INSTANTONS AND BIHERMITIAN METRICS

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Coxeter Lectures

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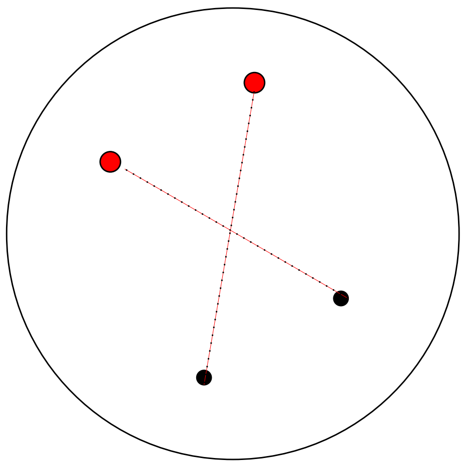
BIHERMITIAN FOUR-MANIFOLDS

- Riemannian 4-manifold M

- self-dual Weyl tensor W_+

- projective spinor bundle $P(V_+) = S(\Lambda^2_+)$



- $W_+ \in S^4 V_+$: quartic polynomial

- almost complex structure $I \Rightarrow \omega \in \Lambda^2_+$

- integrable $\Rightarrow \pm\omega$ roots of the quartic

- *Two* Hermitian structures ω_+, ω_-, all four roots

- *Three* Hermitian structures $\Rightarrow W_+ = 0$

  $\Rightarrow$ (locally) infinitely many (twistor space integrable)

BIHERMITIAN STRUCTURES

- P Z Kobak, *Explicit doubly-Hermitian metrics*, Differential Geom. Appl. **10** (1999), 179–185.

- V Apostolov, P Gauduchon, G Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. **79** (1999), 414–428

- S J Gates, C M Hull, M Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*. Nuclear Phys. B **248** (1984), 157–186.

- $g([I_+, I_-]X, Y) = \Phi(X, Y)$ 2-form
- $\Phi \in \Lambda^{2,0} + \Lambda^{0,2} \subset \Lambda^2_+$ for both complex structures
- $\Rightarrow \sigma \in \Lambda^{0,2} \cong \Lambda^2 T$
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- $\Rightarrow \sigma \in \Lambda^{0,2} \cong \Lambda^2 T$

- $\bar{\partial}\sigma = 0$: holomorphic bivector

- $\Rightarrow M$ is a complex Poisson manifold

GENERALIZED COMPLEX STRUCTURES

- NJH: *Generalized Calabi-Yau manifolds*

  math.DG/0209099 (QJM **54** (2003) 281–308)

- Marco Gualtieri:*Generalized complex geometry*

  math.DG/0401221

- vector field X - section of T

- differential 1-form ξ - section of T^*

- $X + \xi$ - section of $T \oplus T^*$

- *Courant bracket*

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(i_X \eta - i_Y \xi)$$

GENERALIZED COMPLEX STRUCTURE

Manifold M^{2n}

- $(T \oplus T^*) \otimes \mathbf{C} = E \oplus \bar{E}$
- E is isotropic
- sections of E closed under Courant bracket

$=$ complex structure J on $T \oplus T^*$ $+$ integrability condition

EXAMPLES

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

  $E = [\ldots, \partial/\partial z_j \ldots, \ldots, d\bar{z}_k, \ldots]$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

  $E = [\ldots, \partial/\partial x_j + i \sum \omega_{jk} dx_k, \ldots]$

MODULI SPACES

Theorem: (Gualtieri) *There exists a Kuranishi moduli space.* Deformation complex:

- symplectic manifold $(\Omega^\bullet \otimes \mathbf{C}, d)$
- complex manifold $(\Omega^{0,\bullet}(\Lambda^\bullet T), \bar{\partial})$
- $H^0(\Lambda^2 T) + H^1(T) + H^2(\mathcal{O})$

COMPLEX POISSON MANIFOLDS

$$\sigma = \sum \sigma_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \qquad E = [\ldots, \frac{\partial}{\partial \bar{z}_j}, \ldots, dz_k + \sum_\ell \sigma_{k\ell} \frac{\partial}{\partial z_\ell}, \ldots]$$

EXAMPLES:

- $\mathbf{C}P^2$ – since $K^{-1} \cong \mathcal{O}(3)$, σ vanishes on a cubic curve
- Hilbert scheme of points on a Poisson surface
- moduli space of stable bundles on a Poisson surface

  $a, b \in H^1(M, \mathrm{End}\, E \otimes K), \quad \mathrm{tr}(ab)\sigma \in H^2(M, K) \cong \mathbf{C}$

GENERALIZED KÄHLER MANIFOLDS

Kähler $\Rightarrow$ complex structure $+$ symplectic structure

- complex structure $\Rightarrow J_1$ on $T \oplus T^*$
- symplectic structure $\Rightarrow J_2$ on $T \oplus T^*$
- compatibility $J_1 J_2 = J_2 J_1$

A generalized Kähler manifold is M^{2n} with two commuting generalized complex structures J_1, J_2 such that $(J_1 J_2(X+\xi), X+\xi)$ is positive definite.

GUALTIERI'S THEOREM

A generalized Kähler manifold is equivalent to:

- a metric g
- two integrable complex structures I_+, I_-
- a 2-form b, such that...
- $d^c_-\omega_- = db = -d^c_+\omega_+ \qquad d^c = I^{-1}dI$

Equivalently, there are two $U(n)$ connections ∇_+, ∇_- with skew torsion $\pm H = \pm db$ and $\nabla = (\nabla_+ + \nabla_-)/2$.

CONSTRUCTIONS

GENERALIZED COMPLEX – THE GENERIC EVEN CASE

- $\beta = B + i\omega \in \Omega^2$
- $d\beta = 0$
- ω symplectic
- $E = \{X + \xi : (i_X + \xi\wedge)\exp\beta = 0\}$

GENERALIZED KÄHLER – THE GENERIC EVEN CASE

- $\exp \beta_1, \exp \beta_2$ $(\dim_{\mathbf{R}} M = 4k)$

- $J_1 J_2 = J_2 J_1 \Leftrightarrow \dim E_1 \cap E_2 = \dim E_1 \cap \bar{E}_2 = 2k$

- $\Leftrightarrow (\beta_1 - \beta_2)^{k+1} = 0 = (\beta_1 - \bar{\beta}_2)^{k+1}$

EXAMPLE 1. Hyperkähler forms $\omega_1, \omega_2, \omega_3$

$$\beta_1 = \omega_1 + \frac{i}{2}(\omega_2 - \omega_3) \quad \beta_2 = \frac{i}{2}(\omega_2 + \omega_3)$$

EXAMPLE 2. (after D Joyce)

- function $f \Rightarrow$ Hamiltonian vector field X (rel.ω_1) $\Rightarrow$ one-parameter group of Hamiltonian diffeomorphisms F_t

$$\beta_1 = \omega_1 + \frac{i}{2}(\omega_2 - F_t^* \omega_3) \quad \beta_2 = \frac{i}{2}(\omega_2 + F_t^* \omega_3)$$

EXAMPLE 3.

- $J_1 =$ Poisson structure $\sigma = (dz_1 \wedge dz_2)^{-1}$ on $\mathbf{C}P^2$
- ... cubic curve = triple line $z_0^3 = 0$
- $J_2 \sim \exp \beta_2$
- $SU(2)$-invariance

- invariant forms $v_1 = (\bar{z}_1 dz_1 + \bar{z}_2 dz_2)/r^2, v_2 = (z_1 dz_2 - z_2 dz_1)/r^2$

- $\beta_1 = r^2 v_1 v_2 = dz_1 dz_2$

- $\beta_2 = \sum H_{ij} v_i \bar{v}_j + \lambda v_1 v_2 + \mu \bar{v}_1 \bar{v}_2.$

- $(\beta_1 - \beta_2)^2 = 0 = (\beta_2 - \bar{\beta}_1)^2 \Rightarrow$

- $\lambda = \mu, \qquad \det H = \lambda(\lambda - r^2)$

Differential equations $d\beta_1 = 0 = d\beta_2 \Rightarrow$

$$H_{12} = \lambda - 4\frac{1}{r^2}\int_a^r s\lambda ds, \quad H_{21} = -\lambda + 4\frac{1}{r^2}\int_b^r s\lambda ds$$

$$H_{22}^2 = \int_c^r s^{-1}\lambda(\lambda - s^2)ds, \quad H_{11}H_{22} - H_{12}H_{21} = \lambda(\lambda - r^2)$$

One complex function $\lambda(r)$ (cf Kähler $\partial\bar{\partial}\log\phi(r)$)

INSTANTONS

ANTI-SELF-DUAL YANG-MILLS EQUATIONS

- $\Lambda^2 = \Lambda^2_+ + \Lambda^2_-$
- ASDYM connection, curvature $F \in \mathfrak{g} \otimes \Lambda^2_-$
- Hermitian $\Lambda^2_- = \Lambda^{1,1}$ + primitive ($\omega \wedge \alpha = 0$)
- Bihermitian $\Rightarrow F$ type $(1,1)$ relative to I_+ and I_-.

STABILITY

- $d^c\omega = db \Rightarrow dd^c\omega = 0$

- degree of a holomorphic line bundle with curvature F:

$$\deg L = \int_M F \wedge \omega$$

- change connection, $F \mapsto F + dd^c \log f$

$$\int_M dd^c \log f \wedge \omega = -\int_M \log f dd^c\omega = 0$$

Theorem: *On a stable holomorphic bundle there exists a unique ASDYM connection.*

- J Li and S-T Yau, *Hermitian Yang-Mills connections on non-Kähler manifolds*, in "Mathematical aspects of string theory", World Scientific (1987)

- N P Buchdahl, *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces*, Math. Ann. **280** (1988) 625–648.

- M Lübke and A Teleman, " The Kobayashi-Hitchin correspondence", World Scientific (1995)

MODULI SPACE

- ASD connections modulo gauge equivalence: $\mathcal{M}$

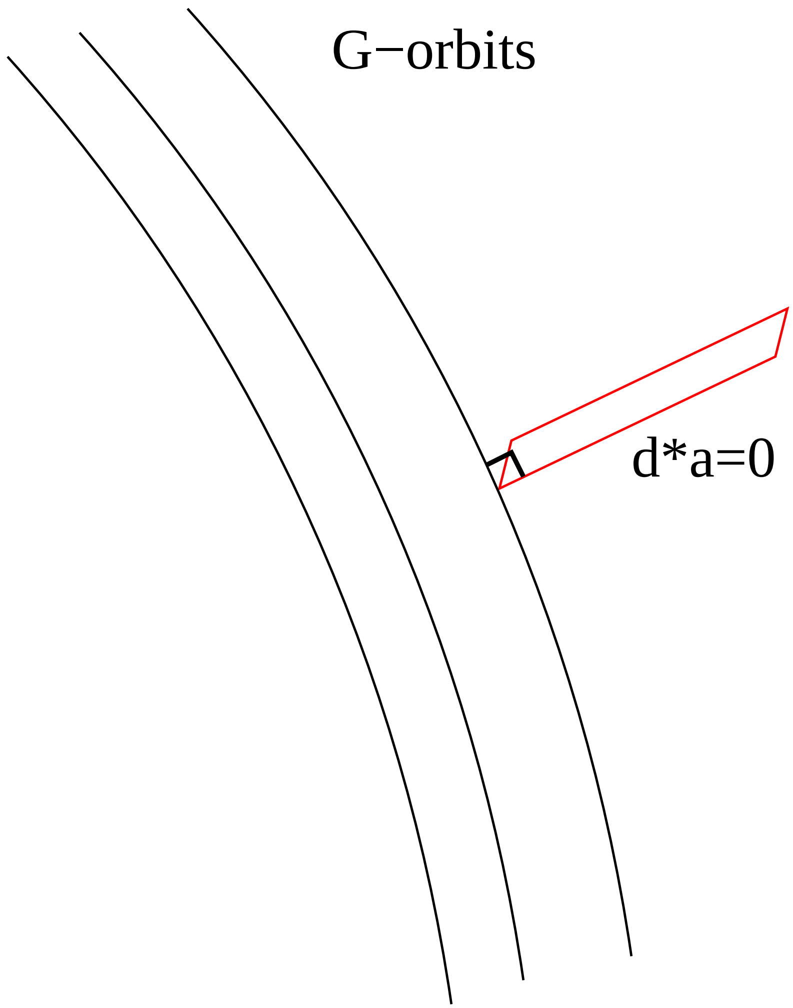
- $=$ moduli space of stable bundles for I_+ and I_-

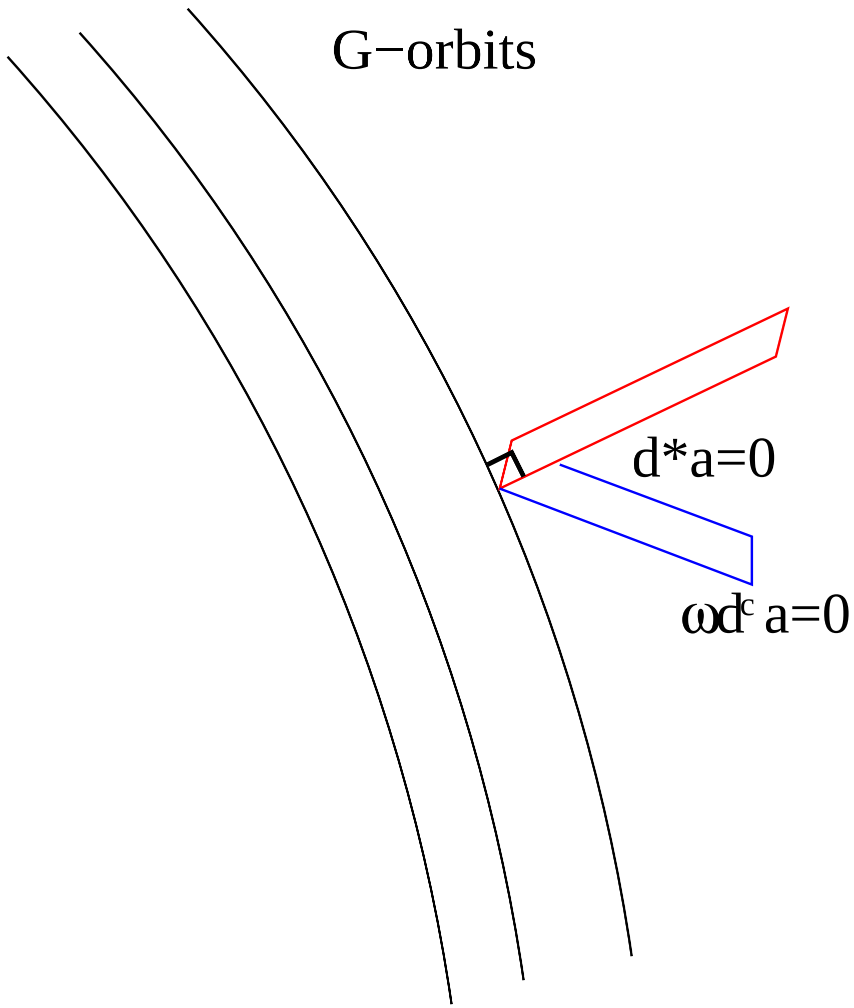
- two complex structures on $\mathcal{M}$

AIM

Show that the moduli space of instantons is a generalized Kähler manifold

METRICS ON THE MODULI SPACE


G−orbits
d*a=0


G−orbits
d*a=0
ωd^c a=0

- moduli space metric is Hermitian

- G-connection has curvature of type $(1,1)$

- if $dd^c\omega = 0$ on M^4, the same is true for the moduli space.

(see M Lübke and A Teleman, " The Kobayashi-Hitchin correspondence", World Scientific (1995))

- tangent vector to $\mathcal{M}$ represented by $\pm$-horizontal forms a and $a + d\psi \in \Omega^1(\mathfrak{g})$

- $\omega_- d^c_- a = 0 \quad \omega_+ d^c_+ (a + d\psi) = 0$

- Required to prove: $(a, a) = (a + d\psi, a + d\psi)$

$$(d\psi, d\psi) = \int \omega_+ \,\mathrm{tr}(d\psi I_+ d\psi) = -\int \omega_+ \,\mathrm{tr}(d\psi d_+^c \psi)$$

But

$$d_+^c(\omega_+ \,\mathrm{tr}(d\psi\psi)) = d_+^c\omega_+ \,\mathrm{tr}(d\psi\psi) + \omega_+ \,\mathrm{tr}(d_+^c d\psi\psi) - \omega_+ \,\mathrm{tr}(d\psi d_+^c \psi)$$

Integrate...

$$0 = \int d_+^c\omega_+ \,\mathrm{tr}(d\psi\psi) + \int \omega_+ \,\mathrm{tr}(d_+^c d\psi\psi) + (d\psi, d\psi)$$

.. but $\mathsf{tr}(d\psi\psi) = d\,\mathsf{tr}\,\psi^2/2$ so

$$\int d^c_+\omega_+\,\mathsf{tr}(d\psi\psi) = \int d^c_+\omega_+ d\,\mathsf{tr}(\psi^2)/2 = \int dd^c_+\omega_+\,\mathsf{tr}(\psi^2)/2 = 0$$

and $\omega_+ d^c_+(a + d\psi) = 0$ so

$$\int \omega_+\,\mathsf{tr}(d^c_+ d\psi\psi) = -\int \omega_+\,\mathsf{tr}(d^c_+ a\psi)$$

and

$$(d\psi, d\psi) = \int \omega_+\,\mathsf{tr}(d^c_+ a\psi)$$

$$d^c_+(\omega_+ \,\mathsf{tr}(a\psi)) = d^c_+\omega_+ \,\mathsf{tr}(a\psi) + \omega_+ \,\mathsf{tr}(d^c_+ a\psi) - \omega_+ \,\mathsf{tr}(a d^c_+ \psi)$$

Now $d^c_+\psi = -I_+ d\psi$ so

$$\int -\omega_+ \,\mathsf{tr}(a d^c_+ \psi) = (a, d\psi)$$

and $d^c_+\omega_+ = -d^c_-\omega_-$ so integrating...

$$0 = -\int d^c_-\omega_- \,\mathsf{tr}(a\psi) + (d\psi, d\psi) + (a, d\psi)$$

Finally...

$$d_-^c(\omega_- \,\mathrm{tr}(a\psi)) = d_-^c\omega_- \,\mathrm{tr}(a\psi) + \omega_- \,\mathrm{tr}(d_-^c a\psi) - \omega_- \,\mathrm{tr}(a d_-^c \psi)$$

but $\omega_- d_-^c a = 0$ so integrating

$$0 = \int d_-^c\omega_- \,\mathrm{tr}(a\psi) + (a, d\psi)$$

$$0 = \int d^c_- \omega_- \,\mathrm{tr}(a\psi) + (a, d\psi)$$

$$0 = -\int d^c_- \omega_- \,\mathrm{tr}(a\psi) + (d\psi, d\psi) + (a, d\psi)$$

so...

$$0 = (d\psi, d\psi) + 2(a, d\psi)$$

and

$$(a + d\psi, a + d\psi) = (a, a)$$

- $\mathcal{M}$ has two complex structures

- $\mathcal{M}$ is bihermitian

- $d^c_+\omega_+ = db = -d^c_-\omega_-$?

- $\mathcal{A}$ = affine space of all connections on V

- tangent vector $a \in \Omega^1(\mathfrak{g})$

$$\Omega(a,b) = \int_M \omega \wedge \mathrm{tr}(a \wedge b)$$

is a closed and gauge-invariant 2-form on $\mathcal{A}$

Hermitian forms $\bar{\omega}_\pm$ on $\mathcal{M}$ are defined by

$$\bar{\omega}_\pm(a,b) = \Omega_\pm(a,b) = \int_M \omega_\pm \wedge \mathrm{tr}(a \wedge b)$$

where a, b are *horizontal*.

$$3d\alpha(a,b,c) = a \cdot \alpha(b,c) - \alpha([a,b],c) + \ldots$$

Ω closed $\Rightarrow$

$$3d\bar{\omega}(a,b,c) = -\int_M \omega \wedge \mathrm{tr}(d_A\psi(a,b),c) + \ldots$$

where $\psi(a,b) \in \Omega^0(\mathfrak{g})$ is the curvature of the G-connection.

$$
\begin{aligned}
3d\bar{\omega}(a,b,c) &= -\int_M \omega \wedge \mathrm{tr}(d_A\psi(a,b),c) + \ldots \\
&= \int_M d\omega \wedge \mathrm{tr}(\psi(a,b)c) + \int_M \omega \wedge \mathrm{tr}(\psi(a,b),d_A c) + \ldots \\
&= \int_M d\omega \wedge \mathrm{tr}(\psi(a,b)c) + \int_M \omega \wedge \mathrm{tr}(\psi(a,b),F'_A(c)) + \ldots \\
&= \int_M d\omega \wedge \mathrm{tr}(\psi(a,b)c) + \ldots
\end{aligned}
$$

since $\omega \wedge F = 0$

- $d^c\omega(a,b,c) = -d\omega(Ia,Ib,Ic)$
- curvature of G-bundle of type $(1,1) \Rightarrow \psi(Ib,Ic) = \psi(b,c)$
- $\ldots \Rightarrow d^c_+\bar{\omega}_+(a,b,c) = \int_M d^c_+\omega_+ \wedge \mathrm{tr}(\psi(a,b),c)$
- so $d^c_+\omega_+ = -d^c_-\omega_-$ on M
- $\Rightarrow d^c_+\bar{\omega}_+ = -d^c_-\bar{\omega}_- = H$ on $\mathcal{M}$
- $dd^c\bar{\omega} = 0 \Rightarrow dH = 0$

CONCLUSION

The moduli space of instantons $\mathcal{M}$ *on a generalized Kähler manifold* M^4 *is a (twisted) generalized Kähler manifold.*

QUESTION

Does $(\mathcal{M}, \mathcal{J}_1)$ *parametrize objects canonically associated to the generalized complex structure* J_1*?*