THIS TALK HAS THREE PARTS:

a) SUPERCONDUCTORS

b) FOUR-MANIFOLDS

c) WEAK INTERACTIONS

THE "ELECTROMAGNETIC" FIELD IS DESCRIBED MATHEMATICALLY BY A CONNECTION "A" ON A COMPEX LINE BUNDLE $\mathcal{L}$ OVER SPACETIME

THE CURVATURE

$$F = dA$$

COMBINES THE ELECTRIC AND MAGNETIC FIELDS

SPLIT t = TIME FROM x^i = SPACE (2)

$i = 1..3$

$$F = \sum_i E_i \, dx^i \wedge dt$$

$$+ \underbrace{\sum_{i,j} B_{ij} \, dx^i \, dx^j}_{F_{spatial}}$$

$\vec{E}$ = electric field

and (as we are in 3 dimensions)

$\vec{B}$ $\quad (= * F_{spatial})$

= magnetic field

NOW MANY MATERIALS, SUCH AS LEAD, BECOME "SUPERCONDUCTING" AT LOW TEMPERATURES. TO DESCRIBE A SUPERCONDUCTOR, ALONG WITH MACROSCOPIC VARIABLES LIKE DENSITY, PRESSURE, ... , WE NEED A SECTION $\mathcal{L} \to \text{SPACETIME}$

$$s : (\text{SPACETIME}) \to \mathcal{L}^2$$

$$\begin{tikzcd} \mathcal{L}^2 \arrow[d, shift right, "\pi"'] \\ \text{SPACETIME} \arrow[u, shift right, "s"'] \end{tikzcd}$$

AND SUPERCONDUCTIVITY MEANS THAT IT IS ENERGETICALLY FAVORED TO HAVE

$$|s| = a$$

(WHERE THE POSITIVE CONSTANT a DEPENDS ON THE MATERIAL AND TEMPERATURE)

AND

$$d_A s = 0$$

THE SIMPLEST ENERGY FUNCTION (5)

WITH THESE PROPERTIES IS THE

"LANDAU-GINZBURG" FUNCTION

$$H = \int d^3x \left[\frac{1}{2} |d_A s|^2 + \frac{\lambda}{4} \left(|s|^2 - a^2 \right)^2 + \frac{1}{8\pi} \vec{B}^2 \right]$$

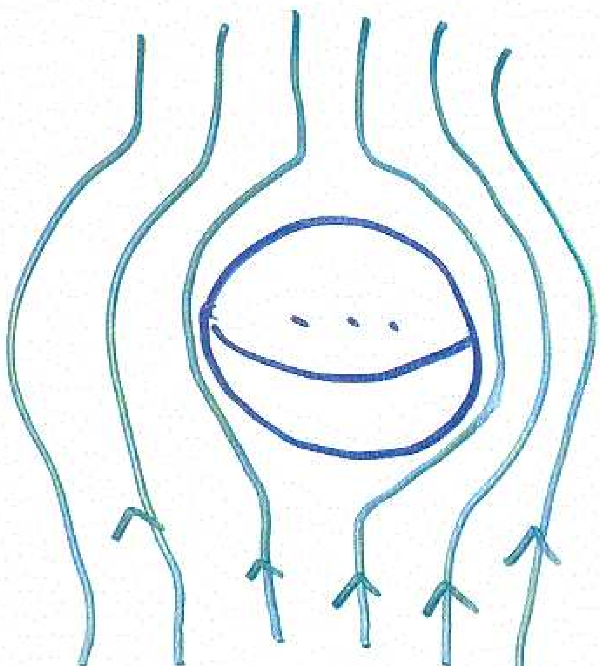
A = CONNECTION, $F = dA$

$\vec{B} = *F$

s : SPACETIME $\to \mathcal{L}^2$

A SUPERCONDUCTOR HAS MANY STRANGE PROPERTIES

MEISSNER EFFECT



MAGNETIC FIELD IS ZERO IN A SUPERCONDUCTOR

AS

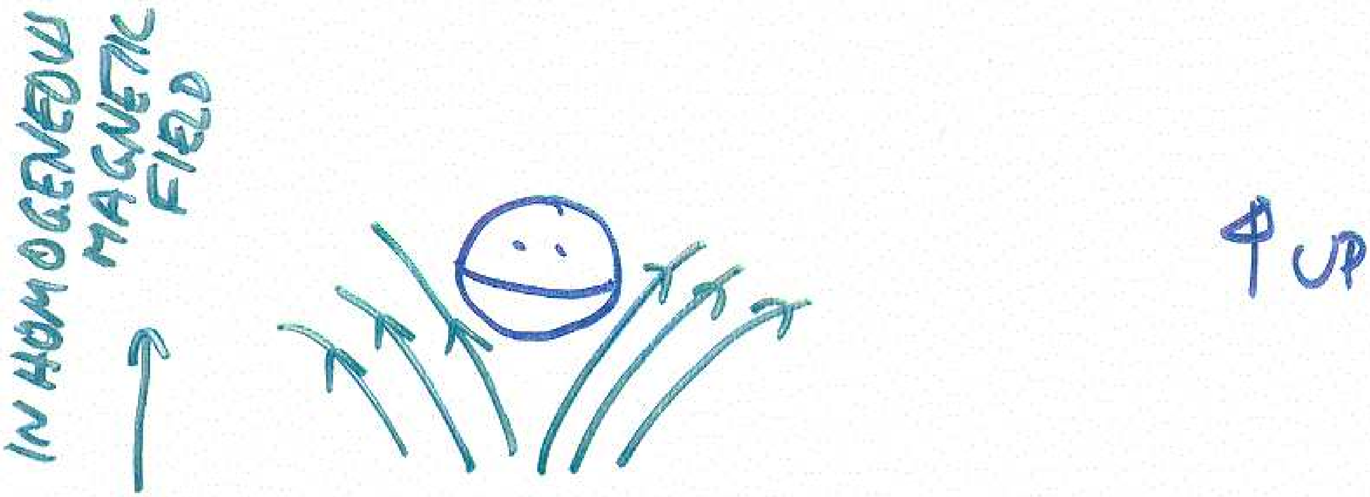
$d_A s = 0, \; s \neq 0$

$\Rightarrow$

$0 = d_A^2 s = F_A s$

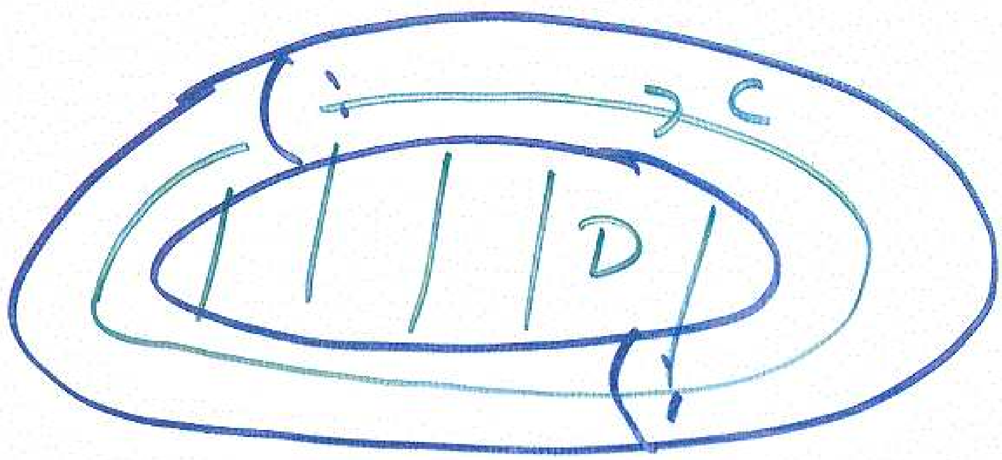
$\therefore \; F_A = 0$

THIS LEADS TO "MAGNETIC LEVITATION"



$$\int \frac{B^2}{8\pi}$$

NOW LET'S UNDERSTAND SUPERCONDUCTIVITY:



RING OF SUPERCONDUCTING MATERIAL

C = CIRCLE INSIDE THE RING

C = BOUNDARY OF A DISC D (NOT DRAWN)

$\mathcal{L}|_D$ IS TRIVIAL BECAUSE

D IS CONTRACTIBLE.

BUT $\mathcal{L}$ IS TRIVIALIZED

ON $C = \partial D$ BY s

AND $(D, \partial D) \simeq S^2$

D 

SO $\mathcal{L}|_D$ HAS A

"RELATIVE FIRST CHERN CLASS"

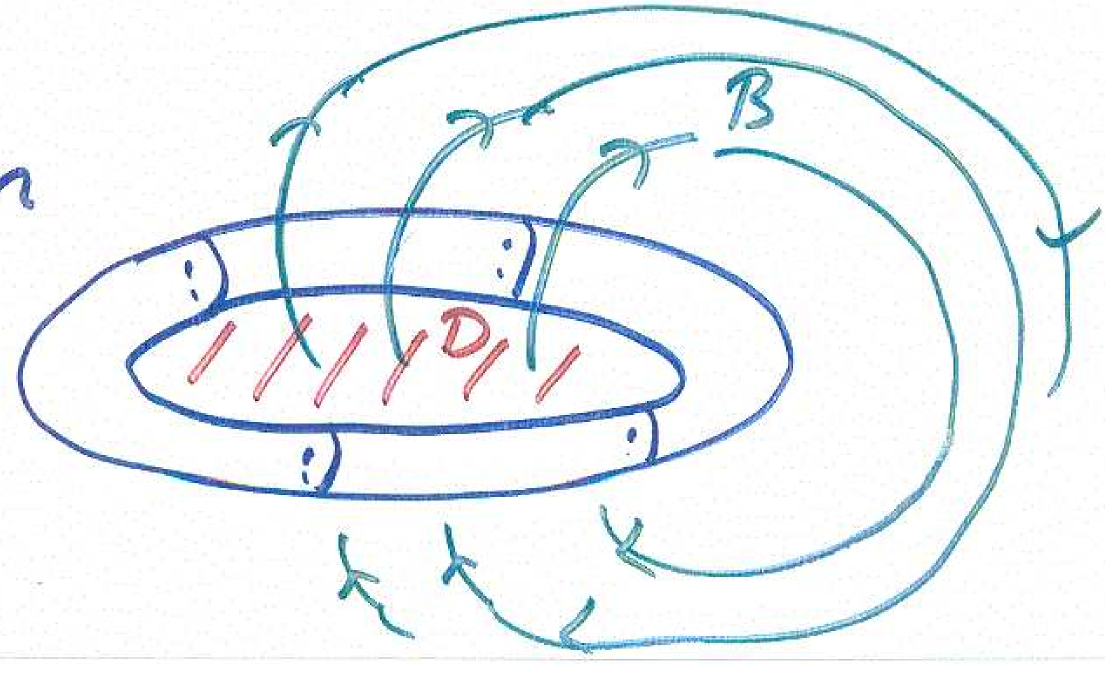
RELATIVE TO s. SET

$n = c_1(\mathcal{L}, s)$

SINCE IT IS AN INTEGER IT CANNOT CHANGE (AS LONG AS IT IS TRUE THAT $S \neq 0$ EVERYWHERE DEEP IN THE SUPERCONDUCTOR)

A STATE WITH $n \neq 0$ HAS MAGNETIC FIELDS AND CURRENTS THAT LAST "FOREVER"

$$\int_D B = 2\pi n$$

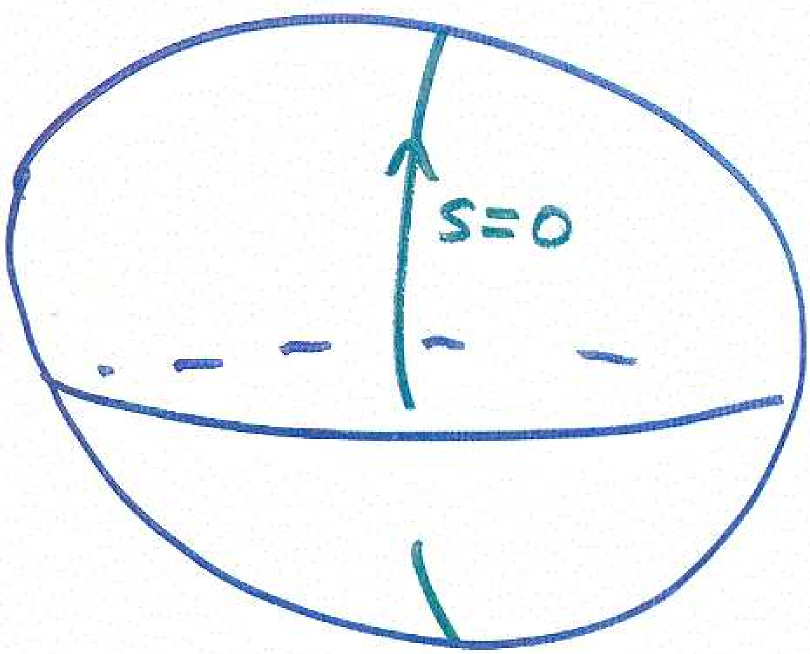


THE REASON n EVENTUALLY DOES CHANGE IS THAT ALTHOUGH THERE IS A COST IN ENERGY TO HAVE $s=0$, THIS WILL SOMETIMES HAPPEN, BY THERMAL (AND QUANTUM) FLUCTUATIONS.

SO LET US CONSIDER A SUPERCONDUCTOR IS WHICH $s=0$ SOMEWHERE

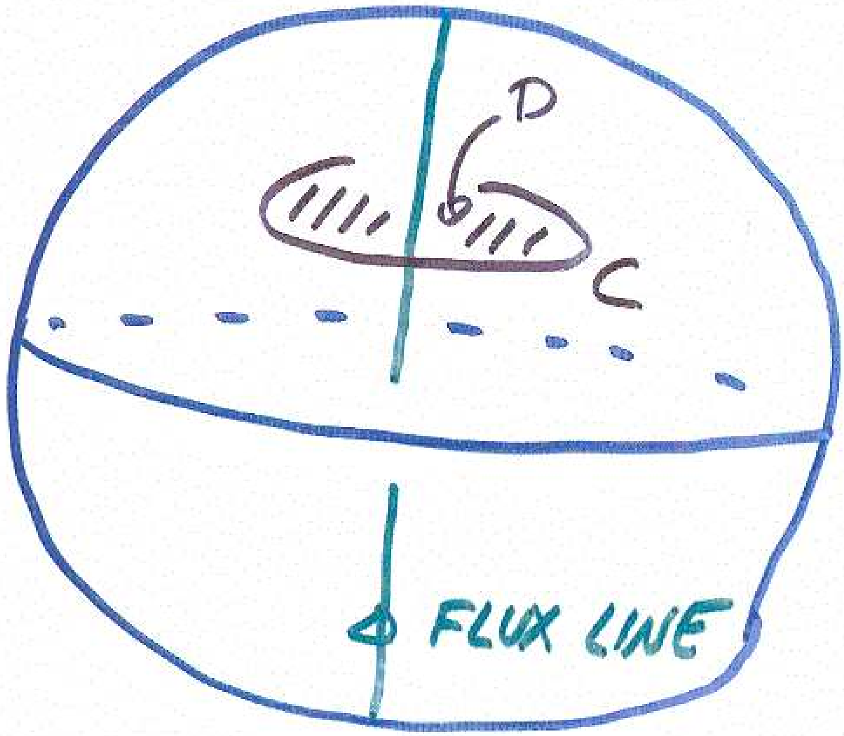
BY TRANSVERSALITY WE EXPECT

$S = 0$ IN CODIMENSION TWO:



LINES ON WHICH $S=0$ ARE OF GREAT SCIENTIFIC AND INDUSTRIAL IMPORTANCE AND ARE CALLED "ABRIKOSOV-GORKOV FLUX LINES"

A FLUX LINE IS CHARACTERIZED BY AN INTEGER:



JUST AS BEFORE

$$n = c_1\left(\mathcal{L}|_{(D,\partial D)}\right)$$

IS AN INTEGER

WE DESCRIBE A FLUX LINE MATHEMATICALLY AS A LOCAL MINIMUM OF THE ENERGY FUNCTION

$$H = \int_{\mathbb{R}^3} d^3x \left[|d_A s|^2 + \frac{\lambda}{4}(|s|^2 - a^2)^2 + \frac{1}{8\pi}\vec{B}^2 \right]$$

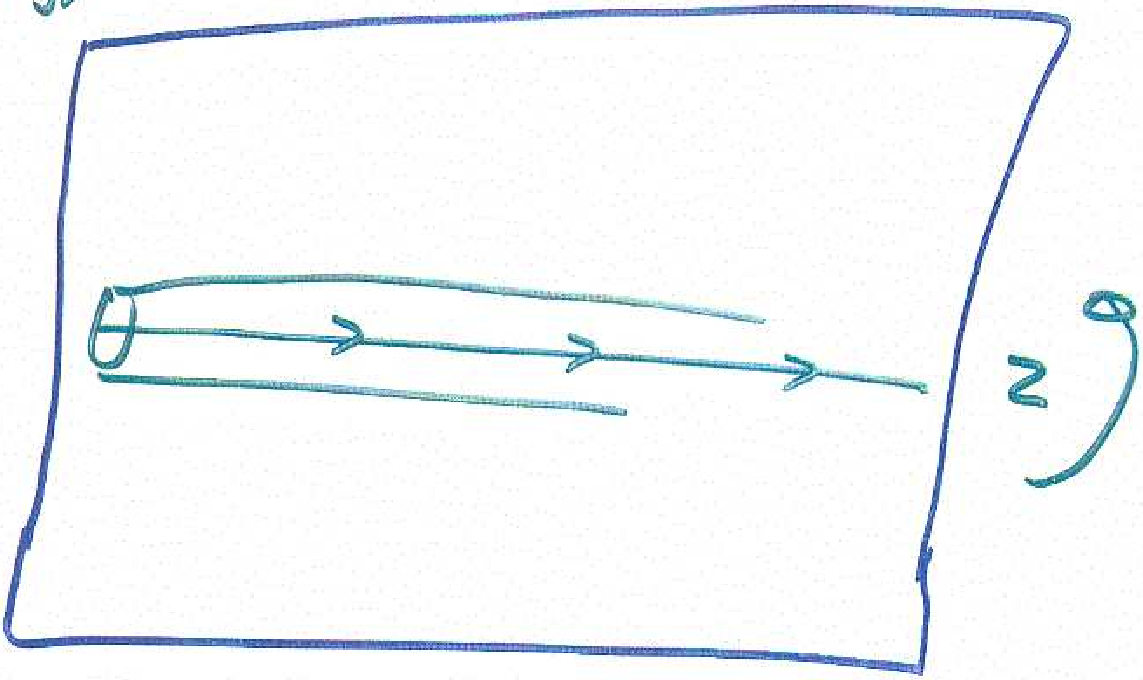
IN FACT WE WORK ON $\mathbb{R}^3$ BUT ASSUME TRANSLATION INVARIANCE IN ONE DIRECTION, SO EULER-LAGRANGE EQN = 2^{nd} ORDER P.D.E. ON $\mathbb{R}^2$

ROUGHLY

$$*d*F = i\left(\bar{s}\, d_A s - (d_A \bar{s})\, s\right)$$

$$*d_A * d_A s = \frac{\lambda}{2} s\left(|s|^2 - a^2\right)$$

$s(x+iy) = x+iy$

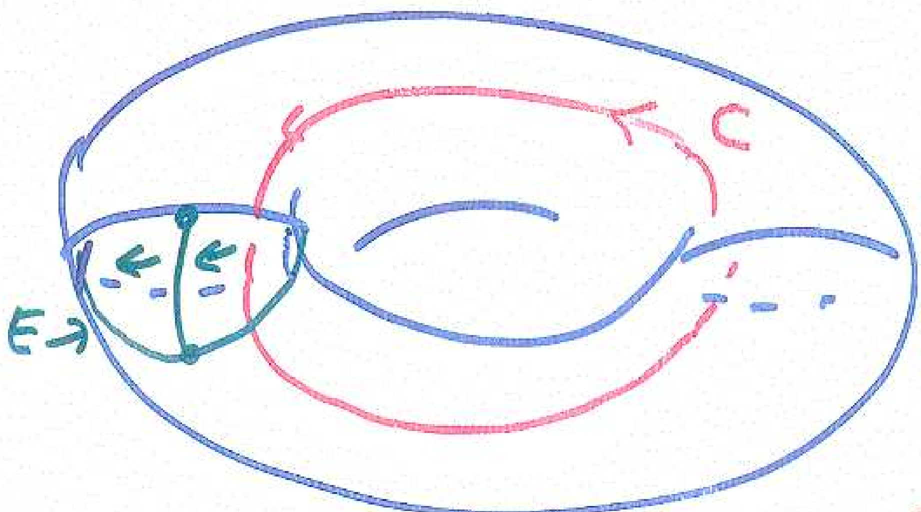


THESE P.D.E.'s DESCRIBE

A FLUX LINE WITH TRANSLATION SYMMETRY

REDUCE TO O.D.E. USING ROTATION SYMMETRY

THE WAY THE CURRENT IN A SUPERCONDUCTING LOOP RELAXES IS TO "NUCLEATE" A FLUX LINE THAT MIGRATES ACROSS THE LOOP



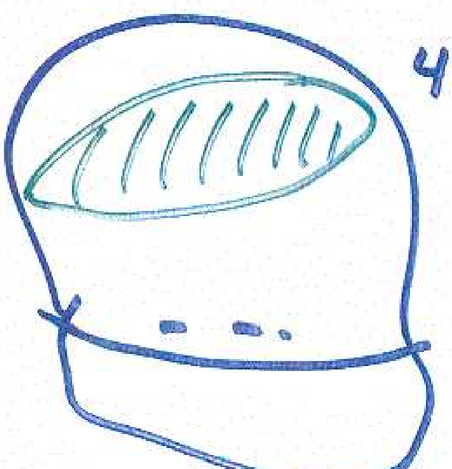
$C \cap E = \pm 1$

n, THE RELATIVE $c_1(D, \partial D)$, CHANGES BY 1 IN THIS PROCESS.

IF I INCLUDE TIME IN THE DESCRIPTION, THE SUPERCONDUCTOR FILLS OUT A FOUR-MANIFOLD

$$D^2 \times S^1 \times \mathbb{R}$$

AND THE FLUX LINE FILLS OUT THE TWO-MANIFOLD D^2



4-MANIFOLD WITH 2-MANIFOLD OF MINIMAL AREA

THE DIFFERENTIAL EQUATION DESCRIBING THE FLUX LINE

$$*d*F = i(\bar{s}\, d_A s - d_A \bar{s}\, s)$$

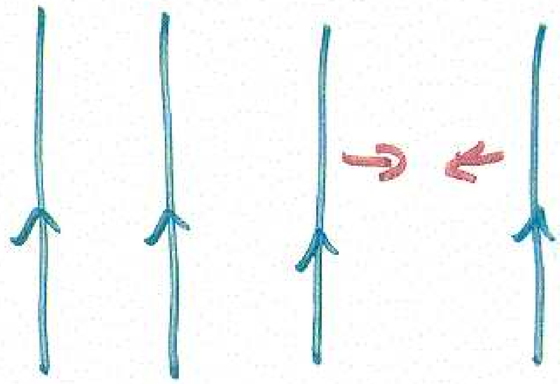
$$*d_A * d_A s = \frac{\lambda}{2} s\,(|s|^2 - a^2)$$

DEPENDS ON A DIMENSIONLESS NUMBER

ROUGHLY λ/a^2

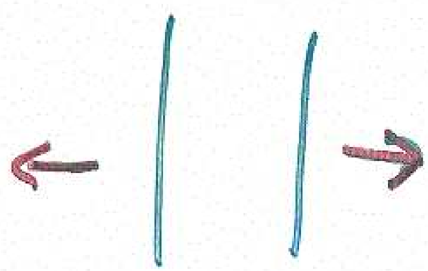
AS WE VARY λ/a^2 WE GO THROUGH A "PHASE TRANSITION"

FOR SMALL λ, THE FLUX LINES ATTRACT



"TYPE 1 SUPERCONDUCTOR"

LARGE λ FLUX LINES REPEL



"Type II SUPERCONDUCTOR"

(MOST SIMPLE MATERIALS ARE Type I BUT MANY Type II ARE KNOWN)

JUST AT THE BOUNDARY

BETWEEN Type I AND Type II

SOMETHING NICE HAPPENS... THE

SECOND ORDER EQUATIONS DESCRIBING

FLUX LINES CAN BE REDUCED TO

FIRST ORDER EQUATIONS

THE COMPLEX STRUCTURE OF

$\mathbb{R}^2 \simeq \mathbb{C}$ APPEARS

1) $\bar{\partial}_A s = 0$

2) $*F_A = 1 - |s|^2$

(I SET $\lambda = a = 1$)

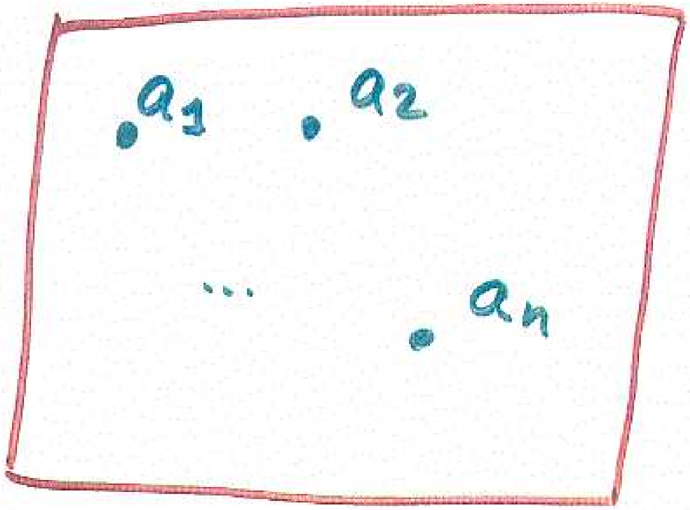
THE LINE BUNDLE $\mathcal{L} \to \mathbb{C}$ IS HOLOMORPHICALLY TRIVIAL, SO AFTER TRIVIALING IT,

1) IS SOLVED WITH

$$s = \prod_{i=1}^{n} (z - a_i)$$

a_i: ARE POINTS IN $\mathbb{C} = R^2$

AT WHICH WE PUT FLUX LINES:



EQUATION 2) IS AN EQUATION THAT DETERMINES THE METRIC ON THE TRIVIAL HOLOMORPHIC BUNDLE — IT IS PROVED TO HAVE A UNIQUE SOLUTION.

ALL THIS HAS AN ANALOG

IN FOUR-MANIFOLD THEORY

X = FOUR-MANIFOLD WITH

Spin^c STRUCTURE V

M = A SECTION OF THE Spin^c BUNDLE V

A = A $U(1)$ "CONNECTION"

($2A$ = A CONNECTION ON $\det(V)$)

SEIBERG-WITTEN EQNs:

$$\not{D} M = 0$$

$$F_A^+ = \bar{M} \Gamma M$$

$\left(V \simeq \mathrm{Spin}^+(M) \otimes \mathcal{L} \right)$

$\bar{\partial}_A S = 0$

$*F = 1 - \bar{S} S$

THE SEIBERG-WITTEN INVARIANTS

ARE, ROUGHLY SPEAKING, THE

NUMBER OF SOLUTIONS OF THE

SEIBERG-WITTEN EQUATIONS

FOR GIVEN V.

THEY CONTAIN INFORMATION
ABOUT A FOUR-MANIFOLD THAT
IS ROUGHLY COMPARABLE TO THAT
CONTAINED IN THE DONALDSON
INVARIANTS.

IT IS EXTREMELY USEFUL TO PERTURB THESE EQUATIONS USING A CLOSED TWO-FORM ON X. ONE IMPORTANT CASE (DEVELOPED BY TAUBES) IS THAT X IS SYMPLECTIC WITH SYMPLECTIC STRUCTURE ω

PERTURB EQUATIONS TO

$$0 = \not{D} M$$

$$F_A^+ = \omega^+ + \bar{M} \Gamma M$$

TAKE $X = \mathbb{R}^4$

AND $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4$

ASSUME TRANSLATION INVARIANCE

IN x_3, x_4

THE SPIN^c STRUCTURE IS TRIVIAL

AND M REDUCES TO A PAIR

(s, t) OF SECTIONS OF

TRIVIAL LINE BUNDLE $\mathcal{L}$

SEIBERG-WITTEN EQUATIONS BECOME

$$0 = \bar{\partial}_A s = \partial_A t$$

i.e. s IS HOLOMORPHIC,

t IS ANTIHOLOMORHIC

AND

$$*F_A = 1 - |s|^2 + |t|^2$$

ONE CAN DEDUCE THAT $t=0$

AND THEN THE EQNS REDUCE TO

THE ONES WE SAW FOR

A SUPERCONDUCTOR

NOW CONSIDER A COMPACT SYMPLECTIC FOUR-MANIFOLD X

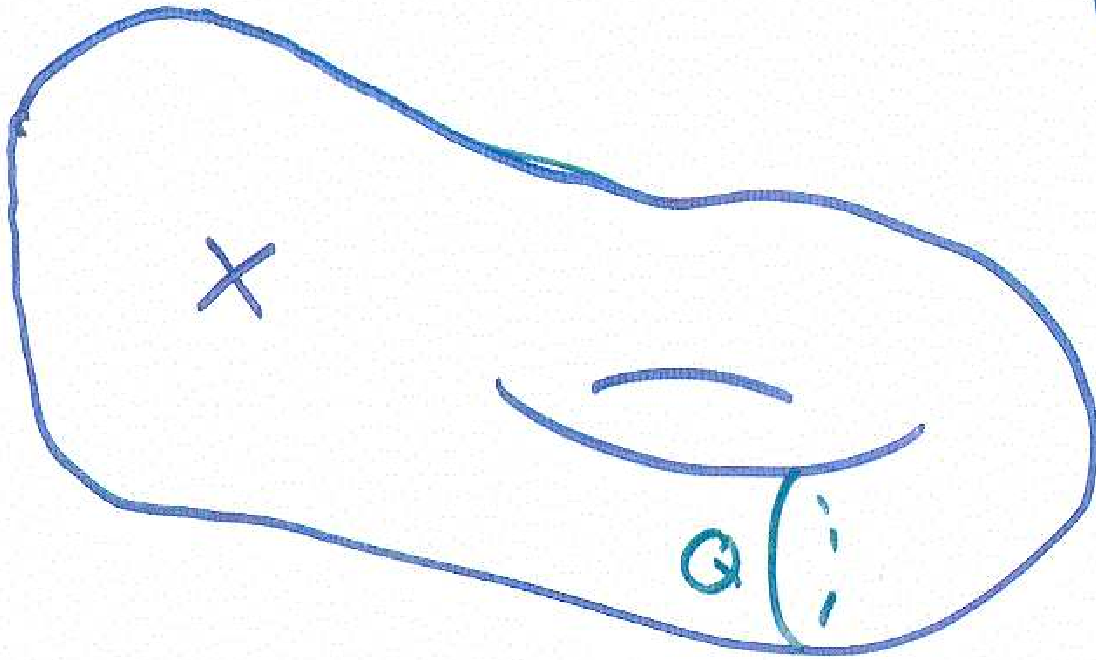
SCALE THE METRIC g LIKE

$g \to tg$ $\quad$ t LARGE

TAUBES GETS A DESCRIPTION OF THE SEIBERG-WITTEN SOLUTIONS

$$0 = \not{D} M$$
$$F^{+} = t\omega^{+} + \bar{M}\Gamma M$$

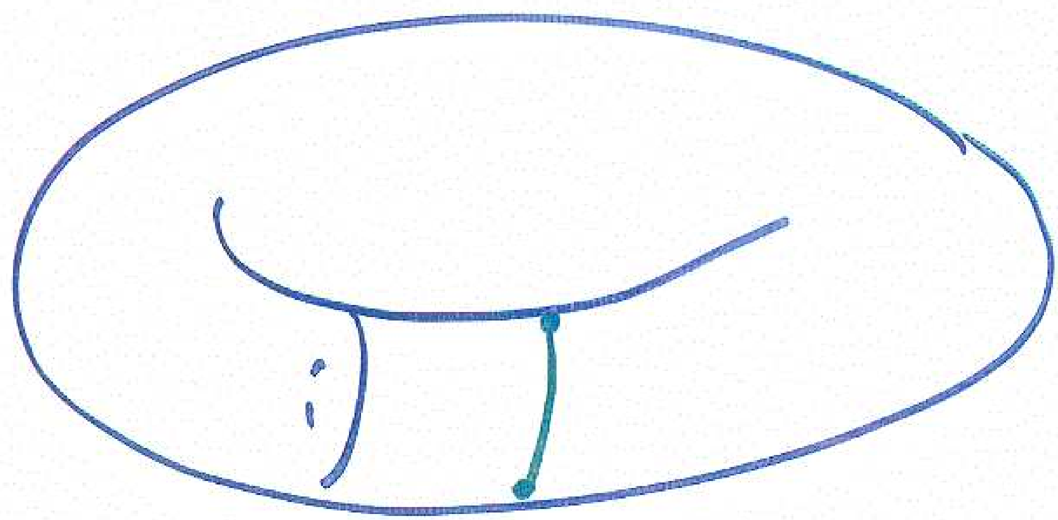
IN TERMS OF PSEUDOHOLO-MORPHIC CURVES



Q = 2-MANIFOLD ... THE CURVATURE F_A OF THE CONNECTION SOLVING SEIBERG-WITTEN EQUATIONS IS SUPPORTED NEAR Q

... THE SOLUTION IN NORMAL PLANE TO Q LOOKS LIKE THE SOLUTION WE DESCRIBED ON $\mathbb{R}^2$

THE PICTURE IS VERY SIMILAR TO THE PICTURE OF CURRENT RELAXATION IN A SUPERCONDUCTING LOOP



ONLY A THREE-DIMENSIONAL SLICE HAS BEEN DRAWN

THE SEIBERG-WITTEN EQNS
WITH THE SYMPLECTIC FORM

$$F_A^+ = t\omega^+ + \bar{M}\Gamma M$$

$$\not{D} M = 0$$

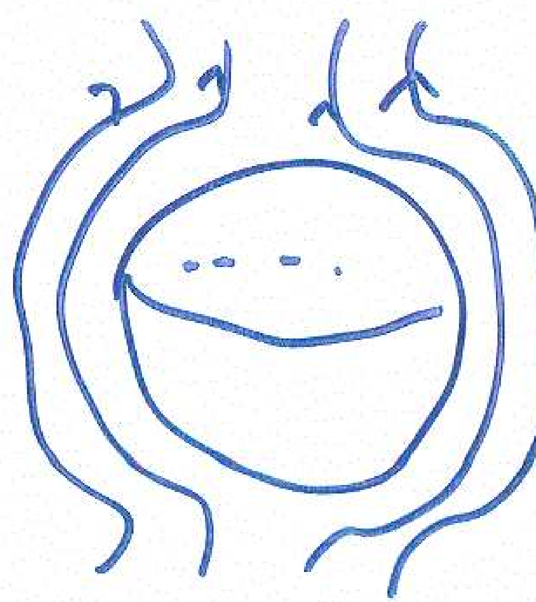
MINIMIZE AN "ACTION" FUNCTIONAL

$$I = \int d^4x \sqrt{g} \Big(|F_A|^2 + |d_A M|^2 + (\bar{M}M - t)^2 \Big)$$

THAT HAS ALL THE ESSENTIAL FEATURES OF THE LANDAU-GINZBURG FUNCTIONAL FOR A SUPERCONDUCTOR

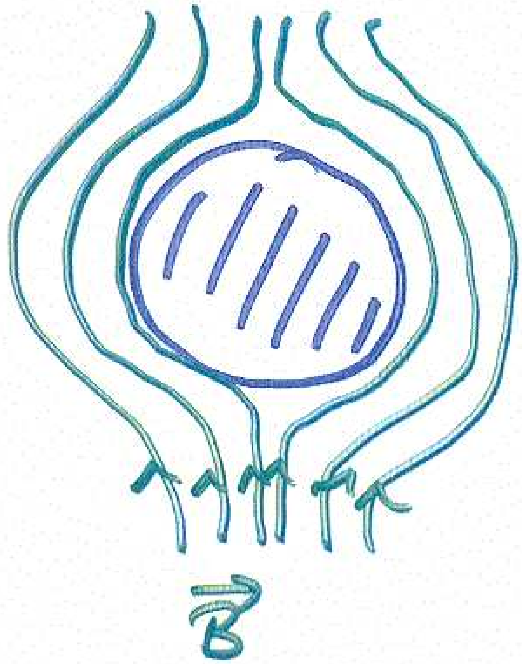
THE FACT THAT THE LINE BUNDLE IS TRIVIALIZED BY A CONFIGURATION OF LEAST ENERGY IS THE ESSENCE OF SUPERCONDUCTIVITY

THE LANDAU-GINZBERG MODEL, WITH JUST A SINGLE S, IS A MINIMAL EXAMPLE.



FOUR-MANIFOLD THEORY, AS STUDIED W TAUBES'S APPROACH TO THE SEIBERG-WITTEN EQUATIONS, INVOLVES A MORE ELABORATE EXAMPLE WITH THE SAME ESSENTIAL PROPERTIES.

NOW – AS A PRELUDE TO THE LAST PART OF THIS TALK – I WANT TO RECONSIDER THE MEISSNER EFFECT:



IT IS AN IDEALIZATION TO JUST SAY $\vec{B}=0$ INSIDE THE SUPERCONDUCTOR

THE MAGNETIC FIELD MUST PENETRATE TO SOME EXTENT

33.8

THE LINEARIZED LANDAU-GINZBURG EQUATIONS ARE A PAIR OF COUPLED EQUATIONS FOR

s AND $\vec{A}$

HOWEVER THIS CAN BE SIMPLIFIED

near

$$\left(s = q_{AS} = \vec{B} = 0 \right)$$

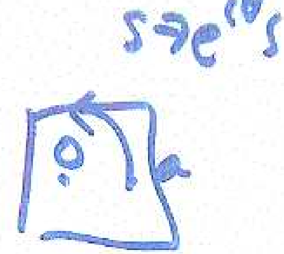
TO DO BETTER, WE MUST RE-EXAMINE THE LANDAU-GINZBURG THEORY

$$H = \int d^3x \left[|d_A s|^2 + \frac{\lambda}{4}\left(|s|^2 - a^2\right)^2 + \frac{1}{8\pi}\vec{B}^2 \right]$$

(where $\vec{B} = *F_A$)

THE "VACUUM" SOLUTION IS

$A = 0, \quad s = a$



AND WE WANT TO LINEARIZE AROUND THIS TO FIND THE NEAR VACUUM BEHAVIOR.

WE PICK THE GAUGE CONDITION THAT s IS REAL AND POSITIVE

SO $\quad s = a + h \quad h$ REAL

FLUCTUATIONS IN A AND h DECOUPLE IN THIS GAUGE.

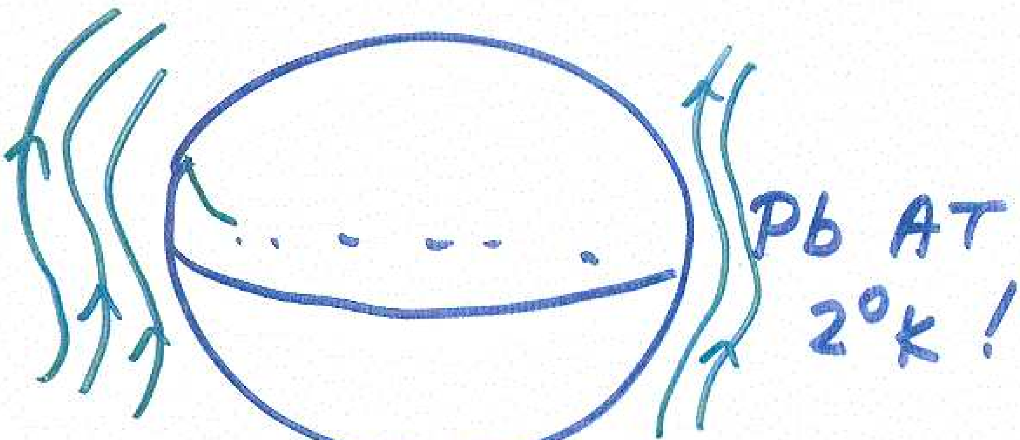
AS $d_A s = ds + iAs = iaA + \ldots$

THE ENERGY FOR A FLUCTUATION IN A IS

$$\int d^3x \left[\frac{\vec{B}^2}{8\pi} + a^2 \vec{A}^2 \right]$$

$|d_A s|^2$

THE EFFECT OF THE $e^2|A|^2$ TERM – ABSENT IN VACUUM – IS THAT INSIDE A SUPERCONDUCTOR



FLUCTUATIONS IN "A" DECAY EXPONENTIALLY IN SPACE, OR OSCILLATE RAPIDLY IN TIME.

"A" IS "MASSIVE"

CORRECTIONS TO MEISSNER EFFECT

AT SHORT DISTANCES, OR FOR SMALL TIMES, THE a^2A^2 TERM ISN'T IMPORTANT; ELECTROMAGNETIC FIELDS BEHAVE "NORMALLY."

THE OTHER EQUATION: $s = a + h$

FLUCTUATIONS IN h OBEY

$$0 = *d*d\,h + \lambda a^2 h$$

AND AGAIN h IS SHORT-RANGED OR RAPIDLY OSCILLATING IN TIME.

NOW I COME TO THE POINT OF THIS LAST PART OF MY TALK ...

APPARENTLY IN PARTICLE PHYSICS THERE IS SOMETHING LIKE THIS ... THE GAUGE GROUP IS NOT JUST U(1) OF ELECTROMAGNETISM, BUT THE LARGER GROUP U(2), WHICH IS "SPONTANEOUSLY BROKEN" TO U(1)

THERE IS A FIELD H IN THE TWO-DIMENSIONAL REPRESENTATION OF $U(2)$ WITH AN ENERGY

$$V(H) = \int d^3x \left(\frac{\lambda}{4} \left(\bar{H} H - a^2 \right)^2 \right)$$

FOR SOME $a > 0$

SO THE ENERGY IS MINIMIZED FOR $H = \begin{pmatrix} 0 \\ a \end{pmatrix}$, WHICH IS INVARIANT ONLY UNDER $U(1) \subset U(2)$

IN VACUUM, $H \neq 0$

REDUCING STRUCTURE GROUP FROM

$U(2)$ TO $U(1)$ — THAT IS WHY

UNDERGRADUATES ONLY LEARN ABOUT $U(1)$.

LET C = THE $U(2)$ CONNECTION

$$F_C = dC + C \wedge C$$

NON ABELIAN

AND

$$I = \int d^4x \Big[|d_C H|^2 + \frac{\lambda}{4}(|H|^2 - a^2)^2 + \frac{1}{4e^2}|F_C|^2 \Big]$$

TO MINIMIZE THE ENERGY

WE TAKE $H \neq 0$, SAY $H = \begin{pmatrix} 0 \\ a \end{pmatrix}$

WHICH "REDUCES THE STRUCTURE GROUP" FROM $U(2)$ TO THE $U(1)$ SUBGROUP

$$\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & 1 \end{pmatrix} \subset U(2)$$

THAT LEAVES H FIXED.

AT LOW ENERGIES, THE $U(2)$ THEORY LOOKS LIKE A $U(1)$ THEORY.

IN VACUUM, SAY

$$C = 0$$

$$H = \begin{pmatrix} 0 \\ a \end{pmatrix}$$ AND WE IMPOSE THE

GAUGE CONDITION

$$H = \begin{pmatrix} 0 \\ a + h \end{pmatrix}$$ h REAL

THE GAUGE FIELD

C = 2×2 HERMITIAN MATRIX OF ONE-FORMS

$$= \begin{pmatrix} A & W \\ \bar{W} & Z \end{pmatrix}$$

NOW A, WHICH GAUGES THE "UNBROKEN" U(1), OBEYS ESSENTIALLY MAXWELL'S EQUATIONS – AND ITS EFFECTS WERE KNOWN PREHISTORICALLY.

W AND Z OBEY, AT THE LINEAR LEVEL, "MASSIVE" EQUATIONS LIKE THOSE WE FOUND IN A SUPERCONDUCTOR

$0 = (*d)^2 W + e^2 a^2 W$, SIMILAR FOR Z

BECAUSE OF THE VERY HIGH FREQUENCIES AND SHORT DISTANCES INVOLVED, W AND Z WERE DISCOVERED JUST ABOUT 20 YEARS AGO AT THE PROTON-ANTIPROTON COLLIDER AT CERN. { ALMOST THE HEAVIEST KNOWN ELEMENTARY PARTICLES

THE LAST PIECE OF THIS PICTURE IS THE "HIGGS PARTICLE" h APPEARING IN

$$H = \begin{pmatrix} 0 \\ a+h \end{pmatrix}$$

A FEW YEARS AGO, THERE WAS A HINT OF DISCOVERY OF h AT THE e^+e^- COLLIDER AT CERN, BUT THIS WAS INCONCLUSIVE. SO THIS PIECE OF THE STANDARD MODEL OF PARTICLE PHYSICS REMAINS TO BE CONFIRMED, AND STUDIED.