GEOMETRY WITH B-FIELDS

Nigel Hitchin (Oxford)

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- NJH: *Generalized Calabi-Yau manifolds*

  math.DG/0209099 (QJM **54** (2003) 281–308)

- Marco Gualtieri:*Generalized complex geometry*

  math.DG/0401221

- Frederik Witt (D Phil student)

- Gil Cavalcanti (D Phil student)

- A Kapustin, *Topological strings on noncommutative manifolds* **hep-th/0310057**

- D Huybrechts, *Generalized Calabi-Yau structures, K3 surfaces, and B-fields* **math.AG/0306162**

- U Lindström, R Minasian, A Tomasiello, M Zabzine, *Generalized complex manifolds and supersymmetry*, **hep-th/0405085**

- O Ben-Bassat, *Mirror Symmetry and Generalized Complex Manifolds*, **math.AG/0405303**

BASIC SCENARIO

- manifold M^n

- replace T by $T \oplus T^*$

- inner product of signature (n, n)

$$(X + \xi, X + \xi) = -i_X \xi$$

- skew adjoint transformations:

$$\mathrm{End}\, T \oplus \Lambda^2 T^* \oplus \Lambda^2 T$$

- *.... in particular* $B \in \Lambda^2 T^*$

TRANSFORMATIONS

- exponentiate B:

$$X + \xi \mapsto X + \xi + i_X B$$

- $B \in \Omega^2$ ... *the B-field*

- natural group $\mathrm{Diff}(M) \ltimes \Omega^2(M)$

SPINORS

- Take $S = \Lambda^\bullet T^*$

- $S = S^{ev} \oplus S^{od}$

- Define Clifford multiplication by

$$\begin{aligned}(X+\xi)\cdot\varphi &= i_X\varphi + \xi\wedge\varphi \\ (X+\xi)^2\cdot\varphi &= i_X\xi\varphi = -(X+\xi, X+\xi)\varphi\end{aligned}$$

- $\exp B(\varphi) = (1 + B + \frac{1}{2}B\wedge B + \ldots)\wedge\varphi$

DIFFERENTIAL FORMS = SPINORS FOR $\mathrm{T} \oplus \mathrm{T}^*$

- Invariant bilinear pairing *(Mukai pairing)* with values in $\Lambda^n T^*$:

  $\langle \varphi, \psi \rangle = [\varphi \wedge \sigma(\psi)]_n$

  $\sigma(\varphi_0 + \varphi_2 + \varphi_4 + \ldots) = \varphi_0 - \varphi_2 + \varphi_4 - \ldots$

- $\langle e^B \varphi, e^B \psi \rangle = \langle \varphi, \psi \rangle$

- cf. Chern character $\mathrm{ch} : K(M) \to H^{ev}(M)$

  $\mathrm{ch}(E^*) = \sigma\,\mathrm{ch}(E)$

  $\mathrm{ch}(LE) = e^{c_1(L)}\,\mathrm{ch}(E)$

- Lie bracket:

  $$2i_{[X,Y]}\alpha = d([i_X, i_Y]\alpha) + 2i_X d(i_Y\alpha) - 2i_Y d(i_X\alpha) + [i_X, i_Y]d\alpha$$

- $A = X + \xi$, $B = Y + \eta$ use Clifford multiplication $A \cdot \alpha$ to define a bracket $[A, B]$:

  $$2[A, B] \cdot \alpha = d([A, B]_{\text{Cliff}} \cdot \alpha) + 2A \cdot d(B \cdot \alpha) - 2B \cdot d(A \cdot \alpha) + [A, B]_{\text{Cliff}} \cdot d\alpha$$

- *COURANT bracket*

  $$[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)$$

  (using $\mathcal{L}_X\alpha = d(i_X\alpha) + i_X d\alpha$)

Apply a 2-form B...

- $d \mapsto e^{-B} d e^{B} = d + dB$

- $[X+\xi, Y+\eta] \mapsto [X+\xi, Y+\eta] - 2 i_X i_Y dB$

- ... more generally *twist* by a closed 3-form H:

  $d \mapsto d_H = d + H : \Omega^{od} \to \Omega^{ev}$

  $[X+\xi, Y+\eta] \mapsto [X+\xi, Y+\eta] - 2 i_X i_Y H$

GENERALIZED GEOMETRIC STRUCTURES

- $SO(n,n)$ compatibility
- integrability $\sim d$ or Courant bracket
- $\sim d + H$ or twisted Courant bracket
- transform by Diff$(M) \ltimes \Omega^2_{closed}(M)$

DIFFERENTIAL FORMS AS SPINORS

- No exterior product — *use Mukai pairing*

- No interior product with vector fields — *use Clifford product*

- No $\mathbf{Z}$-grading — *only even/odd*

- Use exterior differential d or $d + H$

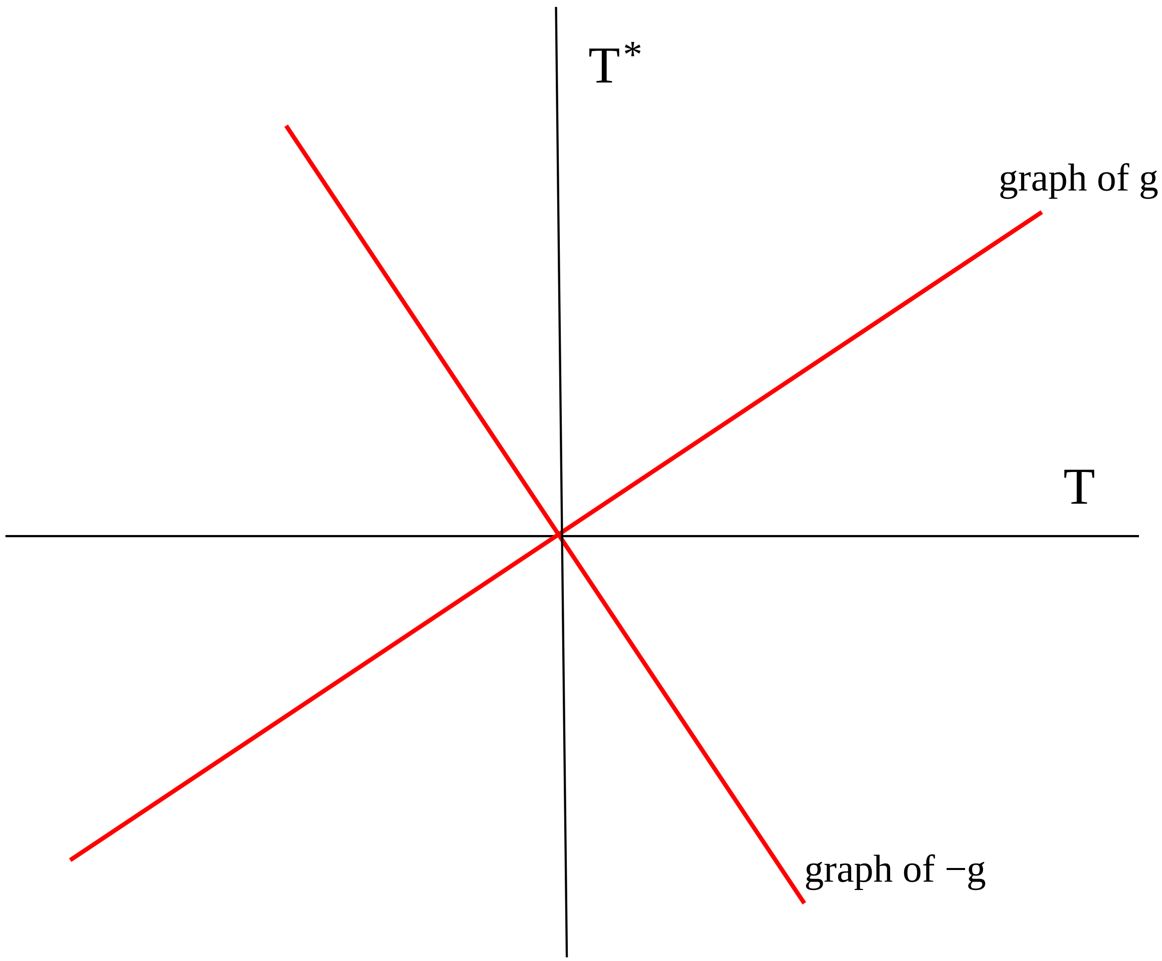
RIEMANNIAN METRIC

- Riemannian metric g_{ij}

- $X \mapsto g(X, -) : g : T \to T^*$

- *graph* of $g = V \subset T \oplus T^*$

- $T \oplus T^* = V \oplus V^\perp$



T^*
graph of g
T
graph of −g

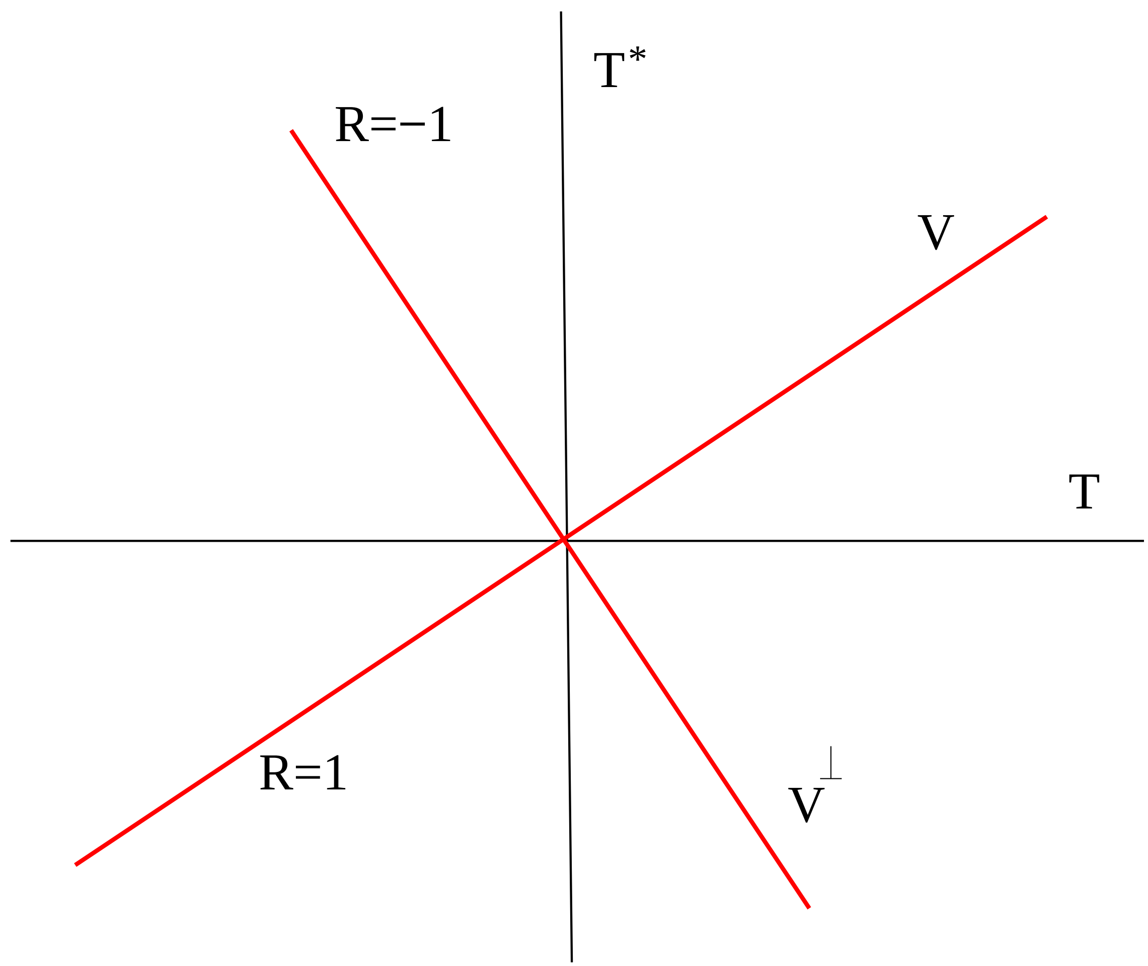
GENERALIZED RIEMANNIAN METRIC

- $V \subset T \oplus T^*$ positive definite rank n subbundle
- $=$ graph of $g + B : T \to T^*$
- $g + B \in T^* \otimes T^*$: g symmetric, B skew
- $=$ B-field transform of a metric

... OR ALTERNATIVELY

- Define $R \in \mathrm{End}(T \oplus T^*)$, $R^* = R, R^2 = 1$ by...

- $R(v) = v$ on V, $R(v) = -v$ on $V^\perp$


T^*
R=−1
V
T
R=1
$V^\perp$

- lift $R \in O(n,n)$ to $\tilde{R} \in Pin(n,n)$

- $\tilde{R} : \Omega^* \to \Omega^*$ *generalized Hodge star operator*

- $(B = 0, \tilde{R} = \sigma *)$

EXAMPLES OF GENERALIZED GEOMETRIES

- Generalized G_2
- Generalized complex structures
- Generalized Kähler structures

Superstrings with Intrinsic Torsion, J P Gauntlett, D Martelli and D Waldram, Phys.Rev. **D69** (2004)

"...A type II geometry will preserve supersymmetry if and only if there is at least one ϵ^+ or ϵ^- satisfying"

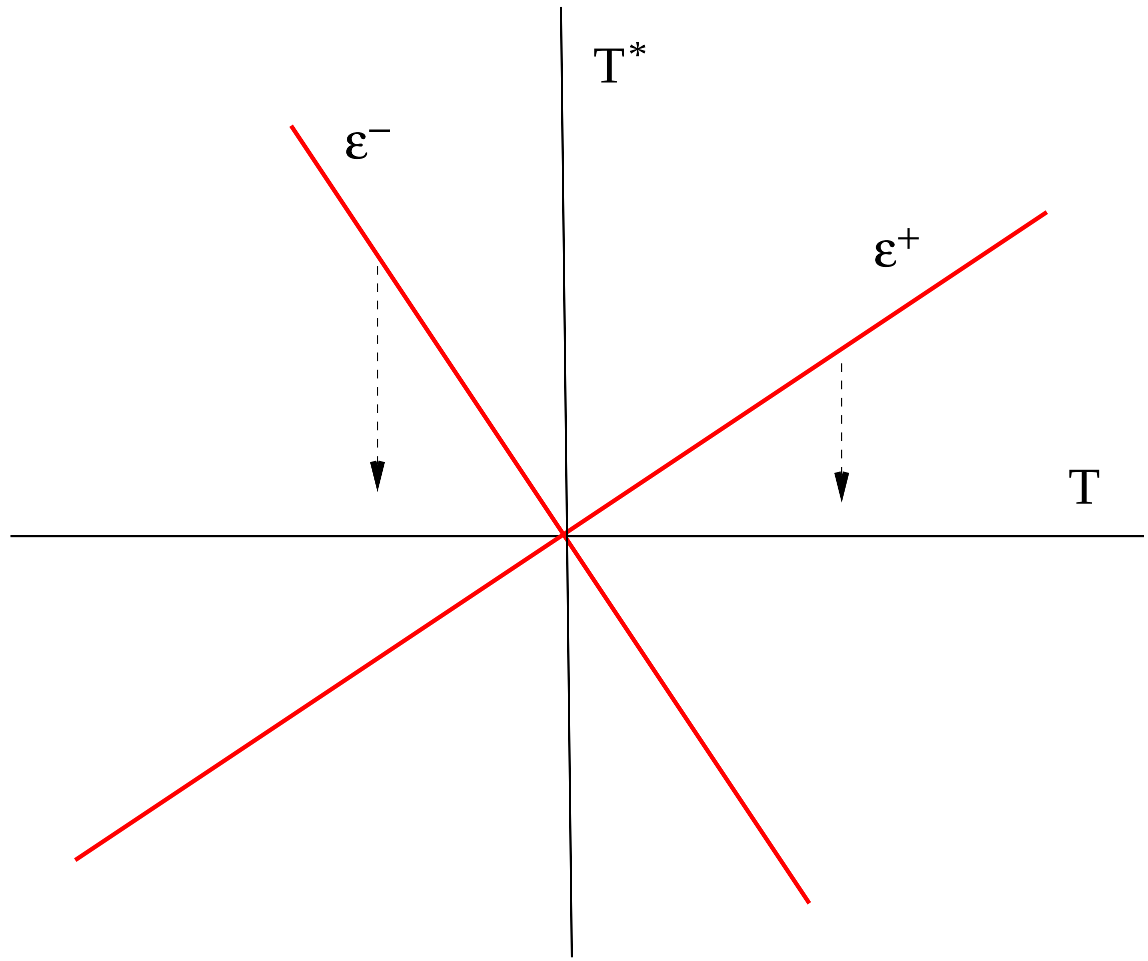
$$\begin{aligned} \nabla^{\pm}_M \epsilon^{\pm} \equiv \left(\nabla_M \pm \frac{1}{8} H_{MNP} \Gamma^{NP} \right) \epsilon^{\pm} &= 0, \\ \left(\Gamma^M \partial_M \Phi \pm \frac{1}{12} H_{MNP} \Gamma^{MNP} \right) \epsilon^{\pm} &= 0, \end{aligned}$$

....locally the three-form is given by $H = dB$

- Riemannian connections $\nabla^{\pm}$ with *skew torsion* $\pm dB \in \Omega^3$

- unit spinors $\epsilon^{\pm}$, covariant constant $\nabla^{\pm}\epsilon^{\pm} = 0$

- function Φ : $(d\Phi \pm dB) \cdot \epsilon^{\pm} = 0$



T*
ε−
ε+
T

DATA

- a metric g
- a 2-form B
- two unit spinors $\epsilon^{+}, \epsilon^{-}$
- a scalar function Φ

7 DIMENSIONS

- Clifford multiplication $\Lambda^{od/ev}T^* \cong \text{End}(S) \cong S \otimes S$
-
-
-
-

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- $\Phi : 1 \quad b : 21 \quad g : 28 \quad \epsilon^{+} : 7 \quad \epsilon^{-} : 7$

  $1 + 21 + 28 + 7 + 7 = 64 = \dim \Lambda^{od}$

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- stabilizer $G_2 \times G_2$ ($\dim 1 + 14.13/2 - 64 = 28$)

EQUATIONS:

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- $d\widehat{\rho} = 0$
-

EQUATIONS:

- $d\rho = 0$
- $d\widehat{\rho} = 0$
- $\widehat{\rho} \wedge \sigma(\dot{\rho}) = D\phi(\dot{\rho}),\ \phi : U \to \Lambda^7$

unfortunately..... if M is compact, then

- $H = 0$
- $\Phi = const.$
- $\nabla^{+} = \nabla^{-} =$ Levi-Civita connection
- standard G_2 manifold + closed B-field.

GENERALIZED COMPLEX STRUCTURES

A *generalized complex structure* is:

- $J : T \oplus T^* \to T \oplus T^*, J^2 = -1$

- $(JA, B) + (A, JB) = 0$

- if $JA = iA, JB = iB$ then $J[A, B] = i[A, B]$ (*Courant bracket*)

- $U(m, m) \subset SO(2m, 2m)$ structure on $T \oplus T^*$

EXAMPLES

(i-*eigenspace of* $J = E \subset (T \oplus T^*) \otimes \mathbf{C}$)

- complex manifold $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

  $E = [\ldots \partial/\partial z_i \ldots, \ldots d\bar{z}_i \ldots]$

- symplectic manifold $J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$

  $E = [\ldots, \partial/\partial x_j + i \sum \omega_{jk} dx_k, \ldots]$

GENERALIZED COMPLEX SUBMANIFOLDS

Submanifold $Y \subset M$,

- $0 \rightarrow TY \rightarrow TM|_Y \rightarrow NY \rightarrow 0$

- $TY \oplus (TY)^o = TY \oplus N^*Y \subset (TM \oplus T^*M)|_Y$

- J preserves $TY \oplus N^*Y \Leftrightarrow$ generalized complex submanifold

THE COMPLEX CASE

Generalized complex submanifold = *complex submanifold*

THE SYMPLECTIC CASE

$$\begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix} = \begin{pmatrix} -\omega^{-1}\xi \\ \omega X \end{pmatrix}$$

- $\omega^{-1}\xi \in TY \Rightarrow Y$ coisotropic $\qquad \omega X \in N^* \Rightarrow Y$ isotropic

- $\Rightarrow Y$ is *Lagrangian submanifold*

- a generalized complex submanifold is not necessarily a generalized complex manifold

- the intersection of two generalized complex submanifolds is not necessarily generalized complex

- the whole manifold may not be a generalized complex submanifold

- a point may not be a generalized complex submanifold

EXAMPLE: COMPLEX SYMPLECTIC MANIFOLD

- complex manifold M
- holomorphic symplectic form $\omega = \omega_1 + i\omega_2$
- $\omega_2 \Rightarrow$ symplectic form $\Rightarrow$ generalized complex structure
- B-field $B = \omega_1$
- a generalized complex submanifold is a *coisotropic* complex submanifold (e.g. any complex hypersurface)

EXAMPLE: K3 SURFACE

Generalized complex submanifolds: complex coisotropic

- the complex manifold M itself
- any holomorphic curve
- **NO** points

EXAMPLE: $\mathbf{C}P^2$ AS A COMPLEX POISSON MANIFOLD

Holomorphic section of $\Lambda^2 T \sim$ cubic curve $C \Rightarrow$ generalized complex structure.

Generalized complex submanifolds:

- the complex manifold M itself
- any holomorphic curve
- all points on C
- cf *non-commutative complex plane*

COHOMOLOGY

DECOMPOSITION OF SPINORS

- spin representation for $U(m,m)$
- $S^+ = (\Lambda^{0,0} + \Lambda^{0,2} + \ldots + \Lambda^{0,2m})K^{1/2}$

  $S^- = (\Lambda^{0,1} + \Lambda^{0,3} + \ldots + \Lambda^{0,2m-1})K^{1/2}$
- differential forms = spinors for $SO(2m,2m)$
- $\Lambda^{ev/od}$ decomposes into eigenspaces under the Lie algebra action of J ($J^* = -J$)
- $U_{m-k} = \Lambda^{0,k}K^{1/2}$ has eigenvalue i$(m-k)$

THE GENERALIZED dd^c-LEMMA

- define $d^c = J^{-1}dJ$ with the Lie group action ($J^* = J^{-1}$) of J
- dd^c-lemma: $d\alpha = 0, \alpha = d^c\beta \Rightarrow \alpha = dd^c\gamma$
- complex structure: usual dd^c-lemma (e.g. Kähler) $\Rightarrow$ generalized dd^c-lemma
- symplectic structure: strong Lefschetz theorem (e.g. Kähler) $\Rightarrow$ generalized dd^c-lemma
- generalized Kähler: generalized dd^c-lemma holds (Gualtieri)

HODGE DECOMPOSITION

If the generalized dd^c-lemma holds, the cohomology has a decomposition

$$H^*(M, \mathbf{R}) = \bigoplus_{-m}^{m} U_k$$

- complex structure:

$$U_k = \bigoplus_{p-q=k} H^{p,q}(M)$$

- symplectic structure:

$$U_k \cong H^{m+k}(M)$$

THE SYMPLECTIC CASE

$$J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$$

Lie algebra action on spinors is

$$(L - \Lambda)\alpha = \omega \wedge \alpha - i_{\omega^{-1}}\alpha$$

L, Λ generate an $SL(2, \mathbf{R})$ action on forms:

$$[\Lambda, L] = H, \quad [H, L] = -2L, \quad [H, \Lambda] = 2\Lambda$$

$H = (m - p)$ **on** p**-forms**

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \Lambda = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $$P = \begin{pmatrix} 1 & i/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \exp i\Lambda/2 \, \exp iL$$

$$L - \Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = P^{-1} iHP$$

HOMOLOGY CLASSES DEFINED BY GENERALIZED COMPLEX SUBMANIFOLDS

- **Prop:** *A compact generalized complex submanifold* $Y^n \subset M^{2m}$ *is Poincaré dual to a class in* $U_0 \cap H^{2m-n}(M, \mathbf{Z})$

- symplectic: $L\alpha = 0, \Lambda\alpha = 0 \Rightarrow H\alpha = [\Lambda, L]\alpha = 0$

- $\Rightarrow \alpha \in H^m(M^{2m}, \mathbf{R})$ primitive class

QUESTIONS

- Is there a generalized Hodge conjecture?
- For what classes $a \in H^*(Y)$ is $i_*a \in H^*(M)$ in U_0?
- What is a generalized holomorphic bundle?
- What has all this to do with noncommutative geometry?
  ....etc.