(cont) Conversely, if $(M, \cdot, 0)$ is an associative magma w/ 0 that admits a D.S.o.I. E, then $Ob(\mathbb{B}M) := E$, $e, e' \in E$;
$Hom_{\mathbb{B}M}(e, e') := \{m \in M \mid e'm = m = me\}$, $m \circ m' := mm'$ yields a category and $\mathbb{B}M_g(C) = C$, $M_g \mathbb{B}(M, \cdot, 0, E) = M$.

NB Gabriel's thm assumes $Hom_C(X, Y)$ forms an abelian group for any X, Y in C. Then $C \mapsto \mathcal{R}C := \bigoplus_{X,Y} Hom_C(X, Y)$, a ring without unit.

Exercise. Given the connexion of small categories with magmas, what is the proper notion of homomorphism of categories?

lecture 24.1 NB No class 7 February

Correction: In Gps, morphism is an epimorphism iff it is surjective.

NB.
- If g, h are epi, then gh is epi
- If gh is epi, then h is epi
- If g has a "section", i.e. $\exists\, h : gh = id$, then g is an epi
  In this case, g is a split epimorphism
  E.g. In Sets, every epi is split
  In k-v.s. every epi or mono is split

f is iso $\Rightarrow$ f is split mono $\Rightarrow$ f is mono
f is iso $\Rightarrow$ f is split epi $\Rightarrow$ f is epi

Def. $f : X \to Y$ is a strong epimorphism if in every diagram

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, "b"'] & Y \arrow[d, "a"] \arrow[dl, dashed, "\exists\alpha"'] \\ U \arrow[r, hook, "k"'] & V \end{tikzcd}$$

with k a monomorphism $\exists\, \alpha : Y \to U$ such that $k\alpha = a$.

NB It follows that $\alpha f = b$ and α is necessarily unique.

Def. k is a <u>strong mono</u> iff for every diagram with f epi α exists.

Properties. • If g, h strong epis, then gh is a strong epi
• If gh strong epi, then g is a strong epi
• Any isomorphism is strong epi and strong mono
$\Rightarrow$ split is stronger than strong
• If $f: A \to B$ is both epi and mono, then TFAE:
(1) f is isomorphism
(2) f is a strong epi
(3) f is a strong mono

Def. C a category, $f: X \to Y$ a morphism
(a). A <u>coimage</u> of f is a factorization $f: X \xrightarrow{g} Z \xrightarrow{k} Y$ such that g is a strong epimorphism and k is a mono
(b). An <u>image</u> of f is a factorization $f: X \xrightarrow{g'} U \xrightarrow{k'} Y$ where g' is epi and k' is a strong mono

Properties. • Let $f: X \to Y$ be a morphism that admits a coimage and assume $f = ab$ is a factorization with a mono and b epi. Then $\exists!\ \alpha$ such that

$$\begin{array}{ccccc} X & \to & \text{Coim } f & \to & Y \\ \| & & \downarrow \alpha & & \| \\ X & \underset{b}{\twoheadrightarrow} & A & \underset{a}{\hookrightarrow} & Y \end{array}$$

commutes; α is an epi and a mono

If b is also a strong epi, then α is isomorphism
i.e. Coim is unique up to unique isomorphism
• f admits image and coimage $\Rightarrow \exists!\ \alpha: \text{Coim } f \to \text{Im } f$, both mono and epi
• If every morphism in C that is mono and epi is an isomorphism then $\text{Coim } f \cong \text{Im } f$ (canonically)

Lemma. For f a morphism, TFAE
(a) $f = hg$, where g is a strong epi and h is a strong mono
(b) f admits coimage and image and the canonical morphism is an iso

Def. The morphism f has an analysis if $f = hg$ with g strong epi, h a strong mono

Properties. • If f has an analysis, then every factorization $f = h'g'$ with h' mono, g' epi is an analysis

Def. C is balanced iff every morphism has an analysis

Prop. For a category C, TFAE
a. C is balanced
b. Each morphism can be written uniquely up to unique iso as an epimorphism followed by a monomorphism
c. Every epimorphism is strong and every monomorphism is strong, and each morphism ...
d. Every morphism has a coimage and all morphisms that are epi and mono are isos.
e. Every morphism has an image and all morphisms ...
f. Every morphism has an image and a coimage and the canonical morphism is an isomorphism

E.g. • Sets, Gps, Ab, k-v.s., Mod-R are all balanced
• (Commutative) Rings is not balanced

Ex. $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epi and a mono
$\mathbb{Z} = \mathbb{Z} \hookrightarrow \mathbb{Q}$ is coimage factorisation
$\mathbb{Z} \hookrightarrow \mathbb{Q} = \mathbb{Q}$ is image factorisation

"If you found stuff from epis, monos, whatever quite confusing, then that's the idea"

More generally,

$R \xrightarrow{f} S$ ring hom

$R \twoheadrightarrow R/\ker f \hookrightarrow S$ coimage

$R \to T \to S$ with $T = \{s \in S \mid s \otimes 1 = 1 \otimes s \text{ in } S \otimes_R S\}$ is image factorisation

I.e. In Comm Rings, every morphism has image and coimage, but usually not isomorphism

Q. In Rings, same holds, except that $S \otimes_R S$ must be replaced by $S *_R S$?

Def. Given C and D categories, a functor $F: C \to D$ consists of

(a) A map $F: \mathrm{Ob}\,C \to \mathrm{Ob}\,D$

(b) For every pair $X, Y \in \mathrm{Ob}\,C$, a map
$$F: \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_D(F(X), F(Y))$$

such that for each pair $f: X \to Y$, $g: Y \to Z$ of morphisms,
$$F(g \circ f) = F(g) \circ F(f) \ \ldots$$

E.g. "Forgetful functors", like Comm Rings $\to$ Rings $\to$ Ab $\to$ Gps $\to$ Sets

- Gps $\to$ Ab via $G \mapsto G/G'$
- Simplicial category Δ: $\mathrm{Ob}\,\Delta$ = finite, linearly ordered sets
  morphisms: non-decreasing maps $a \leq b \Rightarrow f(a) \leq f(b)$

C a category, a simplicial object over C is a functor
$$F: \Delta^{op} \to C$$

A functor $G: \Delta \to C$ is a "cosimplicial" object

$C \ni X$. $h^X := \mathrm{Hom}_C(X, ?): C \to$ Sets $\qquad h_X := \mathrm{Hom}_C(?, X) \ldots$

$$\begin{array}{ccc} Y & \longmapsto & \mathrm{Hom}_C(X,Y) \\ {\scriptstyle f}\downarrow & & \downarrow {\scriptstyle f \circ -} \\ Z & \longmapsto & \mathrm{Hom}_C(X,Z) \end{array}$$

Def. C a category. A *presheaf* on C is a functor $F: C^{op} \to Sets$
F is *representable* if $\exists\, X \in C$: $h_X \cong F$

Morphisms of functors ("natural transformations")
$F, G: C \to D$. A morphism $\alpha: F \to G$ consists of a morphism $\alpha_X: F(X) \to G(X)$ in D for each object $X \in C$ such that for each morphism $f: X \to Y$ in C,

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\alpha_Y]{} & G(Y) \end{array}$$ commutes

In Comm Rings $\ni R$
GL_n: Comm Rings $\to$ Gps
$-^*$: Comm Rings $\to$ Gps
det is a morphism of functors $GL_n \xrightarrow{det} -^*$

Lemma (1) Functors from C to D form a category
(2) A morphism $\alpha: F \to G$ of functors is an isomorphism iff $\alpha_X: FX \to GX$ is an isomorphism (in D) for all X.
In that case $(\alpha^{-1})_X = (\alpha_X)^{-1}$

Lemma. If $F: C \to D$ and $G: D \to E$ are functors, then so is GF.

F is an *equivalence* if $\exists\, G: D \to C$: $FG \cong id_D$, $GF \cong id_C$

Eg. • Main theorem of linear algebra:
fd $\mathbb{K}$-v.s. is equivalent to $C_{\mathbb{K}}$:
$Ob\ C_{\mathbb{K}} = \mathbb{N}$
$Hom_{C_{\mathbb{K}}}(n, m) = M_{m,n}(\mathbb{K})$
• Δ is equivalent to the category of intervals $[0,n]_{\mathbb{N}}$, $n \geq 0$

Def. A category is *essentially small* if it is equivalent to a small category.