No class 25 April

Role of injective modules in commutative algebra: website (send email to ragnar@math for link, or Google snowbird + algebra).

Problem with computing Ext's from projective and injective resolutions: too many choices - how to know which is best?
This is the basis of derived categories: make rigorous the dependence or independence of derived functors on resolutions

Complexes: If $(\mathcal{C}, *)$ is a category with zero object $*$, then a sequence of morphisms

$$\cdots \longrightarrow X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X_{-1} \longrightarrow \cdots$$

is a complex if $f_i \circ f_{i+1} = 0$ for all i
The complex is bounded below if $X_i = *$ for all $i \ll 0$
It is bounded above if $X_i = *$ for all $i \gg 0$
It is bounded if it is bounded below and above
(Note: This is homological notation, where maps decrease indices. In cohomological notation, maps raise indices and indices are written as superscripts)
The f_i are called the differentials of the complex
Better distinction would be: bounded in the direction of the differential (BID) vs. bounded against the direction of the differential (BAD)

A morphism of complexes is a morphism between each pair of objects making all squares commute:

$$\begin{array}{ll} X_\bullet \equiv & \cdots \longrightarrow X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} X_{-1} \longrightarrow \\ & \qquad\quad \uparrow h_1 \quad\ \uparrow h_0 \quad\ \uparrow h_{-1} \\ Y_\bullet \equiv & \cdots \longrightarrow Y_1 \xrightarrow[g_1]{} Y_0 \xrightarrow[g_0]{} Y_{-1} \longrightarrow \end{array}$$

Denote by $C(\underline{C}, *)$ the category of complexes with morphisms as described above

$C^-(\underline{C}, *)$ complexes BID

$C^+(\underline{C}, *)$ complexes BAD

$C^b(\underline{C}, *)$ bounded complexes

If $\underline{A}$ is abelian, then $f_i f_{i+1} = 0 \iff \operatorname{Im} f_{i+1} \subseteq \ker f_i$

Thus one can form $H_i(X_\bullet) = \ker f_i / \operatorname{im} f_{i+1} \in \underline{A}$, the i^{th} homology of $X_\bullet$

Rmk. $H^i(X_\bullet) := H_{-i}(X_\bullet)$ is the i^{th} cohomology

If $(X^\bullet, d)$ is a complex over $\underline{A}$, then one defines the translate $X^\bullet[1]$ as

$$X^\bullet \quad \cdots \longrightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \longrightarrow \cdots$$
$$X^\bullet[1] \quad \cdots \longrightarrow X^0 \xrightarrow{-d^0} X^1 \xrightarrow{-d^1} X^2 \longrightarrow \cdots$$

$$X^\bullet[1]^n = X^{n+1}, \quad d_{X^\bullet[1]} = -d_{X^\bullet}$$

$$H^i(X^\bullet) = H^0(X^\bullet[i])$$

$H^0 : C(\underline{A}) \longrightarrow \underline{A}$ is a functor

$$\begin{array}{lcc} X^\bullet & & H^i(X^\bullet) \\ \uparrow f^\bullet \text{ morphism of complexes} & \text{induces} & \uparrow H^i(f^\bullet) \\ Y^\bullet & & H^i(Y^\bullet) \end{array}$$

$T := ?[1] : C(\underline{A}) \longrightarrow C(\underline{A})$ is an equivalence of categories

$T^i := \underbrace{T \circ T \circ \cdots \circ T}_{i \text{ times}} =: ?[i]$

Given complexes X and Y, a homotopy from Y to X is a family of maps h^i:

$$\begin{tikzcd} \cdots \arrow[r] & X^{-1} \arrow[r] & X^0 \arrow[r] & X^1 \arrow[r] & \cdots \\ \arrow[r] & Y^{-1} \arrow[r] & Y^0 \arrow[r] \arrow[ul, "h^0"] & Y^1 \arrow[r] \arrow[ul, "h^1"] & \cdots \end{tikzcd}$$

NB Commutativity not required

Lemma. If $f: Y \to X$ is a morphism of complexes, h a homotopy, then $f^i + h^{i+1} d_Y + d_X h^i =: \tilde{f}^i$ is again a morphism of complexes and $H^\bullet(f) = H^\bullet(\tilde{f})$

Assume $y \in Y^i$ is a cycle, i.e. $d_Y y = 0$.

$$\tilde{f}(y) = f(y) + 0 + \underbrace{d_X h^i(y)}_{\text{a boundary}}$$

Thus in $H(X)$, $\tilde{f}(y)$ and $f(y)$ define the same class

Given two complexes $X, Y \in A$, one can form the Hom-complex (of Abelian groups or k-modules):

$$\mathrm{Hom}^i(Y^\bullet, X^\bullet) = \underbrace{\prod_{j \in \mathbb{Z}} \mathrm{Hom}_A(Y^j, X^{i+j})}_{\text{families of morphisms}}, \quad i \in \mathbb{Z}$$

Define $\underset{\mathrm{Hom}^i(X^\bullet, Y^\bullet)}{D((h^j)_{j\in\mathbb{Z}})} = (d_X h^j - (-1)^i h^{j+1} d_Y)_j \in \prod_j \mathrm{Hom}_A(Y^j, X^{j+i+1})$

$$\mathrm{Ker}\, D^0 = \{(h^j : Y^j \to X^j) \mid d_X h^j - h^{j+1} d_Y = 0\}$$
$$= \{\text{morphisms of complexes}\}$$

$\mathrm{Hom}^{-1}(Y, X) =$ homotopies

$f \in \mathrm{Hom}^0(Y, X)$, $h \in \mathrm{Hom}^{-1}(Y, X)$

$f + D(h) = f + h d_X + d_Y h$

Thus, $H^0(\mathrm{Hom}(Y, X)) = \{\text{morphisms of complexes / homotopies}\}$
$= \text{homotopy classes of morphisms of complexes}$

Notation: $D = [d, -]$; $D(h) = [d, h] = d_X h - (-1)^{\deg h} h d_Y$

To show $\tilde{f} = f + [d, h]$ is a morphism of complexes, we must show $[d, \tilde{f}] = 0$:

$$[d, \tilde{f}] = \underset{\substack{0 \text{ as } f \text{ is} \\ \text{a morphism of} \\ \text{complexes}}}{[d, f]} + [d, [d, h]] = d \circ [d, h] - (-1)^{\deg [d,h]} [d, h] \circ d$$
$$= d \circ (d h - (-1)^{\deg h} h d) - (-1)^{\deg h + 1} (d h - (-1)^{\deg h} h d) \circ d$$
$$= 0$$

Cor. $D^2 = [d, [d, ?]] = 0$

Homotopy category $K^*(\underline{A})$:

Objects are $*$-complexes

Morphisms: $\mathrm{Hom}_{K(\underline{A})}(Y, X) := H^0(\mathrm{Hom}^\bullet(Y, X))$

Note: • $K(\underline{A})$ is additive but not abelian

• The translation functor descends to an auto-equivalence

$$T: K(\underline{A}) \to K(\underline{A})$$
$$X \longmapsto X[1]$$
$$\mathrm{Hom}^\bullet(Y, X) \cong \mathrm{Hom}^\bullet(Y[1], X[1])$$

A morphism $f: Y^\bullet \to X^\bullet$ of complexes is a <u>quasi-isomorphism</u> (or <u>homologism</u>) if $H^i(f): H^i(X) \to H^i(Y)$ is an isomorphism for each i.

E.g. $\underline{A} = Ab$

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 & \cong \mathbb{Z}/2[0] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[u] & \mathbb{Z} \arrow[r] \arrow[u] & 0 \arrow[u, equal] \end{tikzcd}$$

$-1 \qquad 0 \qquad H=\mathbb{Z}/2$

This is a quasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r, "2"'] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$H=\mathbb{Z}/2 \qquad H=\mathbb{Z}/2 \qquad H=0$

Also a quasi-isomorphism

Composition: projective resolution $\to$ object $\to$ injective resolution is also a quasi-isomorphism

It cannot be inverted (cannot go from injective resolutions to projective resolutions), so we do not call it isomorphism

NB Having isomorphic cohomology does not imply being quasi-isomorphic

E.g. $0 \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to 0$

$0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to 0$

X, Y are quasi-isomorphic if

$$\begin{tikzcd} & Z_1 \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & & \cdots \arrow[dl, "\cong"'] & Z_n \arrow[dr, "\cong"] & \\ X & & Z_2 & & & Y \end{tikzcd}$$

$\cong$ is quism

Given $X \in K(\underline{A})$, define $Quism/X$:

Objects $Y \underset{f}{\cong} X$ f quism

Morphisms
$$\begin{tikzcd} Y \arrow[r, "\cong"] & X \\ Y' \arrow[u, "g"] \arrow[ur, "\cong"', "f'"] & \end{tikzcd}$$
g represents a morphism $f' \to f$

Prop. If X admits a projective resolution
($:= X$ is quasi-isomorphic to a complex $\mathbb{P} \in K^-(\underline{A})$ all of whose terms are projective objects from $\underline{A}$), then

(1) All projective resolutions of X are isomorphic in $K(\underline{A})$

(2) Each of these is an initial object in $Quism/X$.

(1½) If $\mathbb{P}$ is a projective resolution of X, then $\exists$ morphism of cxs $\mathbb{P} \to X$ which is a quism

Dual statement holds for injective resolutions

If $f: Y \to \mathbb{P}$ is a quasi-isomorphism, then $\exists$ $g: \mathbb{P} \to Y$ such that $f \circ g = id_{\mathbb{P}}$ in $K(\underline{A})$

This follows from the following: If $h: \mathbb{P} \to C$ is a morphism of complexes and C is acyclic (i.e. $H^*(C) = 0$), then h is homotopic to 0.

WLOG, $\mathbb{P}$ stops at degree 0.

$$\begin{tikzcd} \cdots \arrow[r] & C^{-2} \arrow[r] & C^{-1} \arrow[r] & C^0 \arrow[r] & C^1 \arrow[r] & \cdots \\ \cdots \arrow[r] & P^{-2} \arrow[r] \arrow[u, "h^{-2}"] \arrow[ul, "H^{-2}"] & P^{-1} \arrow[r] \arrow[u, "h^{-1}"] \arrow[ul, "H^{-1}"] & P^0 \arrow[r] \arrow[u, "h^0"] \arrow[ul, "H^0"] & 0 \arrow[ul, "H^1"] & \end{tikzcd}$$

h is a morphism of complexes, $H^*(C) = 0$, P^i is projective

$\exists$ morphisms $H^i: P^i \to C^{i-1}$ such that $h = [d, H]$

$H^i = 0$ for $i > 0$ - clear

Want: $\exists!\, H^0 : P^0 \to C^{-1}$; $d_C H^0 = h^0$

$H^0(C) = 0$: $\operatorname{Ker} d^0 = \operatorname{Im} d^{-1}$

$$\begin{tikzcd}
0 \arrow[r] & \ker d^0 \arrow[r] & C^0 \arrow[r, "d^0"] & C^1 \\
& & P^0 \arrow[ul, dashed, "\exists!"] \arrow[u, "h^0"] \arrow[ur, "0"'] \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
& C^{-1} \arrow[r, "d^{-1}"] \arrow[dr] & C^0 \arrow[r] & C^1 \\
& & \ker d^0 \arrow[u] & \\
0 \arrow[urr] & & P^0 \arrow[u, dashed] \arrow[uu, bend right, "h^0"'] \arrow[uul, dashed, "\exists!\, H^0"] \arrow[r] & 0 \arrow[uu]
\end{tikzcd}$$

Choose $H^0 : P^0 \to C^{-1}$ such that $d^{-1} H^0 = h^0$

Thus $h^0 = 0 + dH^0 + H^{-1}d$

Replace h by h': $h'^0 = 0$, $(h')^{-1} = h^{-1} - H^0 d^{-1}$, $(h')^i = h^i$ for $i \leq -2$

Then the same argument as above applies

"Horseshoe construction":

Assume $\underline{A}$ is an abelian category with enough projectives. Assume given a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\underline{A}$ together with 2 out of 3 projective resolutions. Then one can produce a projective resolution for the third one as follows:

$$\begin{tikzcd}
& 0 & & 0 & \\
0 \arrow[r] & X \arrow[r, "a"] \arrow[u] & Y \arrow[r, "b"] & Z \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[u, "p"] & P_0 \oplus Q_0 \arrow[r] \arrow[u, dashed, "ap+\beta"] & Q_0 \arrow[r] \arrow[u, "q"] \arrow[ul, dashed, "\exists\beta"'] & 0 \\
& \Omega' X \arrow[u] & K \arrow[u] & \Omega' Z \arrow[u] & \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
& Y & \\
& P_0 \oplus Q_0 \arrow[u, "ap"] \arrow[u, shift left=2, "\beta"'] &
\end{tikzcd}$$

β – as Q_0 projective and b epi

Properties: • $r := ap + \beta$ is an epimorphism

$\exists\, u \in Q_0 \ni q(u) = b_y \Rightarrow y - r(0,u) = pav \Rightarrow y = r(v,u)$

• The kernels form an exact sequence

Thus we may replace first sequence with $0 \to \Omega' X \to K \to \Omega' Z \to 0$ and repeat

Morphisms of resolution:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & X \arrow[r] \arrow[u] & Y \arrow[r] \arrow[u] & Z \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^0 \arrow[r] \arrow[u] & P^0 \oplus Q^0 \arrow[r] \arrow[u] & Q^0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^{-1} \arrow[r] \arrow[u] & P^{-1} \oplus Q^{-1} \arrow[r] \arrow[u] & Q^{-1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^{-2} \arrow[r] \arrow[u] & P^{-2} \oplus Q^{-2} \arrow[r] \arrow[u] & Q^{-2} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$$\begin{array}{c|cc} & P^{-1} & Q^{-1} \\ \hline P^0 & \multicolumn{2}{c}{\begin{pmatrix} d_X & \beta^{-1} \\ 0 & d_Z \end{pmatrix}} \\ Q^0 & & \end{array}$$

$$\begin{array}{c|cc} & P^{-2} & Q^{-2} \\ \hline P^{-1} & \multicolumn{2}{c}{\begin{pmatrix} d_X & \beta^{-2} \\ 0 & d_Z \end{pmatrix}} \\ Q^{-1} & & \end{array}$$

Claim: $(P^i \oplus Q^i \longrightarrow P^{i+1} \oplus Q^{i+1})$ is a projective resolution of Y

$$\begin{pmatrix} d_X & \beta^i \\ 0 & d_Z \end{pmatrix} \begin{pmatrix} d_X & \beta^{i+1} \\ 0 & d_Z \end{pmatrix} = \begin{pmatrix} d_X^2 = 0 & d_X\beta^{i+1} + \beta^i d_Z \\ 0 & d_Z^2 = 0 \end{pmatrix}$$

$d_X \beta^{i+1} + \beta^i d_Z = 0$: $(\beta^i) \in Hom(Q, P[1])$ is a cycle

Claim: The class of $(\beta^i) \in H^1(Hom(Q^\bullet, P^\bullet))$ is independent of the choices; it only depends on the given exact sequence

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "a"] & Y \arrow[r, "b"] & Z \arrow[r] & 0 \\
\text{proj. res.} & \mathbb{P} \arrow[r, "\tilde{a}"] \arrow[u] & \mathbb{R} \arrow[r, "\tilde{b}"] \arrow[u] & \mathbb{Q} \arrow[r, "\beta"] \arrow[u] & \mathbb{P}[1]
\end{tikzcd}$$

Morale: A short exact sequence of objects produces a "triangle" of complexes

Mapping cone and cylinder:

Start with a morphism $f: C \to D$ of complexes

The mapping cone over f $Cone(f)$ (or $C(f)$) is given by

$$\begin{tikzcd}
\vdots & \vdots \\
C^1 \arrow[r, "f^1"] \arrow[u] & D^1 \arrow[u] \\
C^0 \arrow[r, "f^0"] \arrow[u] & D^0 \arrow[u] \\
C^{-1} \arrow[r, "f^{-1}"] \arrow[u] & D^{-1} \arrow[u] \\
\vdots \arrow[u] & \vdots \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
C(f)^1 = D^1 \oplus C^2 \\
C(f)^0 = D^0 \oplus C^1 \arrow[u] \\
C(f)^{-1} = D^{-1} \oplus C^0 \arrow[u]
\end{tikzcd}$$

$$\begin{array}{c|cc} & D^0 & C^1 \\ \hline D^1 & \multicolumn{2}{c}{\begin{pmatrix} d_D & f^1 \\ 0 & -d_C \end{pmatrix}} \\ C^2 & & \end{array} =: d_{C(f)}$$

$$d_{C(f)}^2 = 0$$

There is a canonical exact sequence of complexes:

$$0 \to D^{\bullet} \xrightarrow{in_1} C(f) \xrightarrow{pr_2} C^{\bullet}[1] \to 0$$

In each degree, the resulting sequence is exact
in_1, pr_2 are morphisms of complexes

In the above situation: $\mathbb{Q} \xrightarrow{\beta} \mathbb{P}[1]$

$$0 \to \mathbb{P}[1] \to C(\beta) \to \mathbb{Q}[1] \to 0$$

$$\begin{pmatrix} d_{\mathbb{P}[1]} & -\beta \\ 0 & -d_{\mathbb{Q}} \end{pmatrix} = \begin{pmatrix} -d_{\mathbb{P}} & -\beta \\ 0 & -d_{\mathbb{Q}} \end{pmatrix}$$

So $C(\beta) = \mathbb{R}[1]$

Given $C \xrightarrow{f} D$:

$$\begin{tikzcd}
 & 0 \arrow[r] & D \arrow[r] & C(f) \arrow[r] & C[1] \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[ur, "f"] \arrow[r, "in_1"] & Cyl(f) \arrow[u] \arrow[r, "pr_2"] & C(f) \arrow[u, equal] \arrow[r] & 0
\end{tikzcd}$$

$Cyl(f) = (C \oplus D \oplus C[1])$, $C(f) = (D \oplus C[1], d_{C(f)})$

$d_{Cyl(f)}$ is exercise

Formalization: $\mathcal{D}$ is a triangulated category if
- additive
- $\exists$ translation auto-equivalence
- family of "distinguished triangles" $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$
  * $Y \xrightarrow{-v} Z \xrightarrow{-w} X[1] \xrightarrow{-u[1]} Y[1]$ is again distinguished
  * Every morphism $f: X \to Y$ embeds into a distinguished triangle (mapping cone construction)
  * $$\begin{tikzcd}
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1] \\
X' \arrow[r] \arrow[u, "f"] & Y' \arrow[r] \arrow[u, "g"] & Z \arrow[r] \arrow[u, "\exists h"] & X[1] \arrow[u, "f[1]"]
\end{tikzcd}$$

    Extending morphisms of triangles
  * Octahedral axiom (dealt with next time)