OPEN ORBITS AND GEOMETRICAL STRUCTURES

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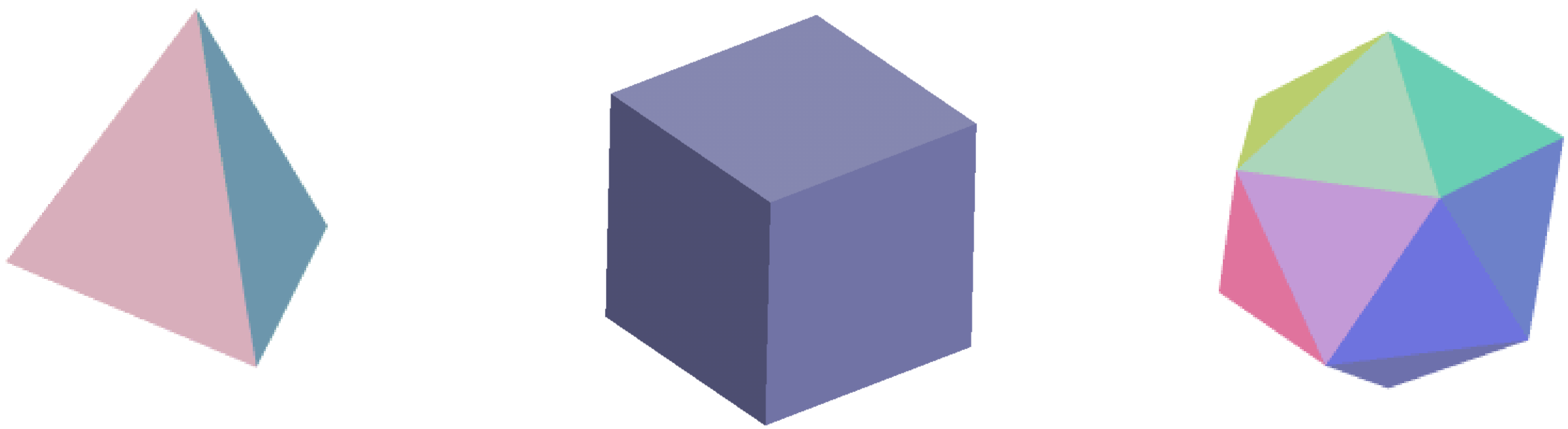
Coxeter Lectures

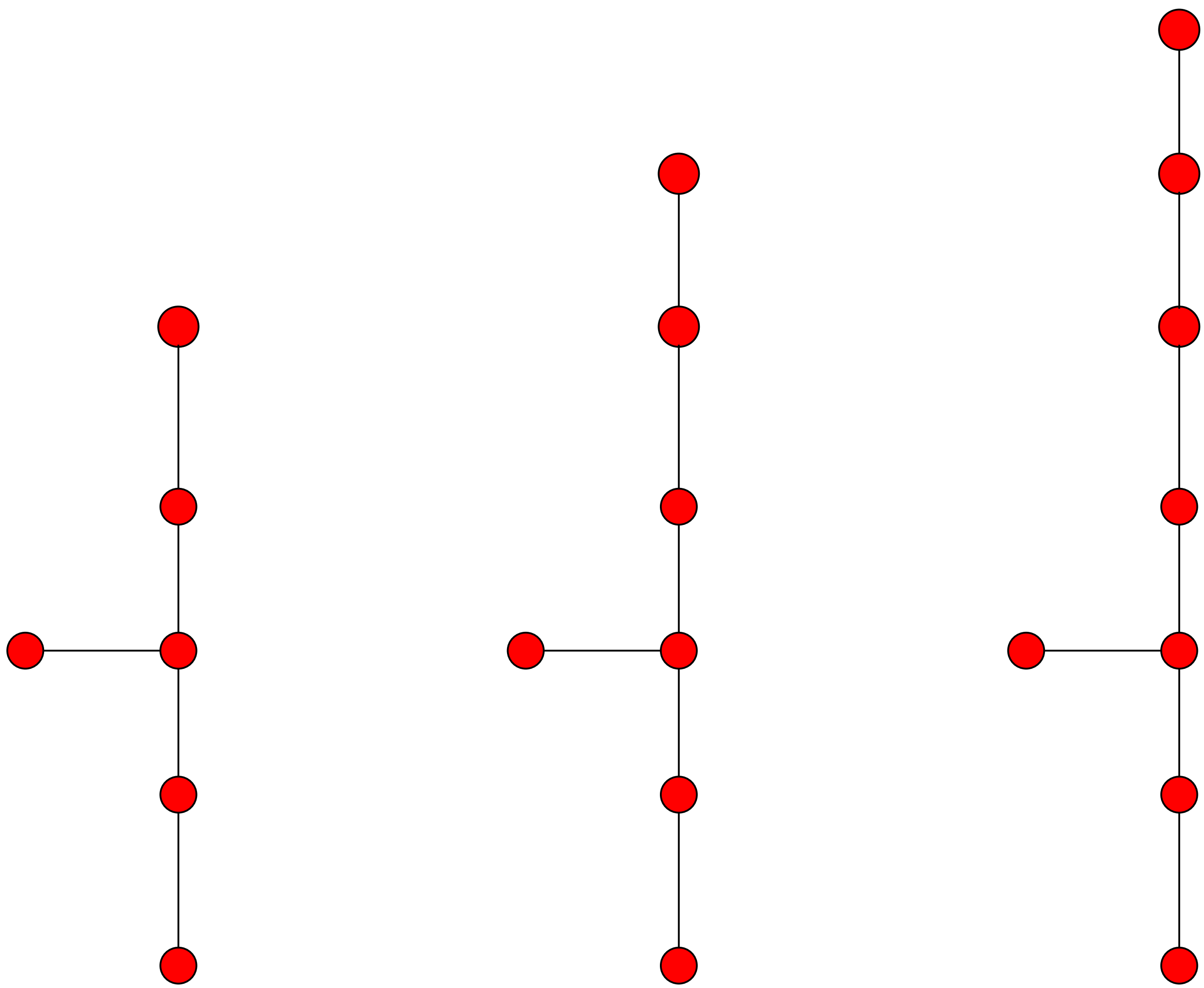
November 15, 2004



  27 lines on a cubic surface (Cayley and Salmon, 1849)

 28 bitangents to a plane quartic curve (Jacobi, 1850)

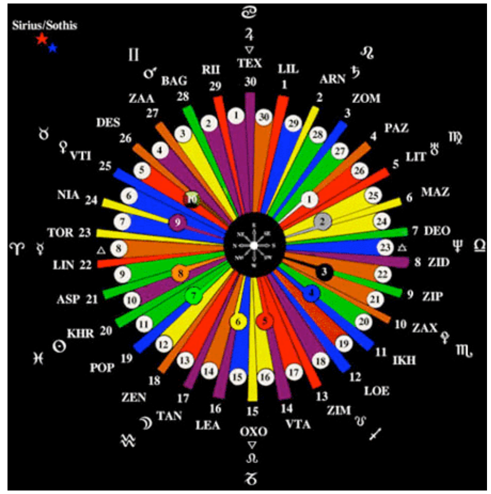
 120 tritangent planes to the intersection of a cubic surface and a quadric surface (Clebsch, 1863)



"Mathematicians thought of them as somehow connected with $A - D - E$ groups.... Dynkin diagrams are all very well and good but they are a method of studying groups, not the other way around, at least for some of us."

Edward Witten

"Small instantons in String Theory" (1999)


Sirius/Sothis
TEX 30
LIL 1
ARN 2
ZOM 3
PAZ 4
LIT 5
MAZ 6
DEO 7
ZID 8
ZIP 9
ZAX 10
IKH 11
LOE 12
ZIM 13
VTA 14
OXO 15
LEA 16
TAN 17
ZEN 18
POP 19
KHR 20
ASP 21
LIN 22
TOR 23
NIA 24
VTI 25
DES 26
ZAA 27
BAG 28
RII 29

$$A_n \qquad D_n \qquad E_6 \qquad E_7 \qquad E_8$$

	Group	Representation	Stabilizer$_o$
1.	$H \times GL(m)$	$\rho \otimes \Lambda^1$	H
2.	$GL(n)$	$Sym^2\Lambda^1$	$O(n)$
3.	$GL(2m)$	Λ^2	$Sp((2)m)$
4.	$GL(2)$	$Sym^3\Lambda^1$	1
5.	$GL(6)$	Λ^3	$SL(3) \times SL(3)$
6.	$GL(7)$	Λ^3	G_2

	Group	Representation	Stabilizer_o
7.	$GL(8)$	Λ^3	$Ad(SL(3))$
8.	$SL(3) \times GL(2)$	$Sym^2\Lambda^1 \otimes \Lambda^1$	1
9.	$SL(6) \times GL(2)$	$\Lambda^2 \otimes \Lambda^1$	$SL(2)^3$
10.	$SL(5) \times GL(3)$	$\Lambda^2 \otimes \Lambda^1$	$PSL(2)$
11.	$SL(5) \times GL(4)$	$\Lambda^2 \otimes \Lambda^1$	1
12.	$SL(3)^2 \times GL(2)$	$\Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$	$GL(1) \times GL(1)$

	Group	Representation	Stabilizer$_o$
13.	$Sp(2n) \times GL(2m)$	$\Lambda^1 \otimes \Lambda^1$	$Sp(2m) \times Sp(2n-m)$
14.	$GL(1) \times Sp(6)$	$\Lambda^1 \otimes \Lambda^3$	$SL(3)$
15.	$SO(n) \times GL(m)$	$\Lambda^1 \otimes \Lambda^1$	$SO(m) \times SO(n-m)$
16.	$GL(1) \times Spin(7)$	$\Lambda^1 \otimes S$	G_2
17.	$GL(2) \times Spin(7)$	$\Lambda^1 \otimes S$	$SL(3) \times SO(2)$
18.	$GL(3) \times Spin(7)$	$\Lambda^1 \otimes S$	$SL(2) \times SO(3)$

	Group	Representation	Stabilizer$_o$
19.	$GL(1) \times Spin(9)$	$\Lambda^1 \otimes S$	$Spin(7)$
20.	$GL(2) \times Spin(10)$	$\Lambda^1 \otimes S$	$SL(2) \times G_2$
21.	$GL(3) \times Spin(10)$	$\Lambda^1 \otimes S$	$SO(3) \times SL(2)$
22.	$GL(1) \times Spin(11)$	$\Lambda^1 \otimes S$	$SL(5)$
23.	$GL(1) \times Spin(12)$	$\Lambda^1 \otimes S$	$SL(6)$
24.	$GL(1) \times Spin(14)$	$\Lambda^1 \otimes S$	$G_2 \times G_2$

	Group	Representation	Stabilizer$_o$
25.	$GL(1) \times G_2$	$\Lambda^1 \otimes \Lambda^2$	$SL(3)$
26.	$GL(2) \times G_2$	$\Lambda^1 \otimes \Lambda^2$	$GL(2)$
27.	$GL(1) \times E_6$	$\Lambda^1 \otimes \Lambda^1 (= 27)$	F_4
28.	$GL(2) \times E_6$	$\Lambda^1 \otimes \Lambda^1 (= 27)$	$SO(8)$
29.	$GL(1) \times E_7$	$\Lambda^1 \otimes \Lambda^6 (= 56)$	E_6
30.	$GL(1) \times Sp(2n) \times SO(3)$	$\Lambda^1 \otimes \Lambda^1 \otimes \Lambda^1$	$Sp(2n-4) \times SO(2) \cdot U(2n-3)$

RIEMANNIAN HOLONOMY GROUPS

(M Berger, 1955)

$O(n)$ $\quad U(n)$ $\quad Sp(n)$ $\quad SU(n)$

$Sp(n) \cdot Sp(1)$ $\quad G_2$ $\quad Spin(7)$

RIEMANNIAN HOLONOMY GROUPS

(M Berger, 1955)

$O(n)$(15) $U(n)$(15) $Sp(n)$(13) $SU(n)$(15)

$Sp(n) \cdot Sp(1)$(13) G_2(25) $Spin(7)$(16)

Product with $\mathbf{R}^*$:

NON-METRIC HOLONOMY GROUPS

(S Merkulov and L Schwachhöfer, 1999)

$$(4), (5), (14), (15), (23), (27), (29)$$

and $Spin(10) \times GL(1)$

OPEN ORBITS

- complex Lie group G
- representation space V
- open orbit $U \subset V$
- relatively invariant polynomial $f : V \to \mathbf{C}$ $(f(gv) = \chi(g)f(v))$

PREHOMOGENEOUS VECTOR SPACES

- M Sato and T Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

- Tatsuo Kimura, "*Introduction to prehomogeneous vector spaces*". Translations of Mathematical Monographs, **215**. American Mathematical Society, Providence, RI, 2003.

OPEN ORBITS OF EXTERIOR FORMS

- (3): $GL(2m, \mathbf{R})$ on Λ^2 ($\deg f = m$ (Pfaffian))
- (5): $GL(6, \mathbf{R})$ on Λ^3 ($\deg f = 4$)
- (6): $GL(7, \mathbf{R})$ on Λ^3 ($\deg f = 7$)
- (7): $GL(8, \mathbf{R})$ on Λ^3 ($\deg f = 16$)
- (14): $Sp(6, \mathbf{R}) \times \mathbf{R}^*$ on Λ^3 ($\deg f = 4$)

- scalar matrices $\lambda I \in GL(n, \mathbf{R})$

- action on Λ^n is λ^n

- $f^{(n/\operatorname{deg} f)}$ defines $\phi \in \Lambda^n$

- $\rho \in U \subset \Lambda^p$ open orbit

- $\phi : U \to \Lambda^n$ invariant map, homogeneous of degree n/p

- derivative $D\phi : \Lambda^p \to \Lambda^n$

- $D\phi(\dot{\rho}) = \widehat{\rho} \wedge \dot{\rho}$

- $\widehat{\rho} \wedge \rho = (n/p)\phi$

complementary form $\widehat{\rho} \in \Lambda^{n-p}$

EXAMPLE (3): SYMPLECTIC

- $\rho = \omega \in \Lambda^2$

- $\phi(\omega) = \omega^m$

- $\widehat{\rho} = m\omega^{m-1}$

EXAMPLE (5): COMPLEX THREE-FORMS

- $\rho \in \Lambda^3$
- $\rho + i\widehat{\rho} = \Omega = \theta_1 \wedge \theta_2 \wedge \theta_3$
- $\Omega \wedge \bar{\Omega} = 2i\widehat{\rho} \wedge \rho = 2i\phi$

EXAMPLE (6): G2 GEOMETRY

- $\rho \in \Lambda^3$

- $i_X\rho \in \Lambda^2 \Rightarrow i_X\rho \wedge i_Y\rho \wedge \rho \in \Lambda^7$

- $g(X,Y) = f^{-1/3} i_X\rho \wedge i_Y\rho \wedge \rho$ metric

- $\widehat{\rho} = *\rho$

- $g(X \times Y, Z) = \rho(X, Y, Z)$

- $\mathbf{R}^7$ = imaginary octonions

- symmetry group G_2

WHAT DOES IT BUY YOU?

$$d\rho = d\widehat{\rho} = 0$$

- (3) symplectic manifold
- (5) Calabi-Yau threefold
- (6) Riemannian manifold with holonomy G_2

MODULI SPACES

$$d\rho = d\hat{\rho} = 0$$

- de Rham cohomology class $[\rho] \in H^p(M, \mathbf{R})$
- moduli space = open set $\mathcal{U} \subset H^p(M, \mathbf{R})$
- $\Phi = (n/p)[\hat{\rho} \wedge \rho]$ function on $\mathcal{U}$
- $D^2\Phi$ defines an (indefinite) metric on $\mathcal{U}$

REDUCED MODULI SPACES

- Restrict the metric to $\Phi = 1$
- For symplectic (e.g. Kähler) and G_2 get a Riemannian connection
- For Calabi-Yau, quotient by $\Omega \mapsto e^{i\theta}\Omega$, get Kähler
- *Conjecture* (P Wilson): For a Kähler manifold the symplectic moduli space has negative curvature.

WHAT ELSE DOES IT BUY?

R Dijkgraaf, S Gukov, A Neitzke and C Vafa

Topological M-theory as Unification of Form Theories of Gravity

hep-th/0411073

A Gerasimov and S Shatishvili

Towards integrability of topological strings I. Three-forms on Calabi-Yau manifolds

hep-th/0409238

- $H^3(M^6, \mathbf{R})$ symplectic, integral symplectic form

- partition function

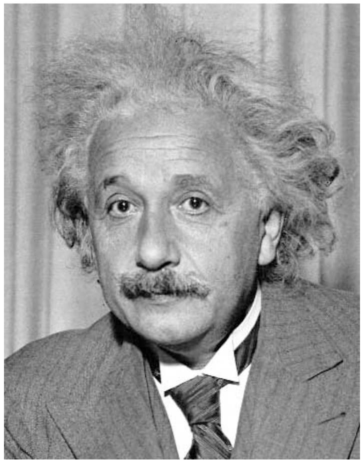
$$Z_H([\rho]) = \int D\beta \, \mathrm{exp}(\phi(\rho + d\beta))$$

- *Conjecture* $Z_H =$ "Wigner function of the B-model".

- M-theory $\sim G_2$ geometry ....

Albert Einstein

"... Since the mathematicians have invaded the theory of relativity, I do not understand it myself."

 **On the method of theoretical physics**: Herbert Spencer lecture, Oxford June 10th 1933

- " ... nature is the realization of the simplest possible mathematical ideas"

OPEN ORBITS OF SPINORS

- (23): $\mathbf{R}^* \times Spin(6,6)$ on S

- $\dim S = 32$, $\deg f = 4$, stabilizer $SU(3,3)$

- (24): $\mathbf{R}^* \times Spin(7,7)$ on S

- $\dim S = 64$, $\deg f = 8$, stabilizer $G_2 \times G_2$

BASIC SCENARIO

- manifold M^n
- replace T by $T \oplus T^*$
- inner product of signature (n, n)

$$(X + \xi, X + \xi) = -i_X \xi$$

- skew adjoint transformations:

$$\mathrm{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- .... *in particular* $B \in \Lambda^2 T^*$

SPINORS

- Take $S = \Lambda^\bullet T^*$

- $S = S^{ev} \oplus S^{od}$

- Define Clifford multiplication by

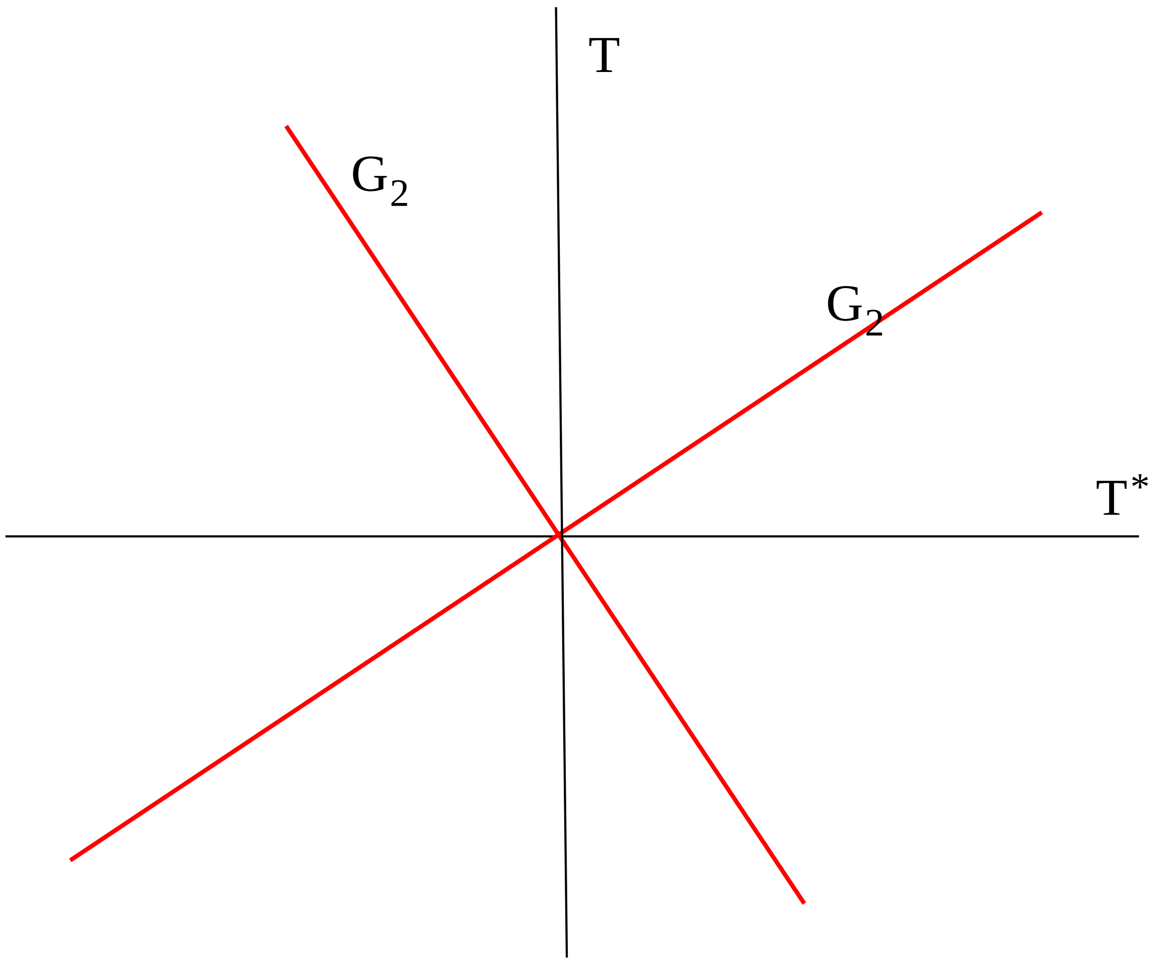
$$\begin{aligned} (X+\xi)\cdot\varphi &= i_X\varphi + \xi\wedge\varphi \\ (X+\xi)^2\cdot\varphi &= i_X\xi\varphi = -(X+\xi, X+\xi)\varphi \end{aligned}$$

- $\exp B(\varphi) = (1 + B + \frac{1}{2}B\wedge B + \ldots)\wedge\varphi$

OPEN ORBITS OF SPINORS

- (23): $\mathbf{R}^* \times Spin(6,6)$ on S
- 6-dimensional manifold M
- $S = \Lambda^{ev/od}T^*$, $\deg f = 4$, stabilizer $SU(3,3)$
- (24): $\mathbf{R}^* \times Spin(7,7)$ on S
- 7-dimensional manifold M
- $S = \Lambda^{ev/od}T^*$, $\deg f = 8$, stabilizer $G_2 \times G_2$

GENERALIZED G2 GEOMETRY



GENERALIZED KÄHLER GEOMETRY