Planar graphs & the discharging method

On 4-choosability of planar graphs

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The Discharging Method

Appel & Haken 77: Four Colour Theorem

Based on Euler's formula: for any planar graph

$$|V| - |E| + |F| = 2$$

The Discharging Method

Appel & Haken 77: Four Colour Theorem

Based on Euler's formula: for any planar graph

$$|V|-|E|+|F| = 2$$

$\mathcal{C}$: a class of planar graphs

P : a specific property

goal: to prove that every graph in $\mathcal{C}$ has property P

The Discharging Method

- minimal counterexample

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- forbidden configurations

  (i) properties of $\mathcal{C}$ (ii) minimal counterexample

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- charging: assign initial charges to vertices & faces

  s. t. total charge is negative (Euler's formula)

The Discharging Method

- minimal counterexample G
- forbidden configurations
  - (i) properties of $\mathcal{C}$ (ii) minimal counterexample
- charging:
  - $ch(v) = 2deg(v)-6$ vertex v
  - $ch(f) = |f|-6$ face f

total charge $\leq \sum_{v \in V} 2deg(v)-6 + \sum_{f \in F} deg(f)-6 = -12$

The Discharging Method

- minimal counterexample
- forbidden configurations

  (i) properties of C (ii) minimal counterexample
- charging: assign initial charges to vertices & faces

  s. t. total charge is negative (Euler's formula)
- discharging (rules): distribute the charge of each element to its neighbours

  preserves total charge

The Discharging Method

- minimal counterexample
- forbidden configurations

  (i) properties of C (ii) minimal counterexample
- charging: assign initial charges to vertices & faces

  s. t. total charge is negative (Euler's formula)
- discharging (rules): distribute the charge of each element to its neighbours

  preserves total charge
- conclusion: recompute total charge

  showing charge of every element is non-negative

Vizing's list chromatic index conjecture

every simple graph is $(\Delta+1)$-edge choosable

Δ is the maximum degree of the graph

Vizing's list chromatic index conjecture

every simple graph is $(\Delta+1)$-edge choosable

Zhang & Wu 04

every planar graph without 4-cycles is $(\Delta+1)$-edge choosable

except for $\Delta = 5$: not settled

every planar graph without 4-cycles & with $\Delta = 5$ is 6-edge choosable every planar graph without 4-cycles & with $\Delta = 5$ is 6-edge choosable

- minimal counterexample

every planar graph without 4-cycles & with $\Delta = 5$ is

6-edge choosable

forbidden configurations in minimal counterexample

(i) properties of C

has no 4-face

has no adjacent 3-faces

every planar graph without 4-cycles & with $\Delta = 5$ is

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forbidden configurations in minimal counterexample

(i) properties of C

has no 4-face

has no adjacent 3-faces

(ii) minimal counterexample

$\delta \geq 3$

every planar graph without 4-cycles & with $\Delta = 5$ is

6-edge choosable

forbidden configurations in minimal counterexample

(i) properties of *C*

has no 4-face

has no adjacent 3-faces

(ii) minimal counterexample

$\delta \geq 3$

$deg(u) + deg(v) \geq 8$ if $(u, v) \in E$

every planar graph without 4-cycles & with $\Delta = 5$ is

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forbidden configurations in minimal counterexample

(i) properties of *C*

has no 4-face

has no adjacent 3-faces

(ii) minimal counterexample

$\delta \geq 3$

$deg(u) + deg(v) \geq 8$ if $(u, v) \in E$

if a 4-vertex is the intersection of two 3-faces

à they are incident to at least two 5-vertices

every planar graph without 4-cycles & with $\Delta = 5$ is

6-edge choosable

discharging rules:

- every 5-vertex

  1 charge to every incident 3-face

  1/5 charge to every incident 5-face

- every 4-vertex

  1 charge to every incident 3-face

  1/5 charge to every incident 5-face

every planar graph without 4-cycles & with $\Delta = 5$ is

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discharging rules:

- every 5-vertex

  3/2 charge to every incident 3-face

  1/5 charge to every incident 5-face

- every 4-vertex

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discharging rules:

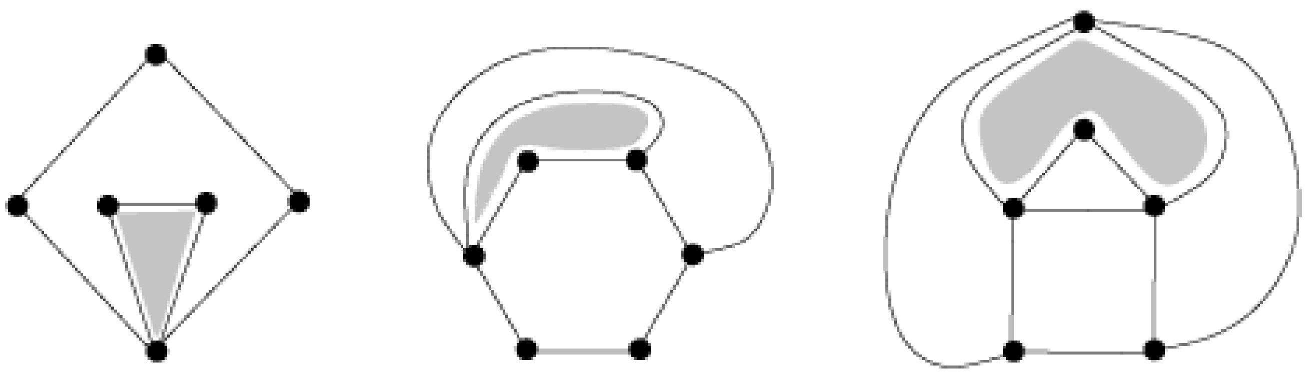
- every 5-vertex
  - 3/2 charge to every incident 3-face
  - 1/3 charge to every incident 5-face

- every 4-vertex
  - 1 charge to every incident 3-face if has no 5-vertex
  - $\frac{3}{4}$ charge to every incident 3-face if has one 5-vertex
  - 1/5 charge to every incident 5-face

Planar graphs without cycles of specific lengths

Planar graphs without cycles of specific lengths

unexpected configurations in planar graphs without 7-cycles



(a) (b) (c)

Planar graphs without cycles of specific lengths

- Lam, Shiu & Xu, *On structure of some plane graphs with application to choosability*, J. Combin. Theory Ser. B, 2001

claimed in planar graphs without 6-cycles:

- a k-vertex, $k \geq 5$, is incident to at most $3k/4$ ≤ 5-faces
- a 5-face is not adjacent to any 3-face
- a 4-face is adjacent to at most one 3-face
- a 3-face is adjacent to at most two 3-faces
- two adjacent 3-faces à none is adjacent to a 4-face

Planar graphs without cycles of specific lengths

- Lam, Shiu & Xu, *On structure of some plane graphs with application to choosability*, J. Combin. Theory Ser. B, 2001
- Wang & Lih, *Choosability and edge choosability of planar graphs without five cycles*, Appl. Math. Lett., 2002
- Wang & Lih, *Structural properties and edge choosability of planar graphs without 6-cycles*, Combin. Probab. Comput., 2001

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every planar graph without 6-cycles is (Δ+1)-edge choosable when $\Delta \neq 5$

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every planar graph without 6-cycles is $(\Delta+1)$-edge choosable when $\Delta \neq 5$

based on: every planar graph without 6-cycles excludes:

a 5-face adjacent to a 3-face

a 4-face adjacent to two non-adjacent 3-faces

a 4-face share one edge with two adjacent 3-faces

Planar graphs without cycles of specific lengths

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F 04, every planar graph without 6-cycles is $(\Delta+1)$-edge choosable

Vizing's list chromatic index conjecture

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every simple graph is (Δ+1)-edge choosable

Conjecture is proved for simple planar graphs

Zhang & Wu 04: without 3-cycles
Zhang & Wu 04 and F 04: without 4-cycles
Wang & Lih 02: without 5-cycles
Wang & Lih 02: without intersecting 3-cycles, $\Delta \neq 5$

Vizing's list chromatic index conjecture

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F 04: without 6-cycles

every simple planar graph without 6-cycles is $(\Delta+1)$-edge choosable

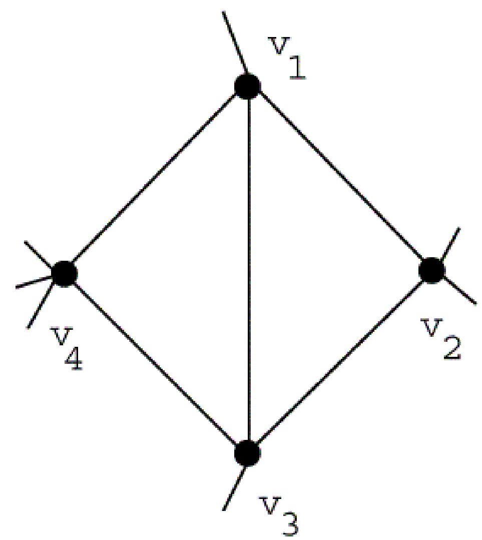
forbidden configurations in a minimal counterexample

- $\delta \geq 3$
- $deg(u) + deg(v) \geq \Delta + 3$ if $(u, v) \in E$
- every 4-face has $\leq$ one 3-vertex & $\geq$ one $\geq$5-vertex

every simple planar graph without 6-cycles is (Δ+1)-edge choosable

forbidden configurations in a minimal counterexample

- $\delta \geq 3$
- $deg(u) + deg(v) \geq \Delta + 3$ if $(u, v) \in E$
- every 4-face has ≤ one 3-vertex & ≥ one ≥5-vertex



deg(v1) = deg(v3) = 4 à
deg(v2) = deg(v4) = 5

deg(v1) + deg(v3) ≤ 9 à
deg(v2) + deg(v4) ≤ 9

every simple planar graph without 6-cycles is (Δ+1)-edge choosable

- cluster: a minimal set of 3-faces s. t. no other 3-face is adjacent to a 3-face in it every simple planar graph without 6-cycles is $(\Delta+1)$-edge choosable

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- size of a cluster: number of 3-faces of the cluster every simple planar graph without 6-cycles is (Δ+1)-edge choosable

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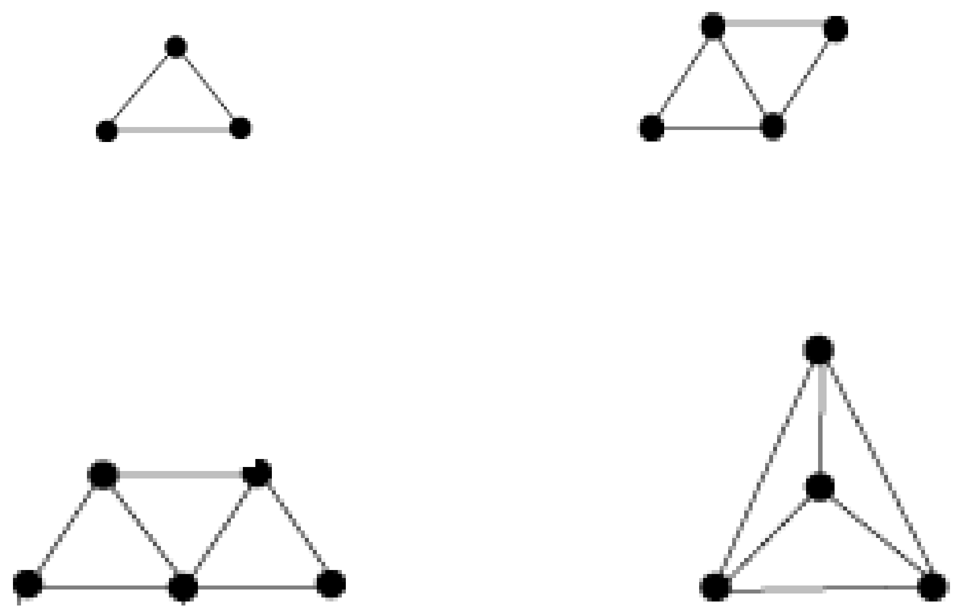

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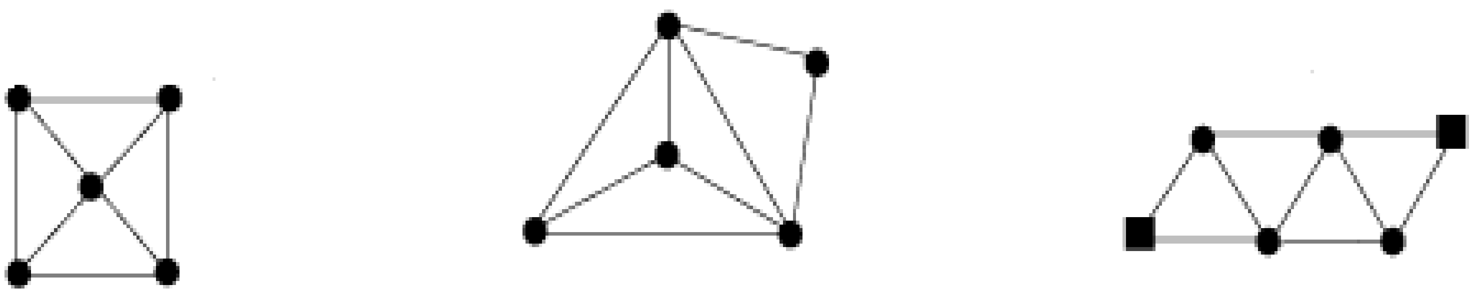

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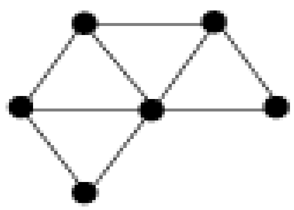

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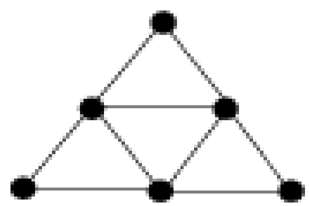
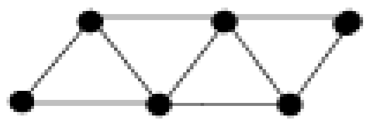


(b)



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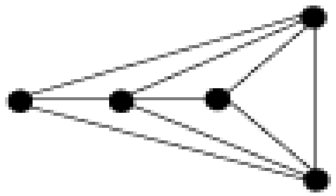

(d)

Choosability of planar graphs

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Thomassen 94: every planar graph is 5-choosable

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Voigt 93: examples of non-4-choosable planar graphs

Choosability of planar graphs

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Voigt 93: examples of non-4-choosable planar graphs

characterizing 4-choosable planar graphs ?

Choosability of planar graphs

easy: every planar graph without 3-cycles is 3-degenerate

à 4-choosable

G is d-degenerate if every subgraph H of G has a vertex of degree at most d in H

Choosability of planar graphs

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Wang & Lih 02:

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planar graph without 4-cycles ?

line graph of dodecahedron

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planar graph without 7-cycles? 3-degenerate ?

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Choudum 77: for each $k \geq 7$

4-regular 3-connected planar graphs without k-cycles

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Conjecture (Wang & Lih 01 - Fijavz, Juvan, Mohar & Skrekovski 02)

every planar graph without 7-cycles is 4-choosable

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Theorem (F 04)

every planar graph without 7-cycles is 4-choosable

planar graphs without 7-cycles are 4-choosable

planar graphs without 7-cycles are 4-choosable

minimum counterexample:

planar graphs without 7-cycles are 4-choosable

minimum counterexample:

- $\delta \geq 4$

planar graphs without 7-cycles are 4-choosable

minimum counterexample:

- $\delta \geq 4$
- has at most six blocks à at most six non-simple faces

planar graphs without 7-cycles are 4-choosable

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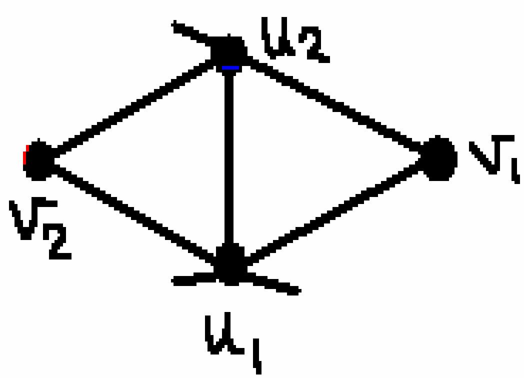

$$deg(u1)+deg(u2) < 10 \Rightarrow deg(v1)+deg(v2) > 8$$

planar graphs without 7-cycles are 4-choosable

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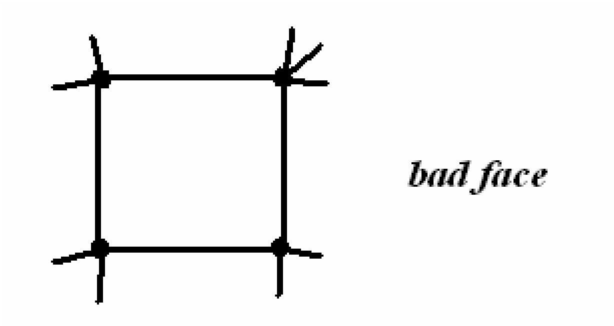

planar graphs without 7-cycles are 4-choosable

minimum counterexample:

- $\delta \geq 4$
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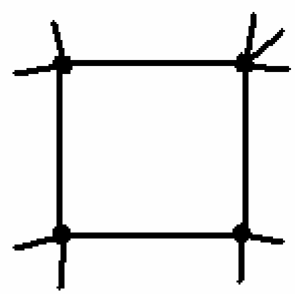
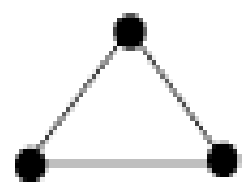
- any 5-vertex is incident to at most one bad face



planar graphs without 7-cycles are 4-choosable

minimum counterexample:

clusters



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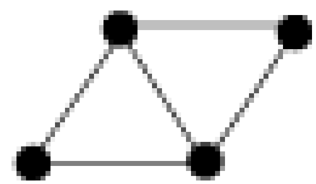

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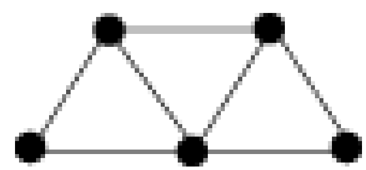

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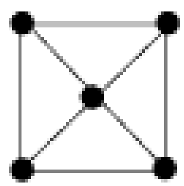
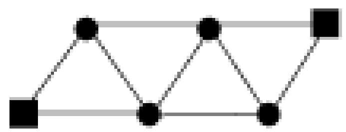


(a)



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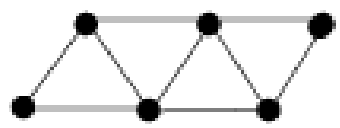
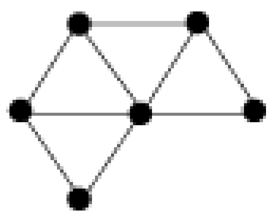
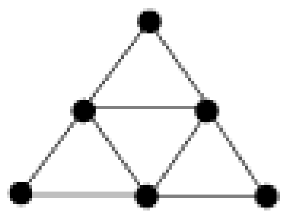


(c)



(d)

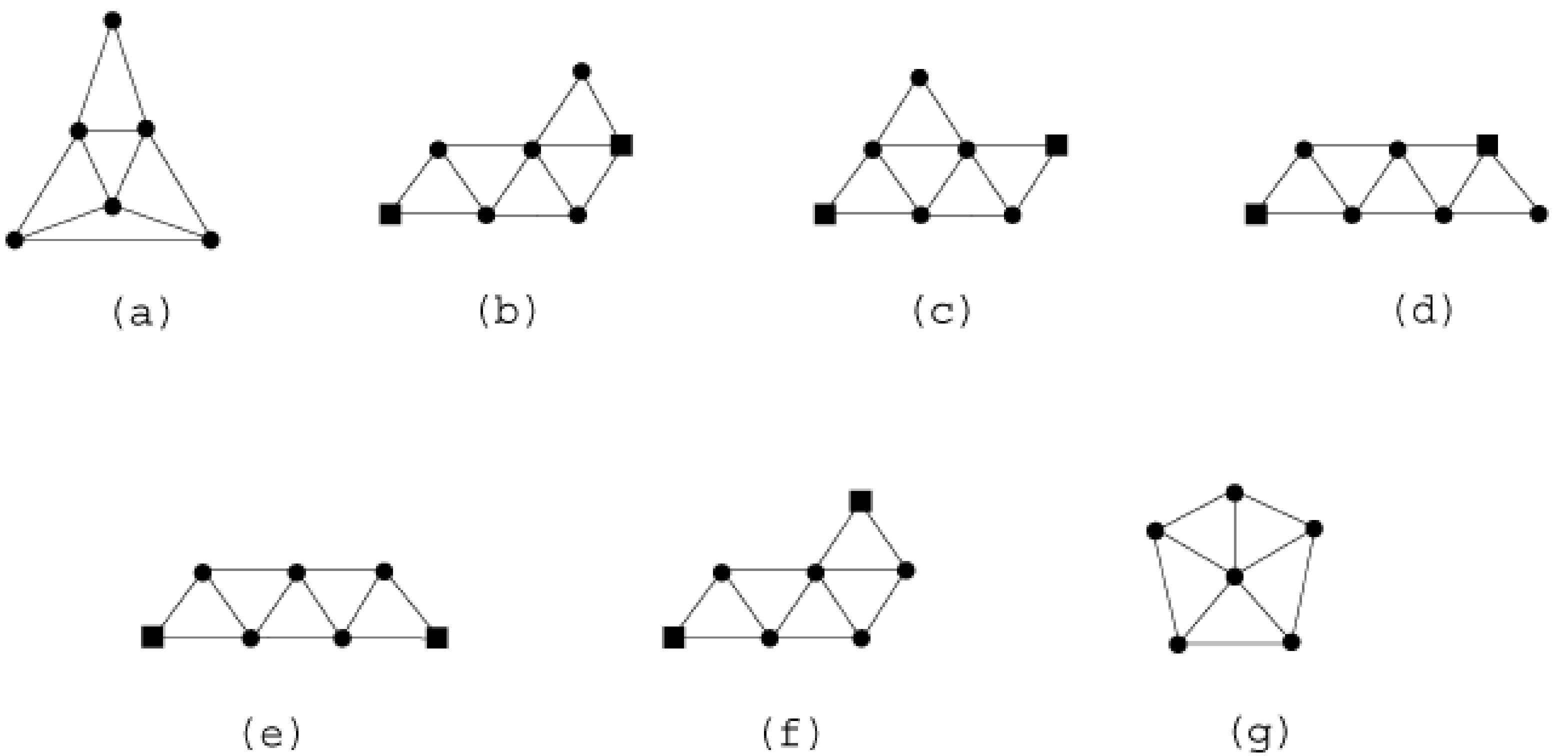


(e)

planar graphs without 7-cycles are 4-choosable

minimum counterexample:

clusters



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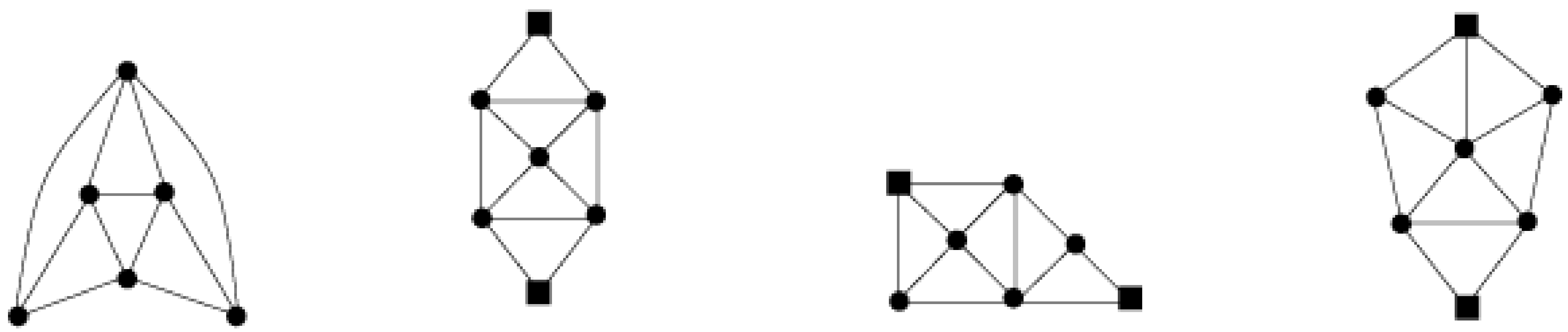
minimum counterexample:

clusters

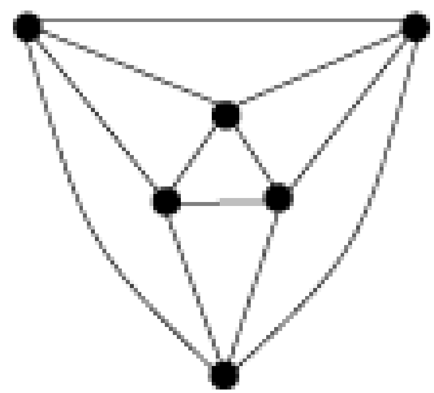


(a) (b) (c) (d)

planar graphs without 7-cycles are 4-choosable

minimum counterexample:

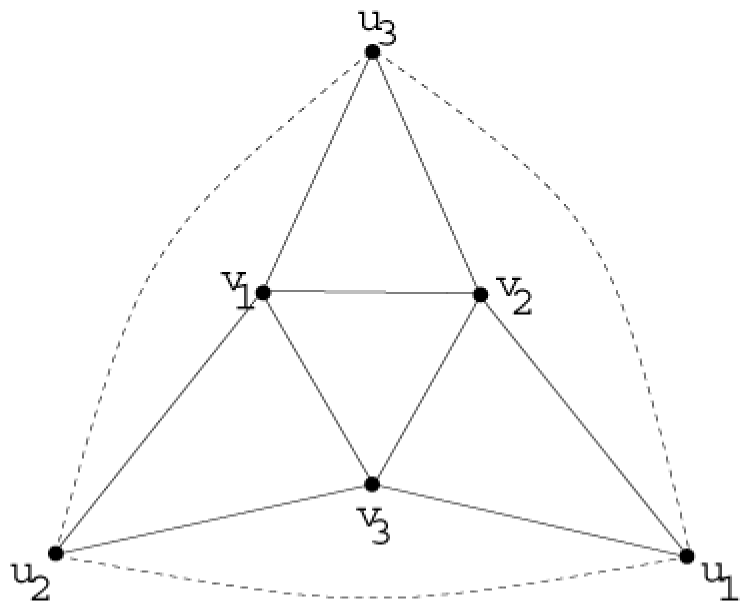
clusters



planar graphs without 7-cycles are 4-choosable

minimum counterexample:

suns



planar graphs without 7-cycles are 4-choosable

Charging:

$ch(v) = 2deg(v)-6$ vertex v

$ch(f) = |f|-6$ face f

planar graphs without 7-cycles are 4-choosable

Charging:

$ch(v) = 2deg(v)-6$ vertex v

$ch(f) = |f|-6$ face f if f is simple

$ch(f) = |f|-6 + 3/2$ if f is non-simple

planar graphs without 7-cycles are 4-choosable

Charging:

$ch(v) = 2deg(v)-6$ vertex v

$ch(f) = |f|-6$ face f if f is simple

$ch(f) = |f|-6 + 3/2$ if f is non-simple

total charge $\leq \sum_{v \in V} 2deg(v)-6 + \sum_{f \in F} deg(f)-6 + 6*3/2 = -3$

Conclusion
4-choosability of planar graphs

belief: can be extended to planar graphs without 8-cycles and to those without 9-cycles

find the maximum k s. t. every planar graph without k-cycles is 4-choosable

$$7 \leq k \leq 63$$

Mirzakhani 63

existence of a non-4-choosable planar graph of size 63