MAT 1103 - Introduction to homological algebra 28.3

Extension groups and how to compute them

Def. A functor $F: \underline{A} \to \underline{B}$ between abelian categories is exact if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\underline{A}$, $0 \to FA \to FB \to FC \to 0$ is exact.

The functor is left exact if $0 \to FA \to FB \to FC$ is exact, i.e. $F(A) \cong \ker(F(B) \to F(C))$

Right exact defined analogously.

The functor is half exact if $FA \to FB \to FC$ is exact, i.e. $\operatorname{Im} F(A) \cong \ker(F(B) \to F(C))$, $\operatorname{Cok}(F(A) \to F(B)) \cong \operatorname{Im} F(B)$

Eg. Ext^1 is half exact

What is more: Fix $X \in \underline{A}$. Then for any short exact sequence $0 \to A \to B \to C \to 0$, we get long exact sequence

$$0 \to \operatorname{Hom}_{\underline{A}}(X,A) \to \operatorname{Hom}_{\underline{A}}(X,B) \to \operatorname{Hom}_{\underline{A}}(X,C) \to$$
$$\to \operatorname{Ext}^1_{\underline{A}}(X,A) \to \operatorname{Ext}^1_{\underline{A}}(X,B) \to \operatorname{Ext}^1_{\underline{A}}(X,C) \to$$
$$\to \operatorname{Ext}^2_{\underline{A}}(X,A) \to \cdots$$

Analogously, $\operatorname{Ext}^1_{\underline{A}}(?, X)$ is half exact (on $\underline{A}^{op}$)

An n-extension of X by Y is an exact sequence
$$0 \to Y \xrightarrow{\alpha_0} Z_1 \xrightarrow{\alpha_1} \cdots \to Z_n \xrightarrow{\alpha_n} X \to 0$$

Addition of n-extensions:

External addition:
$$\left.\begin{array}{l} 0 \to Y \to \cdots \to X \to 0 \\ 0 \to Y' \to \cdots \to X' \to 0 \end{array}\right\} n\text{-extensions}$$
$$0 \to Y \oplus Y' \to \cdots \to X \oplus X' \to 0 \quad \text{again an } n\text{-extension}$$

Functoriality: If $f: X' \to X$, then can form pullback

$$\begin{array}{ccccccccccc} E \equiv & 0 \to & Y & \to & \cdots & \to Z_{n-1} & \to & Z_n & \xrightarrow{\alpha_n} & X & \to 0 \\ & & \| & & \cdots & \| & & \uparrow{\scriptstyle \text{pullback}} & & \uparrow f & \\ E' \equiv & 0 \to & Y & \to & \cdots & \to Z_{n-1} & \to & Z_n \times_X X' & \xrightarrow{\alpha_n'} & X' & \to 0 \end{array}$$

$\ker \alpha_n = \ker \alpha_n'$, so rest stays the same

Push-forward:

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r] & Y' \oplus_{Y} Z_1 \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & X \arrow[r] & 0 \\
0 \arrow[r] & Y \arrow[r] \arrow[u, "g"] & Z_1 \arrow[r] \arrow[u, "\text{push out}"] & Z_2 \arrow[r] \arrow[u, equal] & \cdots \arrow[r] & X \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Now given two n-extensions E, E' of X by Y:
$E + E' := \Delta^Y_* \Delta_X^* (E \oplus E')$ is again an n-extension of X by Y, the "Baer sum"
$\Delta_X : X \to X \oplus X$, $\Delta^Y : Y \oplus Y \to Y$

Exercise. $g_* f^* E \cong f^* g_* E$ (meaning chain map of isos everywhere)
$E + E' \cong E' + E$
$(E + E') + E'' \cong E + (E' + E'')$

Equivalence relation: E equiv. to E':

$$\begin{tikzcd}
E \equiv 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & \cdots \arrow[r] \arrow[d, "\gamma_1"] & X \arrow[r] \arrow[d, equal] & 0 \\
E' \equiv 0 \arrow[r] & Y \arrow[r] & \cdots \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

γ_i's any morphisms such that all squares commute

NB. Unlike 1-extension case, γ_i's need not be isos!
This means that this relation is not symmetric, i.e. not an equivalence relation.

We set the equivalence relation to be the unique equivalence relation generated by the above relation.

Thm. (Yoneda) $\mathrm{Ext}^n_{\underline{A}}(X, Y) =$ n-extensions of X by Y / equivalence is an abelian group with addition induced by Baer sums and is functorial in each argument. The zero object is represented by any extension $0 \to Y \xrightarrow{\alpha_1} Z_1 \to \cdots \to Z_n \xrightarrow{\alpha_n} X \to 0$ with either α_1 a split mono or α_n a split epi. It is half-exact in each object and for any $X \in \underline{A}$ and any short exact sequence $0 \to A \to B \to C \to 0$, there is a long exact sequence

$0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C) \to \mathrm{Ext}^1(X,A) \to \mathrm{Ext}^1(X,B) \to \mathrm{Ext}^1(X,C) \to$
$\to \mathrm{Ext}^2(X,A) \to \mathrm{Ext}^2(X,B) \to \mathrm{Ext}^2(X,C) \to \cdots$

N.B. The connecting map $\mathrm{Ext}^n(X,C) \to \mathrm{Ext}^{n+1}(X,A)$ is given by

$$
\begin{tikzcd}
0 \arrow[dr] \\
& A \arrow[dr] \\
& & B \arrow[dr] \arrow[drr, bend left] \\
& & 0 \arrow[r] & C \arrow[r] \arrow[dr] & Z_1 \arrow[r] & \cdots \arrow[r] & Z_n \arrow[r] & X \arrow[r] & 0 \\
& & & & 0
\end{tikzcd}
$$

$0 \to A \to B \to Z_1 \to \cdots \to Z_n \to X \to 0 \leftarrow$ Concatenation

For any $Y \in \underline{A}$ and $0 \to A \to B \to C \to 0$, one has long exact sequence

$$0 \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(B,Y) \to \mathrm{Hom}(A,Y) \to$$
$$\to \mathrm{Ext}^1(C,Y) \to \mathrm{Ext}^1(B,Y) \to \mathrm{Ext}^1(A,Y) \to$$
$$\cdots$$
$$\to \mathrm{Ext}^n(C,Y) \to \mathrm{Ext}^n(B,Y) \to \mathrm{Ext}^n(A,Y) \to$$
$$\to \mathrm{Ext}^{n+1}(C,Y) \to \cdots$$

The connecting homomorphisms work the same way as above

Def. $\underline{A}$ an abelian category. The global dimension $\mathrm{gl\,dim}\, \underline{A}$ is $\max\{ i \mid \mathrm{Ext}^i_{\underline{A}}(?,??)\}$ is not identically zero.
This may be infinite.

E.g. If $\underline{A}$ is the category of vector spaces over a field, then $\mathrm{gl\,dim}\, \underline{A} = 0$. Such a category is called semi-simple.

E.g. Let k be a field, $n \geq 1$ a natural number, $\underline{A} = \text{Mod-}k^{n\times n}$
Then $\mathrm{gl\,dim}\, \underline{A} = 0$ (This is the basis of quantum mechanics)
This is a consequence of Morita equivalence, i.e. $\text{Mod-}k^{n\times n}$ is equivalent to $\text{Mod-}k$.

E.g. If $R = k[x_1, \ldots, x_n]$, then $\mathrm{gl\,dim}\, \text{Mod-}R = n$. This is Hilbert's syzygy theorem.
The same holds for formal power series and convergent power series.

E.g. $q: V \to k$ a non-degenerate quadratic form, then the global dimension of the Clifford algebra of $q \cong \mathrm{End}_k(\wedge V)$ is 0.

E.g. G a finite group, k a field of characteristic 0, $\mathrm{gl\,dim}(\mathrm{Mod}\text{-}kG) = 0$. This is the content of Maschke's theorem: Every exact sequence of G-representations splits

Categories of gl dim 1: "hereditary categories"
Reiter - van der Bergh JAMS 2002: Classify all these that are noetherian, satisfy Serre duality, and are Ext-finite over some field k

Ext-finite: $\dim_k \mathrm{Ext}^i_{\underline{A}}(X,Y) < \infty$ for any X, Y

Serre duality: The functor $\mathrm{Ext}^1_{\underline{A}}(?,Y)^* : \underline{A} \to k\text{-v.s.}$ for each $Y \in \underline{A}$, $*$ - k-dual, is representable

Thm. If R is a PID, then $\mathrm{gl.dim}\ \mathrm{Mod}\text{-}R = 1$ (if R is not a field).

E.g. Q quiver $(\underset{\text{vertices}}{Q_0}, \underset{\text{arrows}}{Q_1})$ k a field: $\mathrm{gl\,dim}(\mathrm{Rep}(Q,k)) \leq 1$.
If Q is n loops on one element: $\mathrm{Rep}(Q,k) \cong \mathrm{Mod}(k\langle x_1, \ldots, x_n\rangle)$

If $\underline{B} \subseteq \underline{A}$ is a full abelian subcategory closed under extensions, then $\mathrm{gl\,dim}\ \underline{B} \leq \mathrm{gl\,dim}\ \underline{A}$.

V, W two representations of a quiver Q, $\mathrm{Ext}^1_{\mathrm{Rep}(Q,k)}(V,W)$: A morphism $a: V \to W$ is mono/epi/iso iff each a_i is mono/epi/iso.

An extension $E: 0 \to W \to X \to V \to 0$ in $\mathrm{Rep}(Q,k)$:
E exact $\Leftrightarrow$ corresponding sequences of vector spaces are exact
$\Rightarrow X_i \cong V_i \oplus W_i$ for each i
But isomorphism need not commute with arrows

1-extensions $\overset{1:1}{\longleftrightarrow} \prod_{(\to)_i} \mathrm{Hom}_k(V_i, W_j)$

Prop. Given $V, W \in \mathrm{Rep}(Q,k)$, there is an exact sequence of k-v.s.

$$0 \to \mathrm{Hom}_{\mathrm{Rep}(Q,k)}(V,W) \to \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i) \xrightarrow{d_W^V} \bigoplus_{i \xrightarrow{\alpha} j} \mathrm{Hom}_k(V_i, W_j) \to \mathrm{Ext}^1_{\mathrm{Rep}(Q,k)}(V,W) \to 0$$

$$(\tilde{\varphi}_i) \longmapsto (\tilde{\varphi}_j V_\alpha - W_\alpha \tilde{\varphi}_i)_\alpha$$

In particular, $\dim_k \mathrm{Ext}^1_{\mathrm{Rep}(Q,k)}(V,W) = \sum_{i \xrightarrow{\alpha} j} \dim_k V_i \cdot \dim_k W_j - \sum_{i \in Q_0} \dim_k V_i \dim_k W_i$

$$+ \dim_k \mathrm{Hom}_{\mathrm{Rep}(Q,k)}(V,W)$$

This gives a quadratic form on $\mathbb{Z}^{Q_0}$: $x, y \in \mathbb{Z}^{Q_0}$

$$\langle x,y \rangle := \sum_{i \in Q_0} x_i y_i - \sum_{i \xrightarrow{\alpha} j} x_i y_j, \quad q(x) = \langle x,x \rangle$$

This is the Tits form.

Ringel form: bilinear form associated to q: $b_q(x,y) = \langle x,y \rangle + \langle y,x \rangle$

The matrix of the Ringel form in the standard basis of $\mathbb{Z}^{Q_0}$ is the Cartan matrix of Q.

If $\langle d_V, d_W \rangle < 0$, then $\mathrm{Ext}^1_{\mathrm{Rep}(Q,k)}(V,W) \neq 0$.

If $V = W$: $q(d_V) = \langle d_V, d_V \rangle \leq 0$, then $\mathrm{Ext}^1_{\mathrm{Rep}(Q,k)}(V,V) \neq 0$.

Rmk. If q is positive-definite and Q is connected, then

$|Q| =$

$|Q|$ is the underlying graph

If q is semi positive-definite, then the above graphs and their completed counterparts are permitted.

PID: Structure theorem

Any submodule of a free R-module (R a PID) is free.

Lemma. If R is any ring, then $\mathrm{Ext}^n_{\text{Mod-}R}(R, ?) = 0$ for $n \geq 1$

(Any n-extension of R by X splits)

M, N R-modules, M f.g.

$\mathrm{Ext}^i_R(M,N)$: $0 \to R^n \xrightarrow{A} R^m \to M \to 0$

long exact sequence: $0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(R^m, N) \to \mathrm{Hom}_R(R^n, N) \to$

$$\to \mathrm{Ext}^1_R(M,N) \to \underset{\substack{\parallel \\ 0}}{\mathrm{Ext}^1_R(R^m, N)} \to \cdots$$

$\therefore \mathrm{Ext}^i_R(M,N) = 0$ for $i > 1$

and $\mathrm{Ext}^1_R(M,N) = \mathrm{coker}\ \mathrm{Hom}_R(R^m, N) \to \mathrm{Hom}_R(R^n, N)$

$$N^m \xrightarrow{A^T_N} N^n$$

E.g. Hilbert's syzygy thm.

If P is a direct summand of some $R^{(I)}$ (free module), then $\mathrm{Ext}^i_R(P, ?) = 0$ for $i > 0$. Such a P is called a <u>projective</u> R-module.

R a ring, M an R-module.

$\cdots \to R^{(I_n)} \to \cdots \to R^{(I_1)} \to R^{(I)} \to M \to 0$ free resolution of M

$K_1 \leftarrow$ 1st syzygy module

If K_n is projective for some n, then resolution may stop there, since all we need is a projective resolution.

Then $\mathrm{Ext}^i_R(M,N) = 0$ for all $i > 0$ and all N

Hilbert proved that for $R = k[x_1, \ldots, x_n]$, K_n is always projective (and therefore free by a result proved in 1970s).

gl dim $R \leq n \iff$ each chain of syzygies terminates w/ projective after $\leq n$ steps