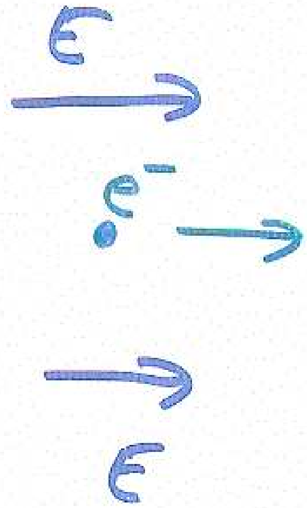
TODAY, I'LL TELL YOU ABOUT THE QUANTUM HALL EFFECT; WE'LL BUILD IT UP IN STAGES:

1) WHAT IS THE HALL EFFECT?
2) WHAT IS THE QUANTUM HALL EFFECT?
3) SOME CONDENSED MATTER PHYSICS
4) SOME GEOMETRY
5) THE JONES POLYNOMIAL
6) THEORY OF THE QUANTUM HALL EFFECT

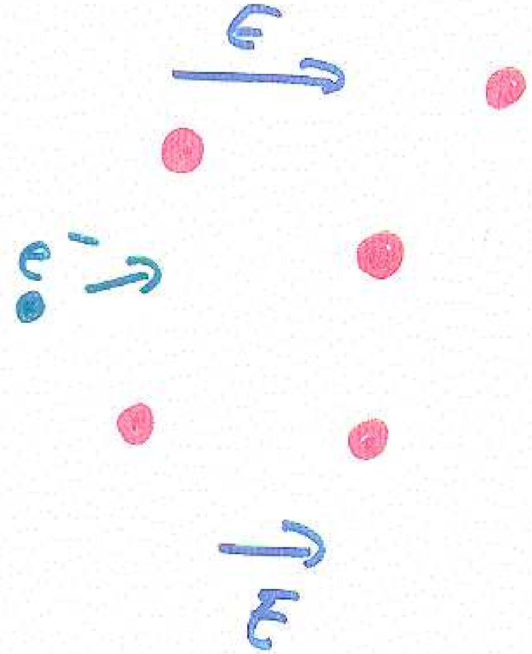


AN ELECTRON IN AN ELECTRIC FIELD IS ACCELERATED

$$m \frac{d^2 \vec{x}}{dt^2} = e \vec{E}$$

SO $\vec{V} = \frac{d\vec{x}}{dt} \sim \frac{e\vec{E}}{m} t$

IF THE ELECTRON IS IN AN OBSTACLE COURSE



WHICH RANDOMIZES ITS VELOCITY, ON THE AVERAGE) AFTER A TIME τ, THEN THE VELOCITY WILL NOT GO TO INFINITY, BUT

TO

$$\langle \vec{v} \rangle \sim \frac{e\vec{E}}{m} \frac{\tau}{2}$$

IF n IS THE NUMBER DENSITY OF ELECTRONS THAT ARE FREE TO MOVE, THE AVERAGE CURRENT WILL BE ROUGHLY

$$\vec{J} = ne\langle \vec{v} \rangle = \frac{ne^2\tau}{2m} \vec{E}$$

WHICH IS OHM'S LAW

$$\vec{J} = \sigma \vec{E}$$

WHERE THE CONDUCTIVITY IS

IN THIS APPROXIMATION

$$\sigma = \frac{n e^2 \tau}{2m}$$

NO TOPOLOGY HERE - σ

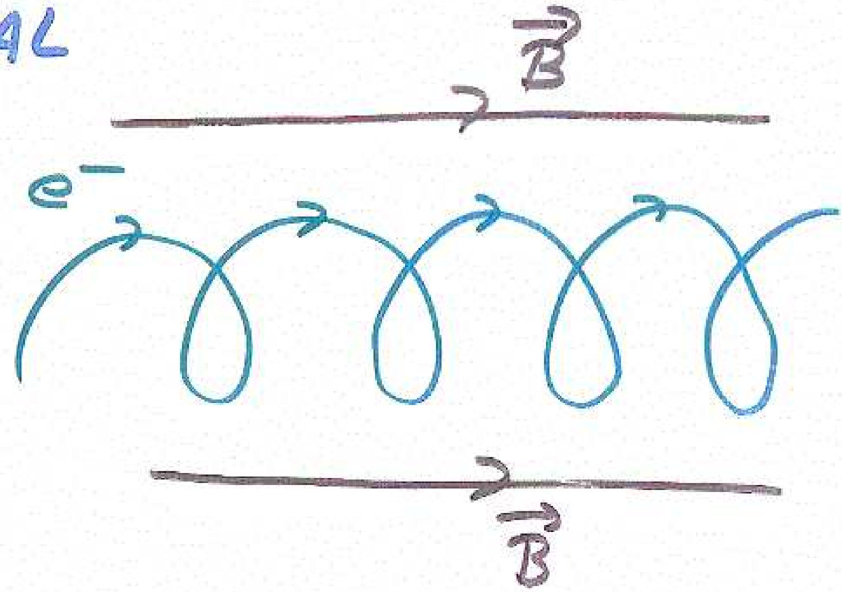
DEPENDS ON ALL THE DETAILS

NOW IN A MAGNETIC FIELD ⑥ $\vec{B}$

$$m \frac{d^2\vec{x}}{dt^2} = \frac{e}{c}\frac{d\vec{x}}{dt} \times \vec{B}$$

THE ELECTRON MOTION IS A

SPIRAL



UNIFORM MOTION ALONG $\vec{B}$ AND

CIRCULAR OCILLATIONS NORMAL

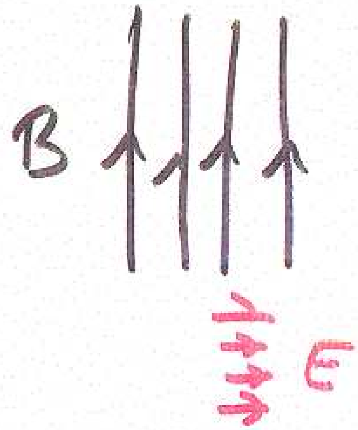
TO $\vec{B}$

TO GET THE HALL EFFECT, 7

WE WANT COMBINED PERPENDICULAR

$\vec{E}$ AND $\vec{B}$ WITH $|\vec{E}| << |\vec{B}|$

$E \perp B$



THE ELECTRON NOW DRIFTS

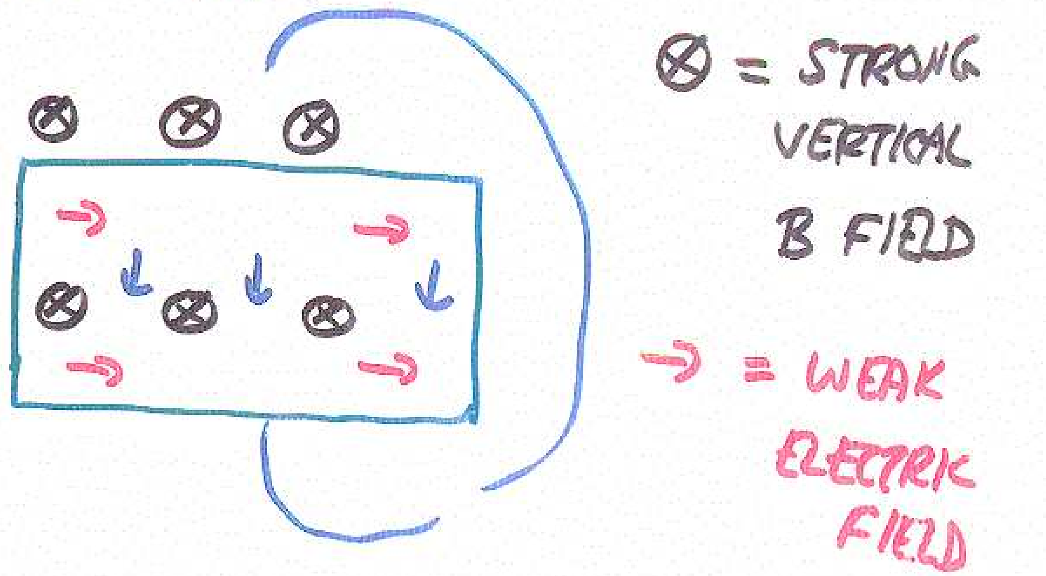
OUT OF THE PAGE, WITH

VELOCITY

$$\vec{v} = c \frac{\vec{E} \times \vec{B}}{|B|^2}$$

WHAT THIS MEANS FOR

SOLIDS IS THIS



WITH NO MAGNETIC FIELD, THE CURRENT J IS IN THE $\vec{E}$ DIRECTION,

BUT WITH A STRONG MAGNETIC FIELD, THE CURRENT FLOW IS PERPENDICULAR TO $\vec{E}$ AND $\vec{B}$

WITH AVERAGE VELOCITY

$$\vec{v} = c\,\frac{\vec{E}\times\vec{B}}{|B|^2}$$ FOR EACH ELECTRON

THE CURRENT IS

$$\vec{J} = ne\vec{v} = nec\,\frac{\vec{E}\times\vec{B}}{|B|^2}$$

$$= \sigma_H\,\hat{B}\times\vec{E}$$

WHERE $\hat{B} = \vec{B}/|B|$ AND THE HALL CONDUCTIVITY IS

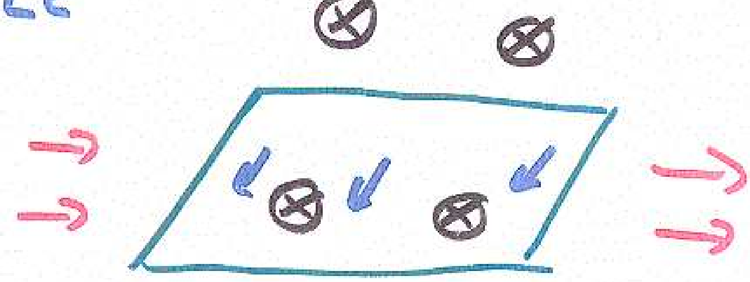
$$\sigma_H = nec/|B|$$

NO TOPOLOGY HERE!

THE QUANTUM HALL EFFECT ARISES

IF WE CONSIDER A VERY THIN SAMPLE

SPECIAL COMPOSITION, LOW TEMP. LARGE B



WHICH EFFECTIVELY IS A TWO-DIMENSIONAL SURFACE

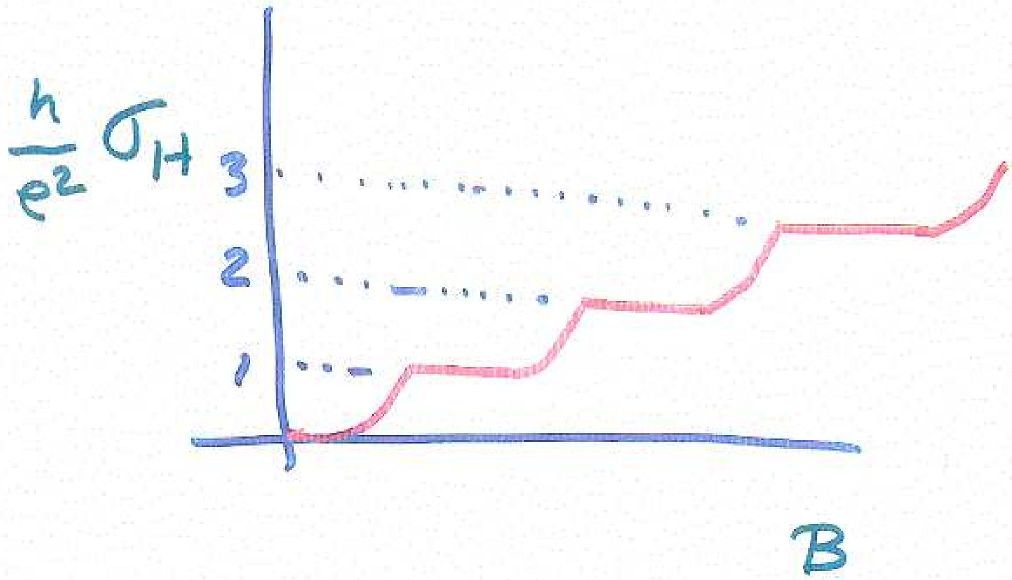
- A MOLECULAR MONOLAYER.

HERE WE FIND, FOR CERTAIN TYPES OF MATERIAL, IN CERTAIN RANGES OF $\vec{B}$ AND TEMPERATURE, THAT THE HALL CONDUCTIVITY DOES HAVE A MAGIC VALUE

$$\sigma_H = \frac{e^2}{h} \cdot n, \qquad n \in \mathbb{Z}$$

THIS VALUE IS UNCHANGED AS $\vec{B}$, TEMPERATURE, CHEMICAL COMPOSITION, IMPURITY CONCENTRATION, ETC., ARE VARIED - WITHIN CERTAIN LIMITS.

h = Planck's constant



THE INTEGRALITY OF $\frac{h}{e^2}\sigma_H$

IS VERIFIED SO PRECISELY THAT IT

HAS BECOME THE MOST ACCURATE

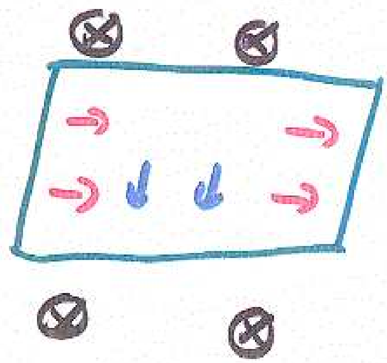
WAY TO MEASURE

$$\frac{e^2}{h}$$

THERE IS AN INTEGER HERE,
SO WE SHOULD SEEK A TOPOLOGICAL
EXPLANATION!

BUT HOW CAN WE HOPE TO
FIND ONE IN THE MESSY
WORLD OF CONDENSED MATTER
PHYSICS?

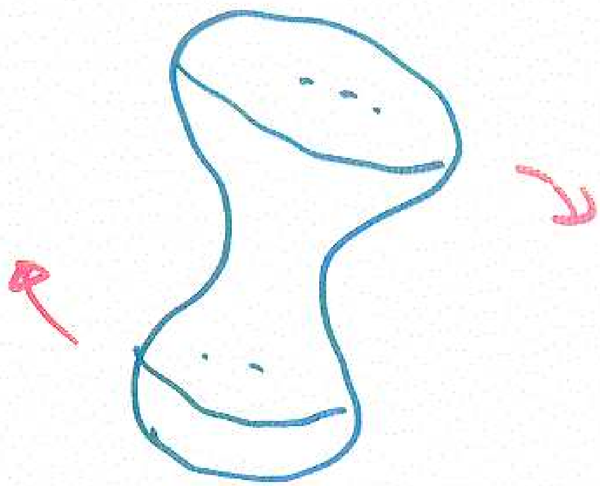
IN THIS MEASUREMENT



WE ARE LOOKING AT VERY LARGE SAMPLES, OBSERVED FOR VERY LONG TIMES, COMPARED TO ATOMIC DISTANCES AND TIMES.

ONE OF THE MAIN REASONS THAT IT IS POSSIBLE TO MAKE PROGRESS IN PHYSICS IS THAT IN ANY OBSERVATION MADE AT A CERTAIN SCALE OF LENGTH AND TIME, MOST OF THE MICROSCOPIC DEGREES OF FREEDOM – NEEDED FOR A PRECISE DESCRIPTION AT MUCH SHORTER LENGTHS AND TIMES – ARE IRRELEVANT.

FAMILIAR EXAMPLE: A RIGID BODY



THE ONLY RELEVANT DEGREE OF FREEDOM IS THE TIME-DEPENDENT EMBEDDING IN SPACE.

SOLIDS VARY WIDELY IN THEIR RELEVANT DEGREES OF FREEDOM:

1. VIBRATIONS ARE ALWAYS RELEVANT FOR CERTAIN QUESTIONS, BUT NOT FOR ELECTROMAGNETIC BEHAVIOR LIKE THE HALL EFFECT

2. ELECTRONS ARE RELEVANT IN CONDUCTORS, WHERE THEY ARE FREE TO MOVE, BUT NOT IN INSULATORS

(3) AN EXOTIC EXAMPLE OF A RELEVANT VARIABLE IS THE TRIVIALIZATION s OF THE LINE BUNDLE $\mathcal{L}^2$ IN A SUPERCONDUCTOR

(4) ANOTHER EXOTIC EXAMPLE ARISES IN THE "FRACTIONAL QUANTUM HALL EFFECT, WHEN

$$\sigma_H \cdot \frac{h}{e^2}$$ IS A RATIONAL NUMBER, NOT AN INTEGER

WHATEVER THE RELEVANT DEGREES OF FREEDOM MAY BE IN A PARTICULAR CASE, THE MACROSCOPIC INTERACTIONS OF ELECTROMAGNETISM WITH THE SOLID ARE DESCRIBED BY EULER-LAGRANGE EQUATIONS + QUANTIZATION FROM AN ACTION

$$\mathcal{L} = \mathcal{L}_{vacuum} + \mathcal{L}_{solid}$$

HERE $\mathcal{L}_{VACUUM}$ IS THE ACTION IN VACUUM OF THE ELECTROMAGNETIC FIELD ... IF $A = U(1)$ CONNECTION

$F = dA$ = CURVATURE

$(F = \vec{E} \cdot d\vec{x} \wedge dt + \vec{B} \cdot d\vec{x} \times d\vec{x})$

THEN

$$\mathcal{L}_{VACUUM} = \frac{h}{4e^2} \int_{\mathbb{R}^{3,1}} F \wedge * F$$

THE PART OF THE ACTION DUE TO THE SOLID IS

$$\mathcal{L}_{SOLID} = \int_{\substack{(SOLID \\ \times TIME)}} d\mu \, W(\Phi, A)$$

WHERE W IS A LOCAL FUNCTIONAL OF THE RELEVANT DEGREES OF FREEDOM Φ AS WELL AS THE CONNECTION A.

"LOCALITY" MEANS THAT THE VARIATION OF W IS A GAUGE-INVARIANT DIFFERENTIAL POLYNOMIAL IN Φ, A.

THE TOTAL ACTION IS THUS

$$\mathcal{L} = \mathcal{L}_{VACUUM} + \mathcal{L}_{SOLID}$$

$$= \frac{h}{4e^2} \int_{\mathbb{R}^{3,1}} F \wedge *F + \int_{SOLID} d\mu \; W(\Phi, A)$$

THE EULER-LAGRANGE EQUATION

FOR A BECOMES

$$d(*F) = -\frac{e^2}{h} \frac{\delta W}{\delta A}(\Phi, A)$$

AND IF WE COMPARE

TO FRESHMAN PHYSICS

$$d(*F) = J$$

$$*J = (\rho, \vec{J})$$

= RELATIVISTIC FORM OF THE CURRENT

WE SEE THAT THE CURRENT IS

$$J = -\frac{e^2}{h}\frac{\delta W}{\delta A}$$

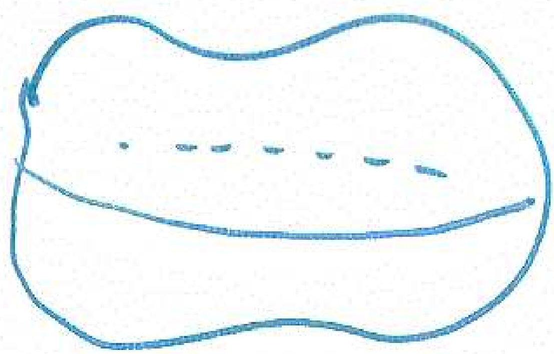
NOW WHAT ARE THE RELEVANT DEGREES OF FREEDOM FOR THE QUANTUM HALL EFFECT?

ANSWER: FOR THE ORDINARY, INTEGER QUANTUM HALL EFFECT, THERE ARE NONE.

MATERIALS WITH NO RELEVANT DEGREES OF FREEDOM IN THEIR INTERACTION WITH ELECTROMAGNETISM ARE NOT UNUSUAL - AN ORDINARY PANE OF GLASS IS AN EXAMPLE.

IT MEANS THAT $W(\Phi, A)$ REDUCES TO $W(A)$.

BEFORE TRYING TO DISCUSS THE QUANTUM HALL EFFECT, LET US DISCUSS A THREE-DIMENSIONAL PIECE OF GLASS:



THE LOCAL STRUCTURE IS INVARIANT UNDER ROTATIONS AND REFLECTIONS, WHICH WILL CONSTRAIN W.

BEFORE WE GET TO THE QUANTUM HALL EFFECT, I AM GOING TO USE THIS FACT TO DESCRIBE SOME PROPERTIES OF GLASS - JUST SO WE'LL FEEL WE KNOW WHAT WE ARE DOING.

A DRASTIC SIMPLIFICATION

COMES FROM THE FOLLOWING.

GRADE THE MONOMIALS IN W

BY "DIMENSION"

A	1
$\frac{\partial}{\partial x}, \frac{\partial}{\partial t}$	1
$\vec{E}, \vec{B}$	2

$\frac{\partial}{\partial x} + c' A$

$F = dA$

IF ALL FIELDS AND

FREQUENCIES ARE SMALL

ON AN ATOMIC SCALE, TERMS

OF HIGHER DIMENSION ARE SMALLER.

THIS IS ESSENTIALLY ALWAYS TRUE IN PRACTICE, AND ANYWAY OTHERWISE THE EFFECTIVE DESCRIPTION IS NOT VALID.

THE VACUUM PART OF THE ACTION

$$\mathcal{L}_{VACUUM} = \frac{\hbar}{4e^2} \int F \wedge * F$$

IS OF DIMENSION FOUR, SO WE CAN NEGLECT IN $W(A)$ TERMS OF DIMENSION GREATER THAN FOUR.

USING <u>ROTATION</u> <u>INVARIANCE</u>

AND <u>REFLECTION</u> <u>SYMMETRY</u>

W(A) CANNOT HAVE ANY TERMS LINEAR IN $\vec{E}$ OR $\vec{B}$ SO THE LOWEST DIMENSION TERMS ARE

$$W(A) = \int d^3x\,dt \left(\epsilon \vec{E}^2 - \mu \vec{B}^2 \right)$$

WHERE ϵ AND μ ARE THE ELECTRIC AND MAGNETIC SUSCEPTIBILITIES

TO FIND HOW GLASS INTERACTS WITH ELECTROMAGNETISM, WE JUST LOOK AT THE COMBINED LAGRANGIAN

$$\frac{h}{4e^2}\int F \wedge * F + \int d^3x\, dt \left(\epsilon \vec{E}^2 - \mu \vec{B}^2 \right)$$

$$= \int d^3x\, dt \left(\left(\frac{h}{4e^2} + \epsilon\right) \vec{E}^2 - \left(\frac{hc^2}{4e^2} + \mu\right) \vec{B}^2 \right)$$

THE RESULTING EULER-LAGRANGE EQUATIONS ARE MAXWELL'S EQUATIONS BUT WITH A REDUCED SPEED OF LIGHT.

IF INSTEAD OF GLASS, WE CONSIDER A CRYSTAL (STILL WITH NO RELEVANT DEGREES OF FREEDOM) THEN W CAN HAVE GENERAL QUADRATIC TERMS IN $\vec{E}$ AND $\vec{B}$

$$\int d^3x\, dt \left(\epsilon_{ij} E_i E_j - \mu_{ij} B_i B_j \right)$$

$$\left(\vec{E} = (E_1, E_2, E_3),\ \vec{B} = (B_1, B_2, B_3) \right)$$

WE GET AN ANISOTROPIC SPEED OF LIGHT AND "BIREFRINGENCE"

IF THE CRYSTAL LACKS REFLECTION SYMMETRY, THEN $W(A)$ CAN CONTAIN TERMS LINEAR IN $\vec{E}$ AND $\vec{B}$

$$\Delta W = \int d^3x \, dt \left(\vec{n} \cdot \vec{E} + \vec{m} \cdot \vec{B} \right)$$

(THE VECTORS $\vec{n}$, $\vec{m}$ DEPEND ON THE CRYSTAL AXES)

SUCH MATERIALS ARE "FERROELECTRIC" AND "FERROMAGNETIC", AS ONE CAN LEARN BY DERIVING AND SOLVING THE EULER-LAGRANGE EQUATIONS.

BY NOW WE KNOW HOW TO STUDY MATERIALS WITH NO RELEVANT DEGREES OF FREEDOM:

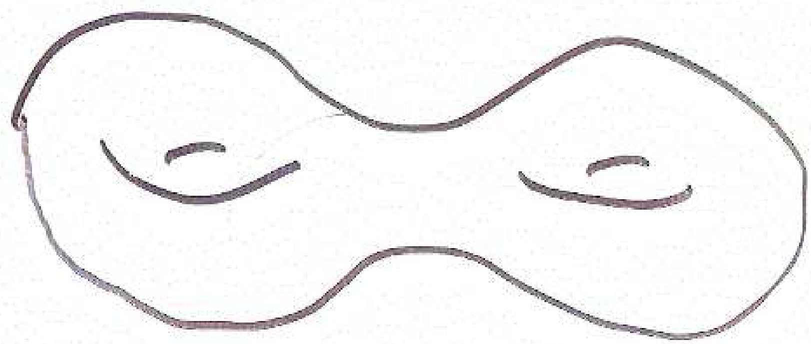
WRITE THE POSSIBLE TERMS IN W(A) OF LOW DIMENSION.

DERIVE AND SOLVE THE EULER-LAGRANGE EQUATIONS.

THE HYPOTHESIS THAT THERE ARE NO RELEVANT DEGREES OF FREEDOM ENSURES THAT <u>W(A) IS A LOCAL FUNCTIONAL OF A ONLY</u>

NOW WE ARE READY TO STUDY THE QUANTUM HALL EFFECT, WHICH INVOLVES AN ATOMIC MONOLAYER OR A TWO-DIMENSIONAL SURFACE IN SPACE

Σ



WHICH, IN SPACETIME, SWEEPS OUT A THREE-MANIFOLD $Q = \Sigma \times R$ WHERE R PARAMETRIZES TIME.

SO WE ARE DOING $U(1)$
GAUGE THEORY ON A THREE-MANIFOLD
Q, AND THE REASON THAT WE WILL
GET SOMETHING INTERESTING IS
THAT THERE IS AN UNUSUAL
TERM, OF TOPOLOGICAL INTEREST,
THAT CAN APPEAR IN $W(A)$.
THIS IS THE "CHERN-SIMONS FORM."

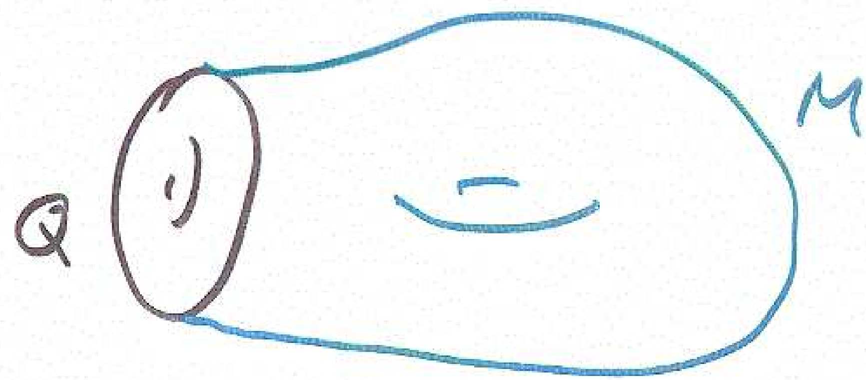
WE CAN ASSUME THAT Q

IS THE BOUNDARY OF A

FOUR-MANIFOLD M OVER WHICH

THE ELECTROMAGNETIC LINE BUNDLE $\mathcal{L}$

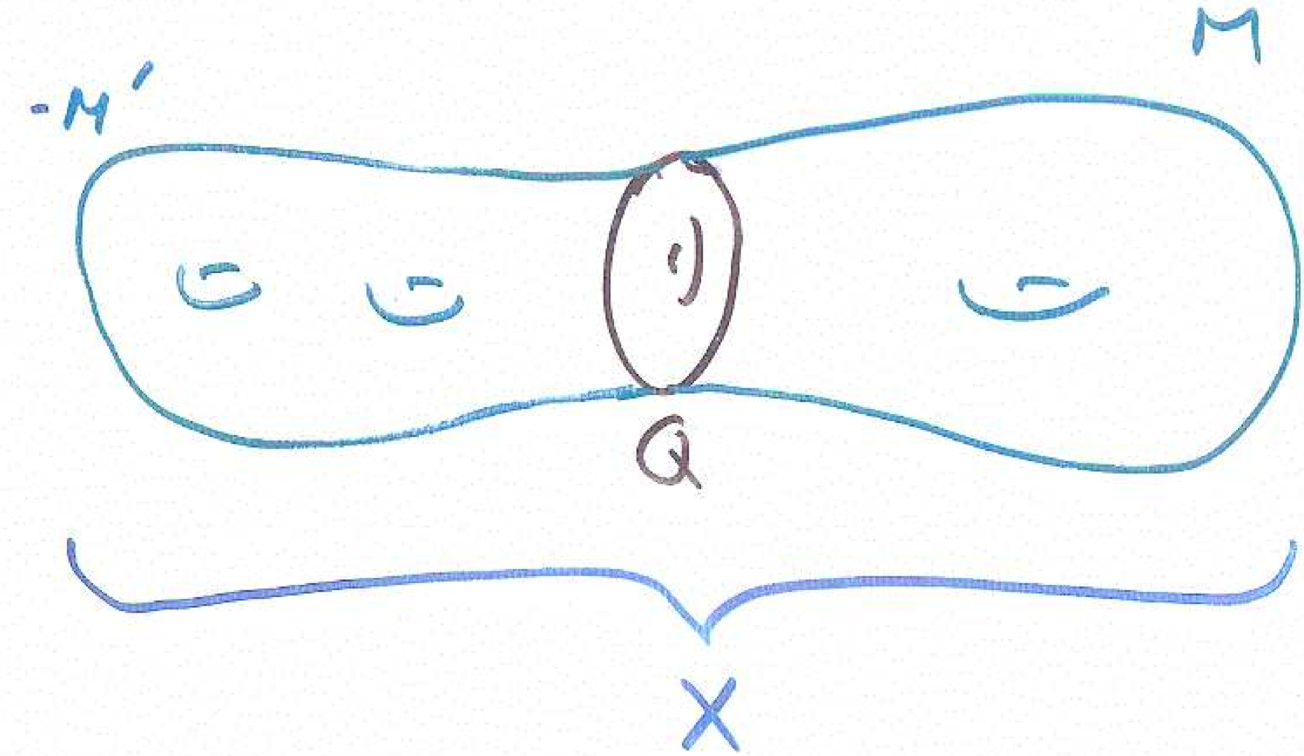
AND CONNECTION A EXTEND



AND WE WRITE

$$S_M(A) = \frac{1}{4\pi^2} \int_M F \wedge F$$

THIS DEPENDS ON M, BUT ONLY SLIGHTLY:



$$S_M(A) - S_{M'}(A) = \frac{1}{4\pi^2}\int_X F \wedge F$$

$$= \int_X c_1(\mathcal{L}) \wedge c_1(\mathcal{L})$$

$$\in \mathbb{Z}$$

SO AS A MAP TO $\mathbb{R}/\mathbb{Z}$,

$S_M(A)$ IS INDEPENDENT OF M

AND WE JUST CALL IT $S(A)$.

(CONCRETELY, FOR A TRIVIAL LINE

BUNDLE $\mathcal{L}$, WE HAVE

$$S(A) = \frac{1}{4\pi^2} \int_Q A \wedge dA \,\Big)$$

S(A) HAS A LOW DIMENSION – THREE – SO IF WE CAN INCLUDE IT IN W(A), THE EFFECTIVE ACTION, THEN THIS WILL BE IMPORTANT.

DOES IT MAKE SENSE TO DO THIS, GIVEN THAT S(A) HAS AN ADDITIVE INTEGER INDETERMINACY?

IN CLASSICAL MECHANICS, ADDING $\mathcal{L}$

A CONSTANT TO THE ACTION I

DOES NOT MATTER AS IT DOES NOT AFFECT THE EULER-LAGRANGE EQUATION.

IN QUANTIZATION, ONE NEEDS TO BE ABLE TO DEFINE

$$\exp(iI)$$

TO COMPUTE QUANTUM TRANSITION AMPLITUDES. SO I CAN HAVE AN ADDITIVE AMBIGUITY, BUT THIS MUST BE OF THE FORM

$$2\pi k$$

FOR $k \in \mathbb{Z}$.

SO A CONCEIVABLE MATERIAL MAY HAVE IN THE ELECTROMAGNETIC EFFECTIVE ACTION $W(A)$ A MULTIPLE OF $S(A)$, BUT IT MUST BE A VERY SPECIAL MULTIPLE:

$$W(A) = 2\pi k\, S(A)$$

$$= \frac{k}{2\pi} \int A \wedge dA$$

FOR SOME $k \in \mathbb{Z}$ (WHICH DEPENDS ON THE MATERIAL, THE MAGNETIC FIELD, ETC.)

THIS ALSO WORKS

IN THE NONABELIAN CASE

A = CONNECTION ON A G-BUNDLE OVER Q = 3-fold

FOR COMPACT SIMPLE G

$$W(A) = \frac{k}{2\pi}\int \underbrace{\mathrm{Tr}\left(A\, dA + \frac{2}{3} A\wedge A\wedge A\right)}_{\text{"CHERN-SIMONS FORM"}}$$

W MAPS TO $\mathbb{R}/2\pi\mathbb{Z}$

SO WE CAN CONSIDER A

QUANTUM THEORY IN WHICH

W(A) IS THE COMPLETE ACTION

THE QUANTUM PATH INTEGRAL ON A THREE-MANIFOLD M IS FORMALLY

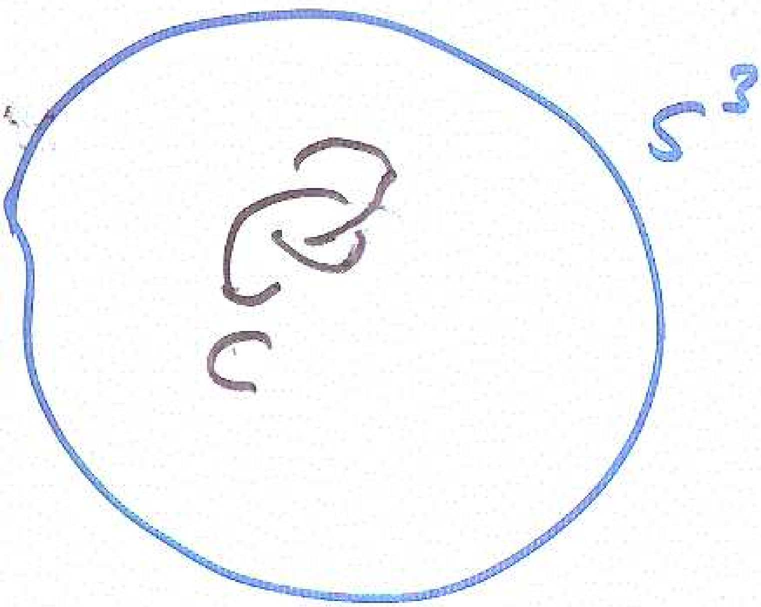
$$Z(M) = \frac{1}{\mathrm{vol}(\hat{G})} \int_{\mathcal{A}} \mathcal{D}A \, \exp i W(A)$$

$\mathcal{A}$ = THE SPACE OF CONNECTIONS

$\hat{G}$ = THE GAUGE GROUP
= maps (Q, G)

THIS IS THE QUANTUM THREE-MANIFOLD INVARIANT $\hat{G}$ $\quad q = \exp \frac{2\pi i}{k+h}$

THE SAME THEORY ALSO
LEADS TO THE JONES POLYNOMIAL
OF KNOTS AND ITS GENERALIZATIONS



$$\Phi_R(C) = \frac{1}{Z(M)} \frac{1}{\mathrm{vol}(\mathcal{G})} \int_{\mathcal{a}} \mathcal{D}A \; \exp iW(A) \; \mathrm{Tr}_R \, \mathrm{Hol}_R(A;C)$$

some R = rep of G

WHEN THE GAUGE GROUP IS

ABELIAN, ONE STILL GETS KNOT

AND THREE-MANIFOLD INVARIANTS

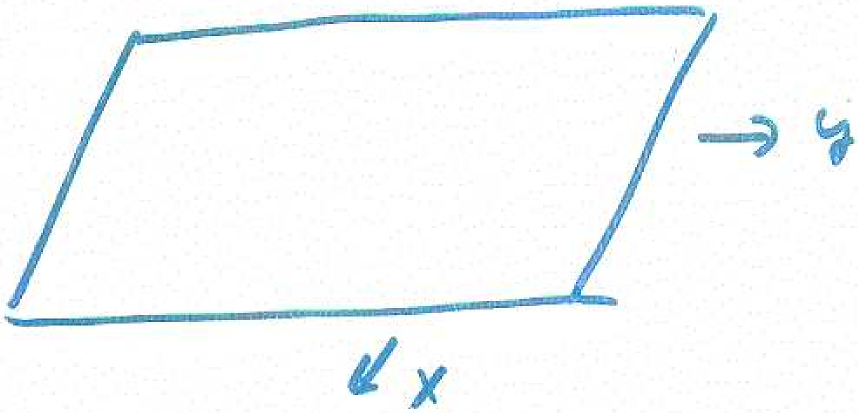
BUT THEY ARE "CLASSICAL" ONES

LINKING NUMBERS, TORSION,

ETA-INVARIANT

NOW LET US GO BACK TO THE QUANTUM HALL EFFECT, WHERE THE GAUGE GROUP (FOR THE CLASS OF MATERIALS WE ARE CONSIDERING) IS ABELIAN...

WE CONSIDER A FLAT SAMPLE IN THE X-Y PLANE WITH $k \neq 0$



$$\mathcal{L} = \frac{h}{4e^2}\int_{\substack{\text{space}\\\text{time}}} F \wedge * F + \frac{k}{2\pi}\int_{\text{sample}} A \wedge dA$$

THE ELECTROMAGNETIC CURRENT

IS

$$J = 4\pi \frac{e^2}{h} \frac{\delta W}{\delta A} = \frac{e^2}{h} k *_3 F$$

WHERE $*_3$ IS THE HODGE $*$

IN THE THREE-MANIFOLD OF THE

SAMPLE.

IN A NONRELATIVISTIC LANGUAGE

THIS BECOMES

$$J_x = \frac{e^2}{h} k \, E_y$$

$$J_y = -\frac{e^2}{h} k \, E_x$$

AND WE HAVE FOUND

THE QUANTUM HALL EFFECT.

$$\sigma_H = \frac{e^2}{h} k$$

old $U(1)$ A

a new $U(1)'$ gauge field B

$$\int F_A \wedge * F_A + k_1 \int A \wedge dA + k_2 \int A \wedge dB + k_3 \int B \wedge dB$$

$U(1) \times U(1)'$

$$\sigma_H = k_1 - \frac{k_2^2}{k_3}$$

$$\int_{M_4} F \wedge *F \quad + \quad \int_{Q_3} A \wedge dA$$

$$d * F + \delta(Q_3) * F = 0$$