MAT 1103 - Introduction to homological algebra 18.4

No class Monday, 25 April
Presentations start Monday, 2 May

Diagram Lemmata

① 5 Lemma A_i, B_i in an abelian category

$$\begin{tikzcd}
A_0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & A_4 \\
B_0 \arrow[r] \arrow[u, "f_0"] & B_1 \arrow[r] \arrow[u, "f_1"] & B_2 \arrow[r] \arrow[u, "f_2"] & B_3 \arrow[r] \arrow[u, "f_3"] & B_4 \arrow[u, "f_4"]
\end{tikzcd}$$

Top, bottom are exact
All squares commute

For simplicity, we assume that we have elements with which to work

If f_1, f_3 are mono and f_0 is epi, then f_2 is mono.
Dually, if f_1, f_3 are epi and f_4 is mono, then f_2 is epi.
Conclusion: If f_1, f_3 are isos, f_0 is epi and f_4 is mono, then f_2 is iso.

Special case:

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r] & A_3 \arrow[r] & 0 \\
0 \arrow[r] \arrow[u, equal] & B_1 \arrow[r] \arrow[u, "f_1"] & B_2 \arrow[r] \arrow[u, "f_2"] & B_3 \arrow[r] \arrow[u, "f_3"] & 0 \arrow[u, equal]
\end{tikzcd}$$

If f_1, f_3 are mono, then f_2 is mono
Similarly for epi, iso.

② Snake lemma (reformulation in view of result of Beilinson, Bernstein, Deligne)

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \ker v'' \arrow[r, "\exists!"] \arrow[d] & \ker v \arrow[r, "\exists!"] \arrow[d] & \ker v' & \\
& A \arrow[r, "u"] \arrow[d, dashed, "\exists! v''"] & B \arrow[r] \arrow[d, "v"] & \operatorname{coker} u \arrow[r] \arrow[d, dashed, "\exists! v'"] \arrow[u] & 0 \\
0 \arrow[r] & \ker w \arrow[r] \arrow[d] & C \arrow[r, "w"] \arrow[d] & D \arrow[d] & \\
& \operatorname{coker} v'' \arrow[r, "\exists!"] \arrow[d] & \operatorname{coker} v \arrow[r, "\exists!"] \arrow[d] & \operatorname{coker} v' \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}$$

$\exists \delta$

$wvu = 0$

$\exists! v'$ (universal property of cokernel)

Snake lemma: Assume $wvu = 0$ in an abelian category. Then $\exists$ $\delta: \ker v' \to \operatorname{coker} v''$ $\ni$

$\ker v'' \to \ker v \to \ker v' \xrightarrow{\delta} \operatorname{coker} v'' \to \operatorname{coker} v \to \operatorname{coker} v'$

is exact (δ: connecting morphism)

Another variant: If u is mono, then $\ker v'' \to \ker v$ is mono, too, and if w is epi, then $\operatorname{coker} v \to \operatorname{coker} v''$ is epi too

Formulation of δ:

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & z \arrow[r, mapsto] \arrow[d, mapsto] & \ker v' \ni x \arrow[d] & \\
A \arrow[r, "u"] \arrow[d, "v'"] & B \arrow[r] \arrow[d, "v"] & \operatorname{coker} u \ni y \arrow[r] \arrow[d, "v''"] & 0 \\
a \in \ker w \arrow[r] \arrow[d, mapsto] & C \ni v(z) \arrow[r, "w"] & D & \\
\delta(x) := b \in \operatorname{coker} v'' \arrow[d] & & & \\
0 & & &
\end{tikzcd}$$

δ well-defined:
If $z' \mapsto y$, then $z - z' \in \operatorname{coker} u$, so $\exists\ \alpha \in A \ni u\alpha = z - z'$. But then $v'\alpha \mapsto 0 \in \operatorname{coker} v''$.

Exactness: $\delta(x) \in \operatorname{coker} v'' \mapsto 0 \in \operatorname{coker} v$ because $v(z) \mapsto 0$

If $b \in \operatorname{coker} v'' \mapsto 0 \in \operatorname{coker} v$, then run the diagram chase backwards to get $x \in \ker v'$: $\delta(x) = b$

Exactness at $\ker v'$ is dual to above

"Corollary": If $0 \to C'' \xrightarrow{f} C \xrightarrow{g} C' \to 0$ is a short exact sequence of complexes over some abelian category, then $\exists\ \delta: H(C') \to H(C''[1])$ such that
$$H^i(C'') \xrightarrow{H^i(f)} H^i(C) \xrightarrow{H^i(g)} H^i(C') \xrightarrow{\delta^i} H^{i+1}(C'') \xrightarrow{H^{i+1}(f)} H^{i+1}(C) \xrightarrow{H^{i+1}(g)} H^{i+1}(C')$$
is an exact sequence in the given abelian category.

This is the long exact sequence in cohomology defined by a short exact sequence of complexes

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & Z_i(d'') \arrow[r] \arrow[d] & Z_i(d) \arrow[r] \arrow[d] & Z_i(d') \arrow[d] & \\
0 \arrow[r] & C_i'' \arrow[r] \arrow[d, "d''"] & C_i \arrow[r] \arrow[d, "d"] & C_i' \arrow[r] \arrow[d, "d'"] & 0 \\
0 \arrow[r] & C_{i+1}'' \arrow[r] \arrow[d] & C_{i+1} \arrow[r] \arrow[d] & C_{i+1}' \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} d'' \arrow[r] \arrow[d] & \operatorname{coker} d \arrow[r] \arrow[d] & \operatorname{coker} d' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

δ from snake lemma

Claim: $H_{i+1}(C'') \subseteq \operatorname{coker} d''$

$\delta(Z_i(d')) \subseteq H_{i+1}(C'')$

$\delta(B_i(d')) = 0$

$\Rightarrow \delta$ induces a morphism $H_i(C') \to H_{i+1}(C'')$

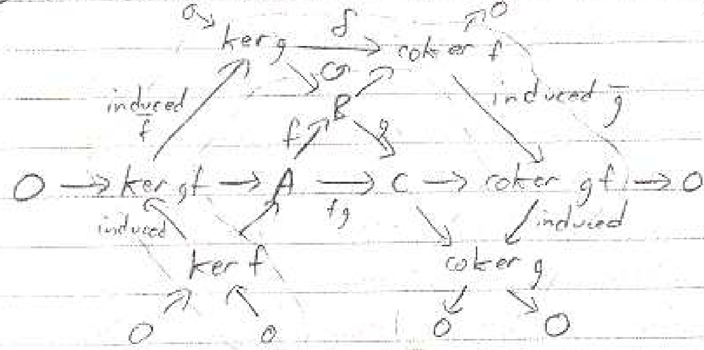
To see $H_{i+1}(C'') \subseteq \operatorname{coker} d''$:

$d''d'' = 0$ as C'' is a complex

$\Rightarrow \exists!\ \tilde{d}'' : \operatorname{coker} d'' \longrightarrow C''_{i+1}$

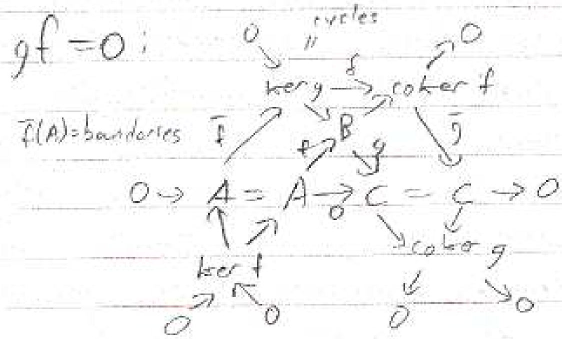
$\rightsquigarrow \ker \tilde{d}'' \cong H_{i+1}(C'')$

③ ker-coker lemma $A \xrightarrow{f} B \xrightarrow{g} C$ morphisms in ab. cat.



The sequence $0 \to \ker f \to \ker gf \to \ker g \to \operatorname{coker} f \to \operatorname{coker} gf \to \operatorname{coker} g \to 0$ is exact

Example: $gf = 0$:



$\ker g =$ cycles at B

$\bar{f}A =$ boundaries at B

$\ker g / \bar{f}A \cong$ Homology at B

$\cong \operatorname{Im} \delta$

$\cong \ker \bar{g} \subseteq \operatorname{coker} f$

Exercise V vector space over a field K. $\sigma : V \otimes V \to V \otimes V$ the switch $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$

(1) $(id - \sigma) \circ (id + \sigma) = 0 = (id + \sigma)(id - \sigma)$; (2) Interpret ker-coker for either composition

$$
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[d] & & & 0 & & \\
 & \mathrm{Alt}^2 V \arrow[r, hook] & \text{Symmetric 2-tensors} \arrow[rr, "\delta"] \arrow[dr] & & \text{Skew-symmetric 2-tensors} \arrow[ur] \arrow[drr] \arrow[r] & \Lambda^2 V = V\otimes V/\langle v\otimes v;\ v\in V\rangle \arrow[d, leftrightarrow] & & \\
 & & & V\otimes V \arrow[ur] \arrow[dr, "id-\sigma"] & & & & \\
0 \arrow[r] & V\otimes V \arrow[uu, Rightarrow, "\text{symmetrization}"] \arrow[uur] \arrow[r] & V\otimes V \arrow[ur, "id+\sigma"] \arrow[rr, "0"'] & & V\otimes V \arrow[r] \arrow[dr] & V\otimes V \arrow[r] \arrow[d] & 0 & \\
 & & \mathrm{Anti}^2(V) \arrow[ul] \arrow[u] & & & \mathrm{Sym}^2(V) = V\otimes V/\langle v_1\otimes v_2 - v_2\otimes v_1,\ v_1, v_2\in V\rangle \arrow[dl] \arrow[dr] & & \\
 & 0 \arrow[ur] & 0 \arrow[u] & & 0 & & 0 &
\end{tikzcd}
$$

$$\ker(id+\sigma) = \{\sum_i x_i\otimes y_i + \sum_i y_i\otimes x_i = 0\}$$
$$= \{\text{antisymmetric tensors}\}$$

$\exists$ sequence $0 \to \Lambda^2 V \to V\otimes V \to \mathrm{Sym}^2 V \to 0$

$$v_1\wedge v_2 \longmapsto v_1\otimes v_2 - v_2\otimes v_1$$

$\mathrm{Im}\,\delta = \{v\otimes v \mid 2v\otimes v = 0\} = H^1(\mu_2; V\otimes V)$

<u>Explicit connecting homomorphism:</u>

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \ker v' \arrow[d, "i'"] & \\
0 \arrow[r] & A \arrow[r, "u"] \arrow[d, "v''"] & B \arrow[r, "u'"] \arrow[d, "v"] \arrow[l, bend right, "r"'] & \mathrm{coker}\, u \arrow[r] \arrow[d, "v'"] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & \ker w \arrow[r] \arrow[d, "p''"] & C \arrow[r, "w"] \arrow[l, bend left, "\rho"] & D \arrow[r] \arrow[l, bend left, "\sigma"] & 0 \\
 & \mathrm{coker}\, v'' \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}
$$

Split exact sequences

$u's = id_{\mathrm{coker}\, u}$

$ru = id_A$

$su' + ur = id_B$

$w\sigma = id_D$

$\vdots$

Existence of sections s, σ gives a preferred choice in construction of δ:

$\delta = p''\rho v s i'$

In particular, for complexes:

$$
\begin{tikzcd}
0 \arrow[r] & C'' \arrow[r, "f"] & C \arrow[r, "g"] & C' \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}
$$

semi-split

$\exists\ s$ - not necessarily morphism of complexes

$gs = id_{C'}$

Then $\delta: H(C') \to H(C''[1])$ is induced by $[d, s]$

$[d,s] = d_C s - s d_{C'} : C' \to C[1]$

$[d,[d,s]]=0 \Rightarrow [d,s]$ factors through $C''[1]$
via $f[1]$
And $[d,s]$ is a morphism of complexes

We do not require that s be in the category, but $[d,s]$ must be in the category

Applications: Differential geometry — maps of vector bundles

Obtain a triangle:

$$C'' \xrightarrow{f} C \xrightarrow{g} C' \xrightarrow{[d,s]} C''[1]$$

Rmk • A short exact sequence of complexes
$$0 \to C'' \xrightarrow{f} C \xrightarrow{g} C' \to 0$$
over an abelian category $\underline{A}$ is semi-split iff it is homotopy equivalent (i.e. isomorphic in the homotopy category $K(A)$) to a sequence
$$0 \to X \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} X[1] \to 0$$
for some morphism $f: X \to Y$ of complexes

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] \\
\bullet\ -1 & 0 \arrow[d] & 0 \arrow[r, "i"] \arrow[d] & X \arrow[r, "p"] \arrow[d, "-f"] & X \arrow[d] \\
0 & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "i"] \arrow[d] & Y \arrow[r, "p"] & 0 \\
1 & 0 & 0 & &
\end{tikzcd}
\quad \Bigg\} X[1]$$

Long exact sequence in cohomology:

$$\begin{tikzcd}
 & X \arrow[r, "-f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & \mathrm{Coker}\, f \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
 & H^{-1}(X[1]) & H^0(0 \to Y \to 0) & H^0(\mathrm{Cone}\, f) & H^0(X[1]) \\
0 \arrow[r] & \ker f \arrow[d, equal] \arrow[uu, bend left] & & & \\
 & H^{-1}(\mathrm{Cone}\, f) & & &
\end{tikzcd}$$

Def A <u>triangulated category</u> consists of

- $\mathcal{D}$, an additive category
- $T: \mathcal{D} \to \mathcal{D}$ auto-equivalence
- A collection of triples $(X \to Y \to Z \to TX)$ called the "distinguished triangles," subject to the axioms

TR1: • Every morphism $u: X \to Y$ occurs in a triangle

• The triangle $X \xrightarrow{id} X \to 0 \to TX$ is distinguished for every X

• If in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "a"] & Y \arrow[r, "v"] \arrow[d, "b"] & Z \arrow[r, "w"] \arrow[d, "c"] & TX \arrow[d, "Ta"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

the top is distinguished and a, b, c are isos, then the bottom is distinguished.

TR2 (Rotation axiom) If $X \to Y \to Z \to TX$ is distinguished then in the sequence

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1] \xrightarrow{-v[1]} Z[1] \xrightarrow{-w[1]} X[2]$$

any three consecutive morphisms form a distinguished triangle

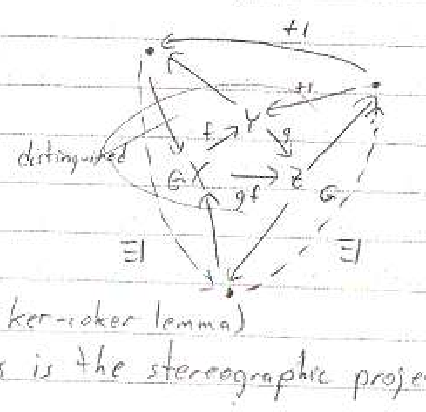
TR3 (Lifting of morphisms)

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & TX \arrow[d, "f{[1]}"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & TX' \end{tikzcd}$$

distinguished triangles

$gu = u'f \Rightarrow \exists\, h: hv = v'g$ and $w'h = Tf \circ w$

TR4 (Octahedral axiom) $f: X \to Y$, $g: Y \to Z$



$\exists$ morphisms so that outside is distinguished

(= ker-coker lemma)

This is the stereographic projection of an octahedron

$\mathcal{D}$ is a triangulated category with arbitrary direct sums if it is a triangulated category also satisfying

TR5 Any (set-indexed) direct sum of distinguished triangles is distinguished

Key example: $\underline{A}$ abelian, $K(\underline{A})$ the homotopy category. Then $\left(K(\underline{A}), T, \underbrace{(X \xrightarrow{f} Y \xrightarrow{i} \mathrm{Cone}(f) \xrightarrow{p} TX)_{X \xrightarrow{f} Y}}_{\text{all semi-split exact sequences of complexes}}\right)$ is a triangulated category.

Also true for subcategories $K^-(\underline{A})$, $K^+(\underline{A})$, $K^b(\underline{A})$

Prop. If $(\mathcal{D}, T, \underbrace{\mathcal{E}}_{\text{collection of dist. triangles}})$ is triangulated, then so is $(\mathcal{D}^{op}, T^{op}, \mathcal{E}^{op})$

For any distinguished triangle $X \to Y \to Z \to TX$ and any object $U \in \mathcal{D}$, the sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{D}}(U,X) \to \mathrm{Hom}_{\mathcal{D}}(U,Y) \to \mathrm{Hom}_{\mathcal{D}}(U,Z) \to \mathrm{Hom}_{\mathcal{D}}(U,TX) \to \cdots$$

obtained by applying $\mathrm{Hom}_{\mathcal{D}}(U,?)$ to the sequence in TR2 is exact.

Pf. ① By TR2, it suffices to prove exactness at one spot, e.g. at Y

② If $g: U \to Y$ is such that $vg: U \to Z$ is zero, then

$$\begin{tikzcd} X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & TX & \text{distinguished by assumption} \\ U \arrow[u, "f"] \arrow[r, "="] & U \arrow[u, "g"] \arrow[r] & 0 \arrow[u] \arrow[r] & TU \arrow[u, "Tf"] & \text{distinguished by TR1} \end{tikzcd}$$

By TR3 applied to a rotated version of above triangle, $\exists f$ as above: $g = uf$ $\square$

Cor. In any distinguished triangle, the composition of two consecutive morphisms is zero (e.g. $vu = 0$: $U = X$, $id_X \mapsto u \mapsto vu = 0$)

Cor. For every $V \in \mathcal{D}$, also $\mathrm{Hom}_{\mathcal{D}}(?, V)$ transforms the sequence in TR2 into a long exact sequence (just the dual statement)

Def. An additive functor $F: \mathcal{D} \to \underline{A}$ from a triangulated category to an abelian one is homological if for every distinguished triangle (u,v,w), applying F to the long sequence in TR2 yields an exact sequence in $\underline{A}$.

One writes $F^i := F \circ T^i$

E.g. • If $\underline{A}$ is abelian, $H = H^0 : K(\underline{A}) \to \underline{A}$ given by $H(C,d) = H^0(C,d)$ is a homological functor

• Each $\mathrm{Hom}_{\mathcal{D}}(U,?) : \mathcal{D} \to \underline{Ab}$ is homological

If $\underline{A} = \text{Mod-}R$, then $\mathrm{Hom}_{K(\underline{A})}(R[0],?) = H^0(?)$

$$\begin{tikzcd} C_{-1} \arrow[r, "d"] & C_0 \arrow[r, "d"] & C_1 \\ 0 \arrow[u] \arrow[r] & R \arrow[u] \arrow[r] \arrow[ul] & 0 \arrow[u] \end{tikzcd}$$

$y \mapsto x$ $\iff$ $dx = 0$ cycles

$x = dy$ boundaries

i.e. H is representable on $K(\text{Mod-}R)$

Prop. Let $\underline{A}$ be abelian, set $K^0(\underline{A}) \subseteq K(\underline{A})$ the full subcategory consisting of all complexes with $H^i(C) = 0$ for $i \neq 0$.

(i) If $\underline{A}$ contains enough projectives, then the functor

$$H^0 : K^0(\underline{A}) \longrightarrow \underline{A}$$
$$C \longmapsto H^0(C)$$

admits a fully faithful left adjoint $\mathbb{P} : \underline{A} \to K^0(\underline{A})$ which assigns to $M \in \underline{A}$ a projective resolution of M.

More precisely, the counit of this adjunction:

$$\mathbb{P}H^0(C) \longrightarrow C$$

constitutes a projective resolution.

(ii) Assuming (i), every short exact sequence $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ is mapped to a unique distinguished triangle in $K(\underline{A})$:

$$\mathbb{P}M' \xrightarrow{\mathbb{P}\alpha} \mathbb{P}M \xrightarrow{\mathbb{P}\beta} \mathbb{P}M'' \xrightarrow{\gamma} T\mathbb{P}M'$$

where $\gamma \in \mathrm{Hom}_{K(\underline{A})}(\mathbb{P}M'', T\mathbb{P}M')$ comes from the horseshoe construction

(iii) The association $(\alpha,\beta) \mapsto \gamma$ factors through $\mathrm{Ext}^1_{\underline{A}}(M'', M')$ and provides a bi-functorial isomorphism $\mathrm{Ext}^1_{\underline{A}}(M'', M') \to \mathrm{Hom}_{K(\underline{A})}(\mathbb{P}M'', T\mathbb{P}M')$

Dual statements hold for categories with enough injectives

Thm. Given a homological functor $F: \mathcal{D} \to A$, set K the full subcategory of $\mathcal{D}$ such that $F \circ T^i|_K = 0$ ("F-acyclic objects"). Then there exists a "quotient" $\mathcal{D}/K$: a triangulated category
$\exists$ natural "projection" $\mathcal{D} \to \mathcal{D}/K$ that commutes with translation and preserves distinguished triangles
Any homological functor (e.g. F itself) G such that $G \circ T^i|_K = 0$ factors uniquely through $\mathcal{D}/K$.

E.g. $K(A) \supseteq$ acyclic complexes, then $\mathcal{D}(A) := K(A)/\text{acyclic complexes}$ is the derived category