chi"] & 1 \end{tikzcd}$$

$\chi \in \mathrm{Hom}_{\mathrm{gps}}(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)$

$$\underbrace{H^2(\mathbb{Z}/2, \mathbb{Z}/2)}_{\cong\, \mathbb{Z}/2} \times \underbrace{\mathrm{Hom}(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)}_{\cong\, \mathbb{Z}/2 \times \mathbb{Z}/2} \longrightarrow H^2(\mathbb{Z}/2 \times \mathbb{Z}/2, \mathbb{Z}/2)$$

$$([E], \chi) \longmapsto (\chi^* E)$$

Image is abelian extensions

Double covers of $\mathfrak{S}_4$: $G \overset{b}{\twoheadrightarrow} \mathfrak{S}_4$, $\ker b \cong \mathbb{Z}/2$

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & \mathbb{Z}/4 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[u, equal] & \mathbb{Z}/4 \times_{\mathbb{Z}/2} \mathfrak{S}_4 \arrow[r] \arrow[u] & \mathfrak{S}_4 \arrow[r] \arrow[u, "\text{sgn}"] & 1 \end{tikzcd}$$

$\sigma \in \mathfrak{S}_4$; identify $\mathbb{Z}/2$ with ± 1, $\mathbb{Z}/4$ with $\pm 1, \pm i$

$G = \{\underbrace{\sqrt{\operatorname{sgn} \sigma}}_{\mathbb{Z}\text{-valued}}\, \sigma \mid \sqrt{\operatorname{sgn}\sigma} \in \mathbb{Z}/4;\ \sigma \in \mathfrak{S}_4\}$, $|G| = 48$

What is the group law? (i.e. cocycle)

multiply signs and permutations

Another double-cover of $\mathfrak{S}_4$:

$\mathfrak{S}_4 \hookrightarrow SO(3)$ as it is rotations of a cube

$SU(2) \twoheadrightarrow SO(3)$ by map defined by quaternions

Take pullback:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] & SU(2) \arrow[r, two heads] & SO(3) \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[u, equal] & 2O \arrow[r, two heads] \arrow[u, hook] & \mathfrak{S}_4 = O \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

Are these two extensions equivalent? No

$2O$ has only one element of order 2; $\operatorname{sgn}^*(\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2)$ has 7 elements of order 2

What is the remaining cover?
Take the two known non-trivial ones and use the group law on $H^2(\mathfrak{S}_4, \mathbb{Z}/2)$ to combine them.

Addition of extensions: (any abelian category / groups)

$E_1 \equiv 1 \to H_1 \to G_1 \to K_1 \to 1$ exact

$E_2 \equiv 1 \to H_2 \to G_2 \to K_2 \to 1$

$E_1 \times E_2 \equiv 1 \to H_1 \times H_2 \to G_1 \times G_2 \to K_1 \times K_2 \to 1$

If $K := K_1 = K_2$, then define $\Delta : K \to K \times K$ and consider $\Delta^*(E_1 \times E_2)$

$$\begin{tikzcd}
1 \arrow[r] & H_1 \times H_2 \arrow[r] & G_1 \times G_2 \arrow[r] & K \times K \arrow[r] & 1 \\
1 \arrow[r] & H_1 \times H_2 \arrow[r] \arrow[u, equal] & G \arrow[r] \arrow[u, "\text{pull-back}"] & K \arrow[r] \arrow[u, "\Delta"] & 1
\end{tikzcd}$$

If $H := H_1 = H_2$ is abelian, then $\nabla : H \times H = H \oplus H \to H$ is sum map (or codiagonal)

$$\begin{tikzcd}
1 \arrow[r] & H_1 \times H_2 \arrow[r] \arrow[d, "\nabla"] & G \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & H \arrow[r] & G' \arrow[r] & K \arrow[r] & 1
\end{tikzcd} \equiv \nabla_{H*} \Delta_K^*(E_1 \times E_2)$$

Def. The "Baer sum" of extensions

$E_1 \equiv 1 \to H \to G_1 \to K \to 1$ H abelian

$E_2 \equiv 1 \to H \to G_2 \to K \to 1$

is $E_1 +_{Baer} E_2 := \nabla_{H*} \Delta_K^*(E_1 \times E_2)$. This is again an extension of K by H.

Exercise. ① What is the Baer sum of $2O$ and $sgn^*(\mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2)$?
Is the middle term isomorphic to one of these we already know?
② $2O \not\cong GL(2, GF(3))$ - show this

Old question by Coxeter: Classify all groups with exactly one elt of order 2

Is $GL(2, GF(3))$ a double-cover of $\mathfrak{S}_4$?

$|GL(2,GF(3))| = \left(\begin{smallmatrix}2\\2\end{smallmatrix}\right)_3 = 48$, so order is okay

$Z(GL(2,F)) = F^*$

$1 \to F^* \to GL(n,F) \to PGL(n,F) \to 1$

$1 \to \mathbb{Z}/2 \to 48 \to 24 \to 1$

Claim: $PGL(2,GF(3)) \cong \mathfrak{S}_4$

$PGL(2,\mathbb{C})$ = group of Möbius transformations
$= \{ z \mapsto \frac{az+b}{cz+d} \mid ad-bc \neq 0 \}$

In general $PGL(2,\mathbb{F}) = \{ z \mapsto \frac{az+b}{cz+d} : \mathbb{P}^1_{\mathbb{F}} \to \mathbb{P}^1_{\mathbb{F}} \mid ad-bc \neq 0 \}$

Also $PGL(2,\mathbb{F})$ is always triply transitive

Thus if $\mathbb{F} = GF(3)$, any permutation on 4 elts may be realized, and claim follows

So $1 \to GF(3)^* \to GL(2,GF(3)) \to PGL(2,GF(3)) \cong \mathfrak{S}_4 \to 1$ is in $H^2(\mathfrak{S}_4, \mathbb{Z}/2)$

Thm. Multiplication of cocycles corresponds to Baer sums of extensions

Extensions in Abelian categories: Fix an Abelian category A

An extension of X by Y ($X, Y \in \mathrm{Ob}\, A$) is a short exact sequence $0 \to Y \xrightarrow{a} Z \xrightarrow{b} X \to 0$ with $\operatorname{coker} b$, $b = \operatorname{coker} a$

Any morphism $f: U \to V$ in A gives rise to two exact sequences:

$$\begin{tikzcd}
0 \arrow[dr] & & & & & & 0 \\
& \ker f \arrow[dr] & & & & \operatorname{Coker} f \arrow[ur] & \\
& & U \arrow[rr, "f"] \arrow[dr] & & V \arrow[ur] & & \\
& & & \operatorname{Im} f \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Two extensions $E: 0 \to Y \to Z \to X \to 0$, $E': 0 \to Y \to Z' \to X \to 0$ are equivalent if $\exists\, \varphi: Z' \to Z \ni$

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, equal] & Z' \arrow[r] \arrow[u, "\varphi"] & X \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

MAT 1103 - Introduction to homological algebra 14.3

$\mathrm{Ext}^1_A(X,Y) = \{$equivalence classes of extensions $0 \to Y \to Z \to X \to 0\}$

Why is this a set? - This may give set-theoretic trouble unless A is small

Lemma. If $f: X' \to X$ is any morphism, then for any extension E, $f^*E \equiv 0 \to Y \xrightarrow{a'} Z \times X' \xrightarrow{\pi_2} X' \to 0$ is an extension of X' by Y'; f^*(equivalent extensions) = equiv. exts.

i.e. $f^*: \mathrm{Ext}^1_A(X,Y) \to \mathrm{Ext}^1_A(X',Y)$

Pf.

$$\begin{tikzcd}
E: 0 \arrow[r] & Y \arrow[r, "a"] & Z \arrow[r, "b"] & X \arrow[r] & 0 \\
f^*E: 0 \arrow[r] & Y \arrow[u, equal] \arrow[r, "a'"'] & Z' \arrow[u, "f'"] \arrow[r, "pr_2"'] & X' \arrow[u, "f"] \arrow[r] & 0
\end{tikzcd}$$

$Z' = Z \times_X X'$

① pr_2 is an epimorphism

② $\exists\, a': Y \to Z'$ $\ni$ $f'a' = a$

③ $a' = \ker pr_2$, $pr_2 = \mathrm{coker}\, a'$

①: Recall that Z' can be obtained as kernel

$$0 \to Z' \to Z \oplus X' \xrightarrow[b\circ in_1 - f\circ in_2]{} X \to 0$$

$eq(b,f) = \ker(b-f)$

$pr_1: Z \oplus X' \to Z$, $pr_2: Z \oplus X' \to X'$

① Clear for sets, so convert into diagram

②:

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r] & Z \oplus X' \arrow[r] & X' \arrow[r] & 0 \\
& & Y \arrow[ul, "\exists! a'"] \arrow[u, "(a{,}0)"] \arrow[ur, "0 = ba - f\cdot 0"'] & &
\end{tikzcd}$$

as $Z' = \ker(Z \oplus X' \to X)$

③:

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r, "a"] & Z \arrow[r, "b"] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[u, equal] \arrow[r, "a'"] & Z' \arrow[u, "f'"] \arrow[r, "pr_2"] & X' \arrow[u, "f"] \arrow[r] & 0 \\
& & U \arrow[ul, "\exists! \tilde{g}"] \arrow[u, "g"] \arrow[ur, "0"'] & &
\end{tikzcd}$$

$pr_2 g = 0 \Rightarrow b f' g = f pr_2 g = 0$

$\Rightarrow \exists!\, \tilde{g}: U \to Y$: $a\tilde{g} = f'g$

$g = a'\tilde{g}$ iff $f'a'\tilde{g} = f'g$ and $pr_2 a'\tilde{g} = pr_2 g$ (by universality of fibred product)

which holds, so $g = a'\tilde{g}$

$\Rightarrow a' = \ker p_2$

$pr_2 = \mathrm{coker}\, a'$ follows by dual argument $\square$

Aside: Thm (Gabriel, Mitchell, Freyd; Lubkin) Any abelian category embeds as a full subcategory into $\mathrm{Mod}\, E$, where E is some ring. Moreover, the embedding is exact, i.e.

$0 \to Y \to Z \to X \to 0$ in $\mathcal{A}$ is exact iff it is exact in $\mathrm{Mod}\, E$.

Given $g: Y \to Y'$, we can form $g_* E$:

$$\begin{tikzcd} g_*E: 0 \arrow[r] & Y' \arrow[r] & Z \oplus Y'/Y \arrow[r] & X \arrow[r] & 0 \\ E: 0 \arrow[r] & Y \arrow[r, "a"'] \arrow[u, "g"] & Z \arrow[r, "b"'] \arrow[u, "=: Z'"'] & X \arrow[r] \arrow[u, equal] & 0 \end{tikzcd} \qquad Z' := \mathrm{coker}\,(Y \xrightarrow{(a,g)} Z \oplus Y')$$

This is compatible with equivalence; thus $g_*: \mathrm{Ext}^1_{\mathcal{A}}(X,Y) \to \mathrm{Ext}^1_{\mathcal{A}}(X,Y)$

$g'_* g_* E \underset{equiv}{\cong} (g'g)_* E$ and $f'^* f^* E \underset{equiv}{=} (ff')^* E$

by transitivity of pullback, pushout.

Thus $\mathrm{Ext}^1_{\mathcal{A}}(?, ??)$ is a bifunctor on $\mathcal{A}^{op} \times \mathcal{A}$

Assume $\mathcal{A}$ is a k-category. Then k acts on $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$:

$$\begin{tikzcd} E: 0 \arrow[r] & Y \arrow[r] & Z \arrow[r] & X \arrow[r] & 0 \\ 0 \arrow[r] & Y \arrow[r] \arrow[u, "m_Y(x)"] & Z \arrow[r] \arrow[u, "m_Z(x)"] & X \arrow[r] \arrow[u, "m_X(x)"] & 0 \end{tikzcd} \qquad x \in k$$

commutes

$m_X(x)^* E \underset{equiv}{\cong} m_Y(x)_* E$.

$m_Z(x)$ uniquely induces a morphism $Z'' := m_Y(x)_* \xrightarrow{\mu} Z' := m_X(x)^*$

$\mu a'' = a'$, $b'\mu = b''$; μ yields an equiv $m_Y(x)_* E \to m_X(x)^* E$

$x \cdot [E] = [m_Y(x)_* E]$

In particular, one gets $-E$: Any of the following will do:

$0 \to Y \xrightarrow{-a} Z \to X \to 0$, $\quad 0 \to Y \xrightarrow{a} Z \xrightarrow{-b} X \to 0$,

$(-1_X)^* E$, $\quad (-1_Y)_* E$

Given E, E' extensions of X by Y

$[E] + [E'] := \nabla_{Y*} \Delta^*_X (E \oplus E')$, the Baer sum of $[E]$ and $[E']$

Thm. $Ext^1_A(?,??)$: A k-cat. takes its values in Mod-k and thus functor is bi-additive $Ext^1_A(X_1 \oplus X_2, Y) \cong Ext^1_A(X_1,Y) \oplus Ext^1_A(X_2,Y)$

Given any exact sequence $0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0$ and $X \in Ob\,A$ apply $Hom_A(X,?)$ to this. Get an exact sequence of abelian groups: $0 \to Hom_A(X,A) \to Hom_A(X,B) \to Hom_A(X,C)$

$$f \mapsto af \qquad g \mapsto bg$$

This is not generally exact on the right, but

$$0 \to Hom_A(X,A) \to Hom_A(X,B) \to Hom_A(X,C) \xrightarrow{\delta = (\ )^*} Ext^1_A(X,A) \xrightarrow{a_*} Ext^1_A(X,B) \xrightarrow{b_*} Ext^1_A(X,C)$$

is exact. Thus Ext measures the extent to which Hom fails to be exact.

The zero element in $Ext^1_A(X,Y)$ is the biproduct:

$$0 \to Y \rightleftarrows X \oplus Y \rightleftarrows X \to 0$$

$$\begin{array}{lccccccccc} E: & 0 & \longrightarrow & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\ & & & \| & & \uparrow f' & & \uparrow f & & \\ f^*E: & 0 & \longrightarrow & A & \longrightarrow & B \times_C X & \underset{s}{\overset{b'}{\rightleftarrows}} & X & \longrightarrow & 0 \end{array}$$

$[f^*E] = 0$ in $Ext^1_A(X,A)$ iff $\exists\, s: X \to B \times_C X$ making diagram commute

$\Rightarrow \beta := f's$ lifts f through b

$b\beta = bf's = fb's = f$

Conversely, if $\exists\, \beta: X \to B$, $b\beta = f$, then s is the unique morphism

$$\begin{array}{ccccc} X & \xrightarrow{s} & B \times_C X & & \\ & \beta \searrow & \swarrow & \searrow & \\ & & B & & X \\ & & & \searrow \ \swarrow & \\ & & & C & \end{array}$$

This implies exactness at $Hom_A(X,C)$

This is the basis of the notion of a triangulated category:
If $0 \to A \to B \to C \to 0$, we may apply a functor $Hom(X,?)$ and obtain a connecting homomorphism δ. This can be incorporated into the structure, yielding derived categories.