MAT 1103 - Introduction to homological algebra

Exercise: $A = \mathbb{Z}$, G finite group. $B = \mathbb{Z}G$ $f: A \to B$
Show each of $f_*, f^*, f^!$ again has both left and right adjoints
What is a property of f that makes this possible?
Think of at least five more examples of left and right adjoint functors

lecture 28.2 If you want credit for this course:
① Send an email to ragnar@math.utoronto.ca
② Tell about background (starting out? finished with thesis? etc.)
③ Tell about research interests (keep this reasonably short)
④ Tell how much time you are willing to invest

Agenda for today:
① Structure of adjoint functors
② Ubiquity of tensor products / left adjoints
$\rightsquigarrow$ Equivalence of categories / Fourier-Mukai transform / noncommutative algebraic geometry
③ "Standard resolutions" / cotriples / Godement construction

① Properties of adjoint of pair of functors $C \underset{L}{\overset{R}{\rightleftarrows}} D$
① R commutes with all limits that exist in D, i.e.
$F: I \to C$ functor such that $\varprojlim_I F$ exists in C
Then $RF: I \xrightarrow{F} C \xrightarrow{R} D$ admits a limit in D, that is,
$\varprojlim_I RF$ exists, and $\varprojlim_I RF \cong R(\varprojlim_I F)$
Dually, L commutes with all colimits that exists in D

E.g. R preserves all products, fibre products, equalizers, final objects, ...

Pf. Recall, for $X \in C$, we set $c_X: I \to C$ the constant functor with value X; $\varprojlim_I F$ is characterized by
$$\mathrm{Hom}_C(X, \varprojlim_I F) \cong \mathrm{Hom}_{\mathrm{Hom}(I,C)}(c_X, F)$$
$\varprojlim_I RF$: $\mathrm{Hom}_{\mathrm{Hom}(I,D)}(c_Y, RF)$, Y an object from D

(cont.) $I \xrightarrow{c_Y} D \xrightarrow{L} C$ So $Lc_Y = c_{LY}$

$$\begin{array}{ccccc} i & \longmapsto & Y & \longmapsto & LY \\ \downarrow & & \| & & \| \\ j & \longmapsto & Y & \longmapsto & LY \end{array}$$

Thus $\mathrm{Hom}_{\mathrm{Hom}(I,D)}(c_Y, RF) = \mathrm{Hom}_{\mathrm{Hom}(I,C)}(c_{LY}, F)$

(1) $\left\{ (a(i) : Y \to RF(i))_{i \in \mathrm{Ob}\, I} \right.$
$\begin{tikzcd} Y \arrow[r, "a(i)"] \arrow[dr, "a(j)"'] & RF(i) \arrow[d, "RF(\alpha)"] \\ & RF(j) \end{tikzcd}$ for each $\alpha : i \to j$ in $I \}$

$\phi_{Y,F(i)} \circ (\,)$ $\quad$ $\phi^{-1}_{Y,F(i)} \circ (\,)$

(2) $= \left\{ (b(i) : LY \to F(i))_{i \in I} \right.$
$\begin{tikzcd} LY \arrow[r, "b(i)"] \arrow[dr, "b(j)"'] & F(i) \arrow[d, "F(\alpha)"] \\ & F(j) \end{tikzcd}$ $\}$

Assuming $\varprojlim_I F$ exists, $\mathrm{Hom}_{\mathrm{Hom}(I,C)}(c_{LY}, F) = \mathrm{Hom}_C(LY, \varprojlim_I F)$

$\cong \mathrm{Hom}_D(Y, R \varprojlim_I F)$

Therefore, $\varprojlim_I RF$ exists and equals $R \varprojlim_I F$. D

Assume C is complete (i.e all limits exist); Enough to say I-complete, i.e. all functors $I \to C$ admit a limit,

Then $c_? : C \to \mathrm{Hom}(I,C)$ admits $\varprojlim_I$ as a right adjoint

Conversely, C admits a right adjoint iff C is I-complete

Pf. $\mathrm{Hom}_{\mathrm{Hom}(I,C)}(c_X, F) \cong \mathrm{Hom}_C(X, \varprojlim_I F)$, an isomorphism of bifunctors. D

Ex. C admits a left adjoint iff C is I-cocomplete iff $\varinjlim_I F$ exists for all $F : I \to C$

Unit and co-unit of an adjunction:

Same situation as before $C \underset{L}{\overset{R}{\rightleftarrows}} D$ adjoint pairs

$Y \in D$, $X \in C$:

$\phi : Hom_C(LY, X) \cong Hom_D(Y, RX)$

$$\begin{tikzcd} f : LY \arrow[r] & X \\ LY \arrow[u, equal] \arrow[ur] & \end{tikzcd}$$

$id_{LY} \in Hom_C(LY, LY) \cong Hom_D(Y, RLY) \ni u_Y : Y \to RLY$

u_Y is the unit of the adjunction at Y.

Naturality of $\phi \to u : id_D \to RL$ is a morphism of functors

If we take $Y = RX$, then $\eta_X = \phi^{-1}(id_{RX}) : LRX \to X$ is the co-unit of the adjunction; $\eta : LR \to id_C$ is a morphism of functors.

Relationship with adjunction: $f : LY \to X$, want $\phi(f) : Y \to RX$

$(LY \xrightarrow{f} X)$ $\quad \mapsto \quad$ $(Y \xrightarrow{u} RLY \xrightarrow{Rf} RX)$

$(RLY \xrightarrow{Rf} RX)$

Claim: $\phi(f) = R(f) \circ u_Y$ and similarly $\phi^{-1}(g) = \eta_X \circ L(g)$ for $g : Y \to LX$.

Moreover, $\underbrace{RX \xrightarrow{u_{RX}} RL(RX) \xrightarrow{R\eta_X} RX}_{id_{RX}}$ and $\underbrace{LY \xrightarrow{Lu_Y} L(RLY) \xrightarrow{\eta_{LY}} LY}_{id_{LY}}$

Thus u_{RX}, Lu_Y are split mono and $R\eta_X$, η_{LY} are split epis

E.g. $f : A \to B$ ring homomorphism. $R = f_* : Mod\text{-}B \to Mod\text{-}A$

$L = f^* : Mod\text{-}A \to Mod\text{-}B$ left adjoint.

$M_A \mapsto M \otimes_A B$

What is u? $u : id_{Mod\text{-}A} \to RL$, i.e. $\forall M$: $u_M : M \to RL(M) = (M \otimes_A B)_A$, $m \mapsto m \otimes_A 1$

Now $ma \mapsto ma \otimes 1$ and $ma \mapsto (m \otimes 1)a$ $\forall a \in A$.

Thus $ma \otimes 1 = (m \otimes 1)a = m \otimes f(a)$

Co-unit: $\eta : LR \to id_{Mod\text{-}B}$, $\forall N$, $\eta_N : L(R(N)) = N \otimes_A B \to N$, $n \otimes b \mapsto nb$

Eg. Forget: Gps $\to$ Sets. Left adjoint is free gp : Sets $\to$ Gps

$u : id_{Sets} \to RL$

$u_S : S \hookrightarrow RL(S) = \text{Free gp}(S)$ is natural inclusion

$\eta : LR \to id_{Gps}$

$\eta_G : \underset{\text{free gp}(G)}{\overset{\shortparallel}{LR(G)}} \to G$ is evaluation in G. NB η is surjective

Forget: Mod-A $\to$ Sets. Left adjoint is free module : Sets $\to$ Mod-A

$S \mapsto A^{(S)}$

$A^{(S)} = \{f : S \to A \mid \text{supp } f = \{x \in S \mid f(s) \neq 0\} \text{ is finite}\}$

$u_S : S \to A^{(S)}$ NB u is injective

$s \mapsto e_s$

$\eta_M : \underset{A^{(M)}}{\overset{\shortparallel}{LR(M)}} \to M$ NB η is surjective

$f \mapsto \sum_m m f(m)$

This gives a free presentation of the module, which can be repeated to obtain a free resolution!

Forget: Ab $\to$ Gps, L : Gps $\to$ Ab $G \mapsto G/[G,G]$

$u : id_{Gps} \to RL$

$u_G : G \to G/[G,G]$

$g \to g \bmod [G,G]$ NB this is surjective

$\eta : LR \to id_{Ab}$

$\eta_A : A \to A$

$g \mapsto g$

Existence and construction of an adjoint!

$R : C \to D$ functor. Given $Y \in D$, consider the category $Y \downarrow R$

Ob $Y \downarrow R$: $(Y \xrightarrow{f} RX, X)$ $X \in \text{Ob } C$, $f \in \text{Hom}_D(Y, RX)$

Mor $Y \downarrow R$

$$\begin{array}{ccc} (Y \xrightarrow{f} RX, & X) \\ \shortparallel \quad\ \ \downarrow Rg & \downarrow g \\ (Y \xrightarrow[f']{} RX', & X') \end{array}$$

$\text{Hom}_{Y \downarrow R}((Y \xrightarrow{f} RX, X), (Y \xrightarrow{f'} RX', X')) \cong \{g : X \to X' \mid Rg \circ f = f'\}$

Thm (Freyd) A functor $R: C \to D$ admits a left adjoint iff

(i) R commutes with all limits that exist in C

(ii) For each Y in D, $Y \downarrow R$ has an initial object $(u_Y : Y \to R(Y))$

(ii') For each Y in D, the limit $L_Y = \varprojlim_{Y \downarrow R} ((Y \to RX, X) \mapsto X)$ exists in C

Warning: $Y \downarrow R$ is not necessarily small

Pf ($\Rightarrow$) All conditions easy to check

($\Leftarrow$) (ii), (ii') define L and (i) verifies that the result is an adjoint. □

Def. Given $R: C \to D$, $Y \in D$, L exists (locally) at Y if the functor
$\mathrm{Hom}_D(Y, R(?)) : C \to \underline{Sets}$
is representable.

② Tensor products: A, B rings (or k-algebras over the same commutative ring k)
${}_A M_B$ is an A-B bimodule, i.e. M is a left A-module and a right B-module, and $(am)b = a(mb)$

($\Leftrightarrow$) Given a ring homomorphism $A \to \mathrm{End}_B(M)$, $a \mapsto \lambda_a(m) := am$

($\Leftrightarrow$) Given a ring homomorphism $B \to (\mathrm{End}_{A^{op}}(M))^{op}$

N_B is any B-module, then $\mathrm{Hom}_B(M,N)$ is naturally a right A-module
$(\varphi \cdot a)(m) = \varphi(am)$

M defines a functor for any A-module L

$\mathrm{Mod}\ B \to \underline{Sets}$

$N \mapsto \mathrm{Hom}_A(L, \mathrm{Hom}_B(M,N))$

Q. Is this functor representable?

More generally, does $\mathrm{Hom}_B(M, ??) : \mathrm{Mod}\text{-}B \to \mathrm{Mod}\text{-}A$ have a left adjoint?

General remark: $\mathrm{Hom}_C(X, ?) : C \to \underline{Sets}$ commutes with all limits

Thm. Left adjoint exists and is given by $(? \otimes_A M)_B$

Pf. We need to define $(? \otimes_A M)_B : \text{Mod-}A \to \text{Mod-}B$

(Recall: If the left adjoint exists, it must commute with all colimits)

Lemma. Given any A-module L, it can be obtained as a cokernel $L \cong \operatorname{coker}(A^{(I)} \to A^{(J)})$.

Pf. ① the counit of the forgetful functor $\text{Mod-}A \to \underline{\text{Sets}}$ gives a surjection $A^{(L)} \xrightarrow{\pi} L$.

② Let $K = \ker \pi$. Then, using the same argument, we get $A^{(K)} \twoheadrightarrow K$. So we get exact sequence

$$\begin{tikzcd} A^{(K)} \arrow[rr] \arrow[dr, two heads] & & A^{(L)} \arrow[r, two heads] & L \arrow[r] & 0 \\ & K \arrow[ur, hook] & & & \end{tikzcd}$$

$$\operatorname{Hom}_A(A^{(K)}, A^{(L)}) = \prod_K \operatorname{Hom}_A(A, A^L) = \prod_K (A^{(L)})$$

i.e. $k \mapsto ( )$ with finitely many nonzero entries $\square$

Pf of thm (cont). Lemma + left adjoints commute with colimits $\Rightarrow$ $(? \otimes_A M)$ is known, once its value on $? = A$ is known

Now define $A \otimes_A M := M$, let $\mathbb{L} = ? \otimes_A M$

① $\mathbb{L}(A) := M$

$$\operatorname{Hom}_B(\mathbb{L}(A), N) \cong \operatorname{Hom}_A(A, \operatorname{Hom}_B(M, N))$$
$$\cong \operatorname{Hom}_B(M, N)$$

$\Rightarrow M \cong \mathbb{L}(A)$ as both represent $\operatorname{Hom}_B(M, ?)$

② $\mathbb{L}(A^{(I)}) = \mathbb{L}(A)^{(I)} = M^{(I)}$

③ Using lemma, if L is arbitrary A-module, choose a free presentation $A^{(I)} \xrightarrow{(a_{ij})} A^{(J)} \to L \to 0$ exact

Applying $\mathbb{L}$: $\mathbb{L}(A^{(I)}) \to \mathbb{L}(A^{(J)}) \to \mathbb{L}(L) \to 0$ exact

$$\begin{array}{ccccccc} \mathbb{L}(A^{(I)}) & \to & \mathbb{L}(A^{(J)}) & \to & \mathbb{L}(L) & \to & 0 \\ \| & & \| & & & & \\ M^{(I)} & \longrightarrow & M^{(J)} & \to & \mathbb{L}(L) & \to & 0 \end{array}$$

$\therefore \mathbb{L}(L) \cong \operatorname{coker}(a_{ij}1_M) =: L \otimes_A M$.

Now $\operatorname{Hom}(\mathbb{L}(L), N) \cong \operatorname{Hom}_B(\operatorname{coker}(a_{ij}1_M), N) \cong \ker(\operatorname{Hom}_B(a_{ij}1_M, N))$

$$\cong \ker\left(\operatorname{Hom}_B(\ ) : \prod_I \operatorname{Hom}_B(M, N) \to \prod_J \operatorname{Hom}_B(M, N)\right)$$
$$\cong \operatorname{Hom}(\operatorname{coker}(a_{ij}), \operatorname{Hom}_B(M, N))$$
$$\cong \operatorname{Hom}(L, \operatorname{Hom}_B(M, N)) \ \square$$

NB. To prove naturality in $\mathcal{C}$ needs further work.

E.g. $?\otimes_{\mathbb{Z}} \mathbb{Z}/n$: A an abelian group, $A\otimes_{\mathbb{Z}} \mathbb{Z}/n = A/nA$

Mimic the same, but as follows: keep Mod-A, but replace Mod-B by some additive category $\mathbb{B}$.

A-$\mathbb{B}$-bimodule: $\lambda: A \to \mathrm{End}_{\mathbb{B}}(M)$ a ring homomorphism

Thus, an A-$\mathbb{B}$-bimodule is an object $M \in \mathbb{B}$ together with a ring homomorphism $\lambda: A \to \mathrm{End}_{\mathbb{B}}(M)$; (M,λ).

If A is a k-algebra and $\mathbb{B}$ is a k-linear category, then M is an A-$_k\mathbb{B}$-bimodule if λ is a k-algebra homomorphism.

Consider the category $\mathrm{Hom}_k(\text{Mod-}A, \mathbb{B})$: k-linear functors Mod-$A \to \mathbb{B}$.
$F \in \mathrm{Hom}_k(\text{Mod-}A, \mathbb{B})$ $F: \mathrm{Hom}_A(F, F') \to \mathrm{Hom}_{\mathbb{B}}(FM, FM')$ is k-linear.

Look at the subcategory $\mathcal{L}$ whose objects are all functors that commute with all colimits. (in particular, all colimits from Mod-A are transformed into such in $\mathbb{B}$, so $\mathbb{B}$ must be cocomplete)

Thm. ($\mathbb{B}$-Mod-$\mathbb{B}$: Eilenberg-Watts Thm) $\mathbb{B}$ cocomplete

① The inclusion $\mathcal{L} \overset{i}{\hookrightarrow} \mathrm{Hom}_k(\text{Mod-}A, \mathbb{B})$ admits a right adjoint e and $e\circ i = id_{\mathcal{L}}$ (i.e. counit is an isomorphism)

(Def. $C \subseteq D$ a full subcategory. Then C is reflexive in D if the inclusion has a right adjoint. $C \subseteq D$ is a localization if it is reflexive and the left adjoint commutes with finite limits)

i.e. $\mathcal{L}$ is a reflexive subcategory.

② A functor F is in $\mathcal{L}$ iff $\exists$ (M,λ) an A_k-$\mathbb{B}$ bimodule such that $F \cong ?\otimes_A M$.

Pf. Step 1. Define $?\otimes_A M$: Mod-$A \to \mathbb{B}$. $A \mapsto M \in \mathbb{B}$,
$L = \mathrm{coker}(A^{(I)} \xrightarrow{(a)} A^{(J)}) \longmapsto L\otimes_A M = \mathrm{coker}(M^{(I)} \xrightarrow{\lambda(a)} M^{(J)})$

Check naturality.

Commuting with colimits: $?\otimes_A M$ is left adjoint of $\mathrm{Hom}_{\mathbb{B}}(M, ?)_A : \mathbb{B} \to \text{Mod-}A$

(cont) Step 2. $F \in \mathcal{L}$, then $F \cong ? \otimes_A F(A)$

$F : \text{Mod-}A \longrightarrow \mathbb{B}$

$\text{End}_A(A) = A \xrightarrow{\quad} F(A) = \text{End}_{\mathbb{B}}(F(A))$

$\lambda_{F(A)}$ is a ring homomorphism

Part (2) follows from $A \otimes_A F(A) = F(A)$

Define $ev_A : \text{Hom}_k(\text{Mod-}A, \mathbb{B}) \to \mathcal{L}$

$F \longmapsto ? \otimes_A F(A)$

Check that ev_A is the left adjoint to the inclusion.

(M, λ) an A-$_k\mathbb{B}$-bimodule,

$\text{Hom}_{\text{Hom}(\ldots)}(? \otimes M, F) \cong \text{Hom}_{A\text{-}_k\mathbb{B}\text{ bimodules}}(M, F(A))$

i.e. $\varphi : ? \otimes M \to F$

$\varphi_A : A \otimes_A M \cong M \to F(A)$ is in

$\text{Hom}_{\substack{A\text{-}_k\mathbb{B} \\ \text{bimodules}}}(M, F(A)) \cong \text{Hom}_{\mathcal{L}}(? \otimes_A M, ? \otimes_A F(A))$

This proves part (1). □

Why is this significant? Problem in string theory, non-commutative alg. geom:

When are $C \cong D$ as categories?

$C \underset{T}{\overset{S}{\rightleftarrows}} D$: $ST \cong id_D$, $TS \cong id_C \Rightarrow S, T$ are adjoint

So look for pair of adjoint functors R, L so that one is an equivalence

In particular, if $D = \text{Mod-}A$, L is a tensor product, so goal is to find modules M such that $- \otimes M$ is an equivalence.

When $C = \text{Sheaves}(Y)$, $D = \text{Sheaves}(X)$, this leads to Fourier-Mukai transform and homological mirror conjecture