Modeling distances between ignitions.

André Dabrowski
University of Ottawa

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Abstract

Forest fires ignitions by lightning are frequently represented as a point process, and one can adopt a variety of methods to study the characteristics of such a process. Here we look at inter-point distances, $\|X_1 - X_2\|$ and the tail index α defined by

$$P[\|X_1 - X_2\| \leq x] \sim x^\alpha$$

as $x \downarrow 0$. This index is can be estimated using the extreme least order statistics of the inter-point distances, and we illustrate those methods on a sample data set of ignitions.

Modeling distances between ignitions.

1. A data set of ignitions.

2. Inter-point distances and a power law.

3. A limit theorem for minimal inter-point distances.

4. Application to data.

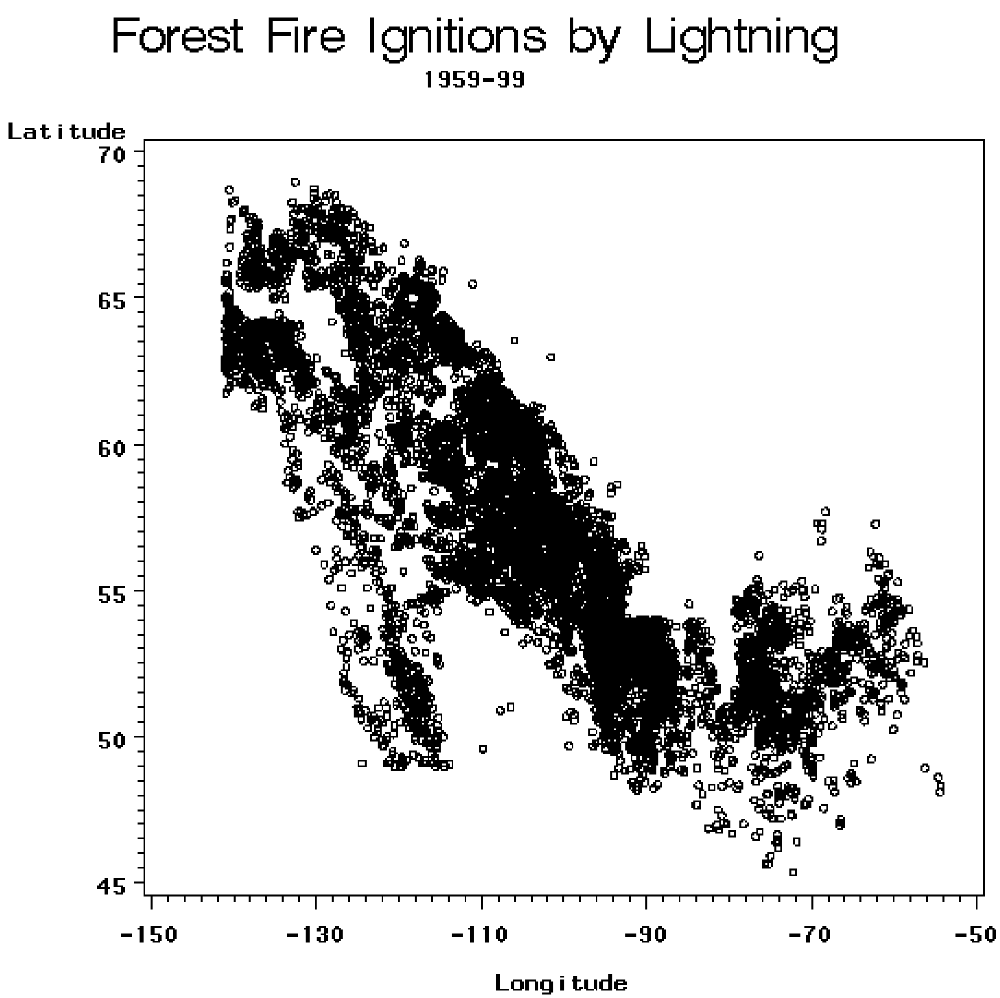
1. A data set of ignitions.

`http://cwfis.cfs.nrcan.gc.ca/en/historical/ha_lfdb_maps_e.php`

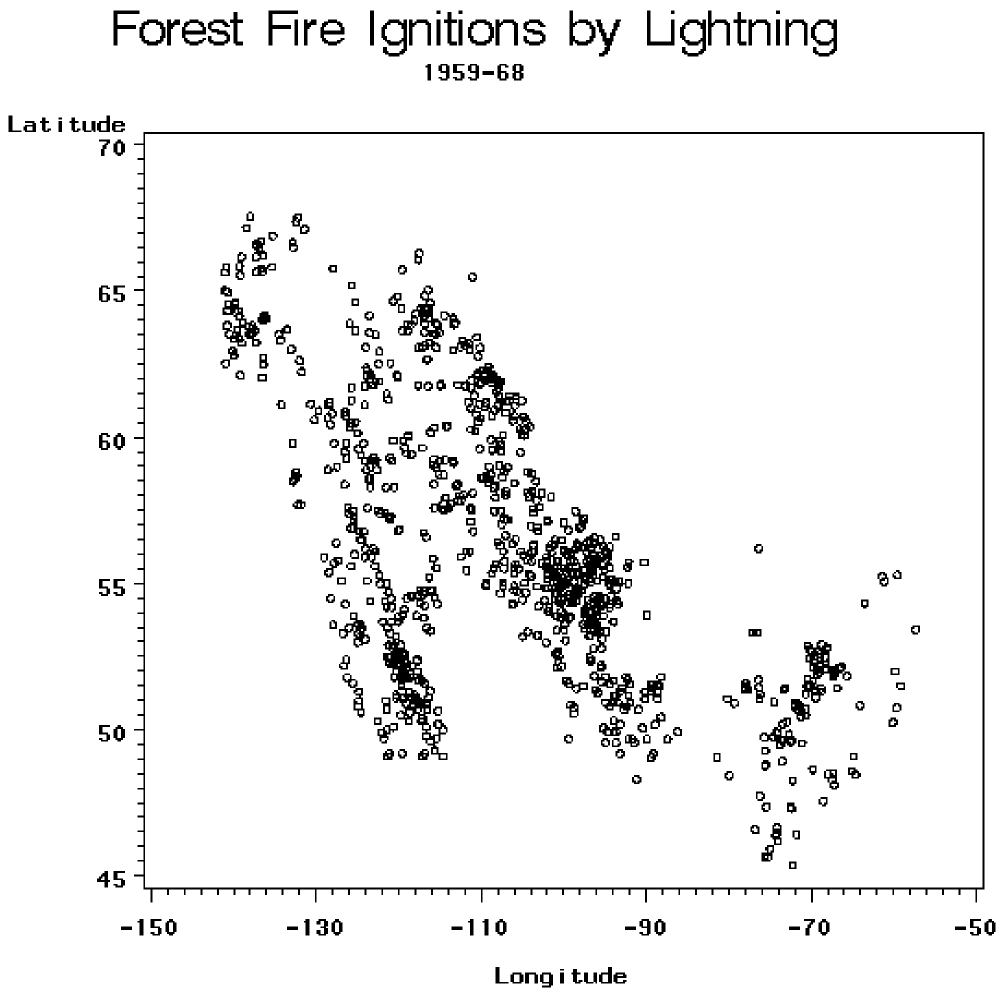
A data set of lightning-induced fires across Canada of at least 200*ha* from 1959 to 1999.

Retained Data:
longitude,
latitude,
detection date

We can see regional variations in numbers of ignitions.



We also see variations in time.



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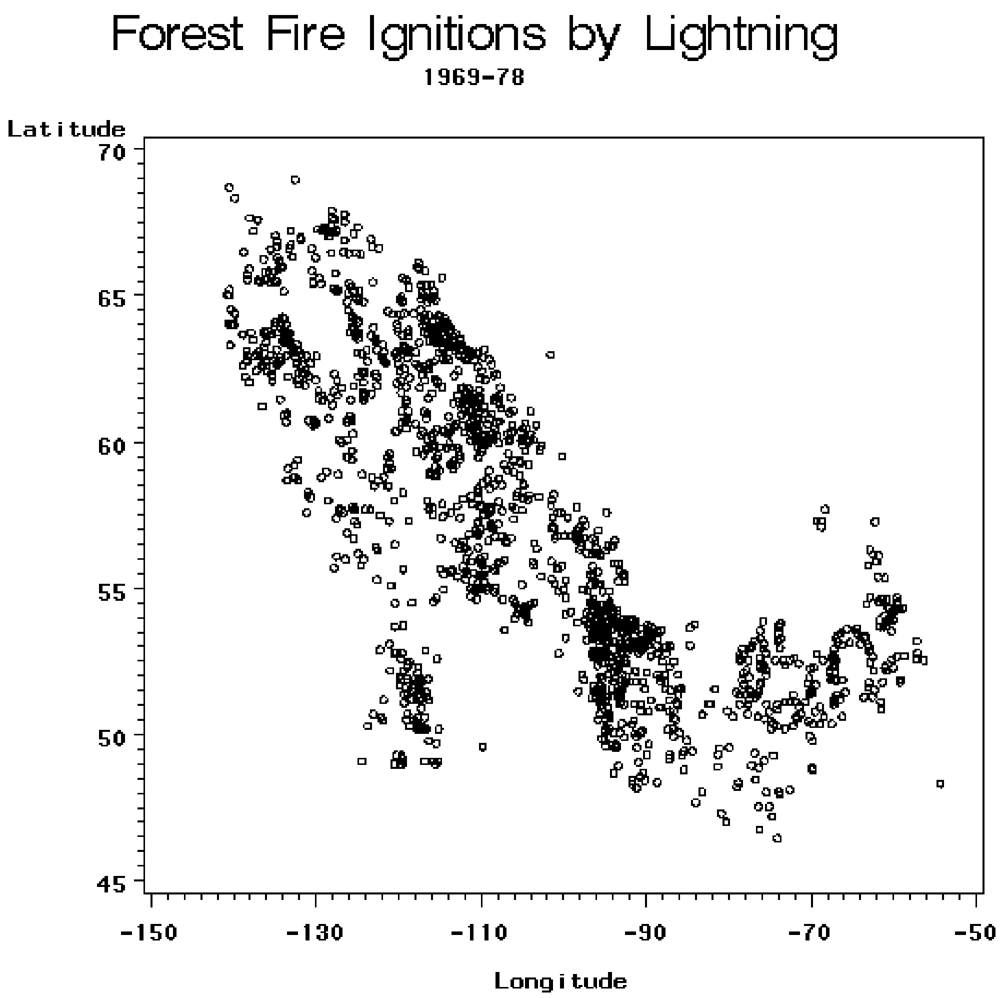


Nonetheless, we can think of the data set as the realization of a spatio-temporal point process, and can seek to characterize some of its properties.

- Poisson-ness (homogeneous, cluster, . . .)
- intensity measure (exogenous, integrate-and-fire, . . .)
- inter-point distances (spatial structure, K function, . . .)

2. Inter-point distances and a power law.

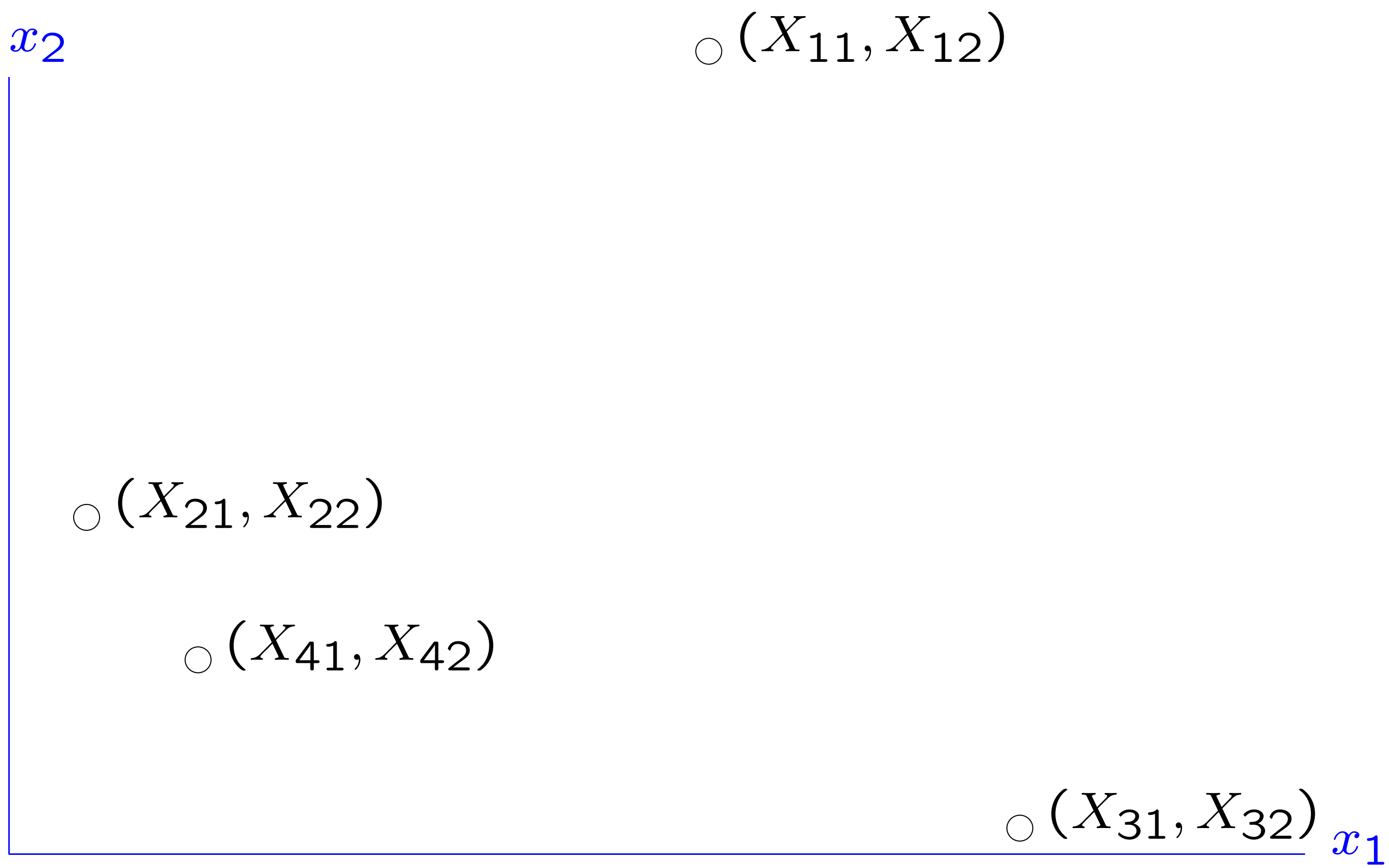
Denote the location of the kth ignition by X_k. In general the location can be in any dimension (e.g. $\Re^d$-valued) but for simplicity we will assume that $X_k = (X_{k1}, X_{k2}) \in \Re^2$. The inter-point distance between X_i and X_j is defined as $h(X_i, X_j)$, where h is a positive symmetric function.

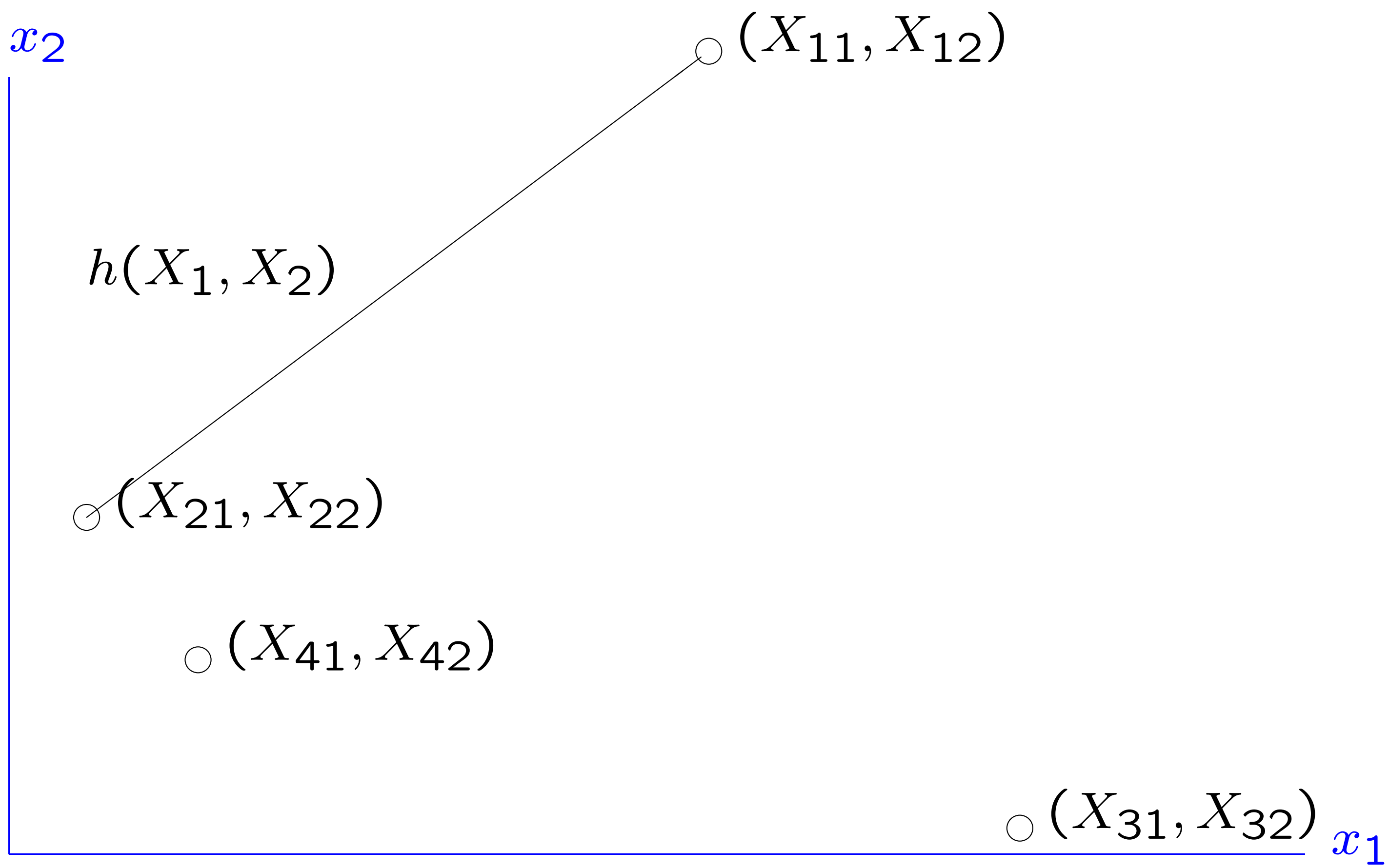
We will assume that $\{X_k \ : \ k \geq 1\}$ are independent and identically distributed random vectors.

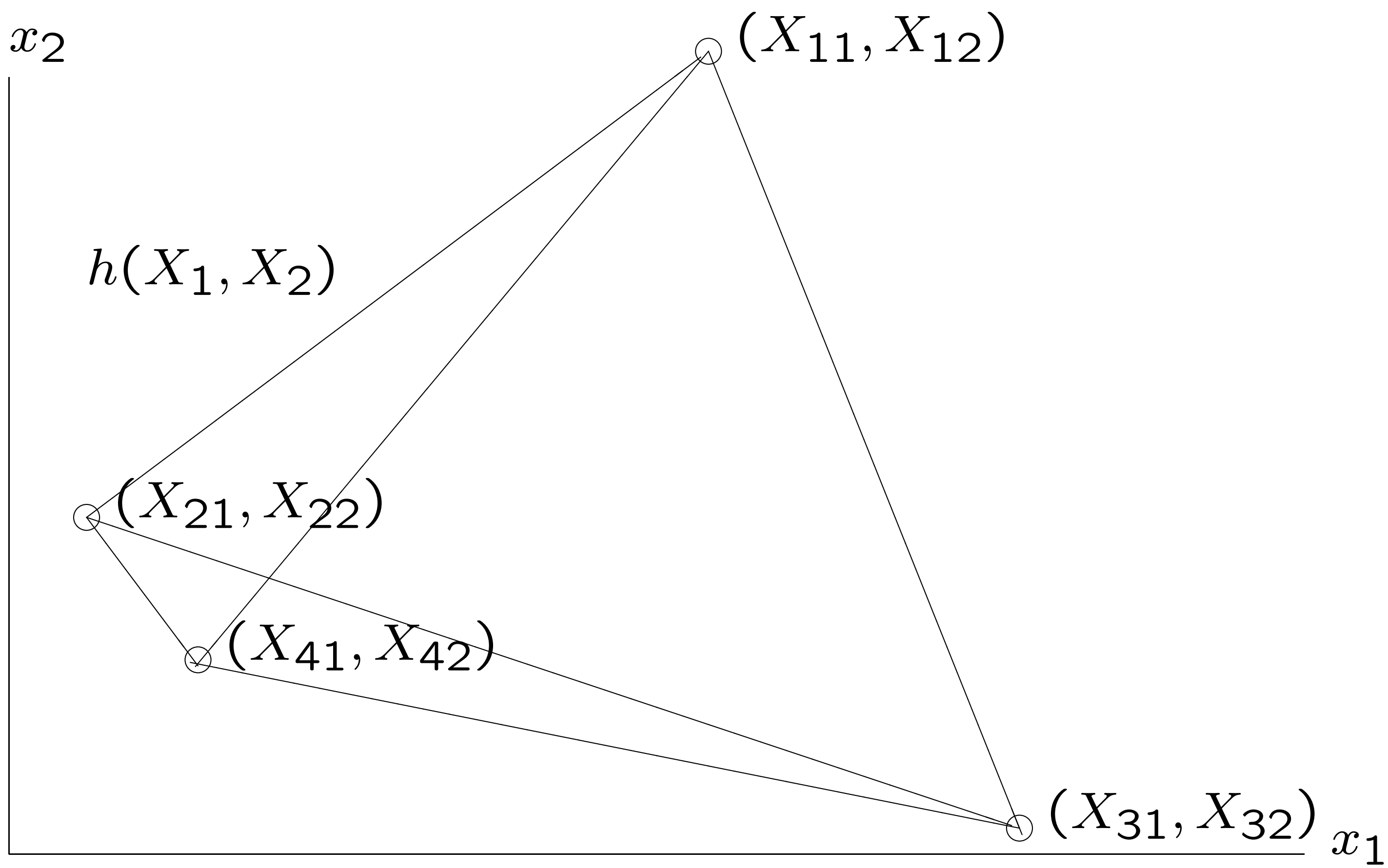
We further assume that

$$P[h(X_1, X_2) \leq x] \ \sim L(x^{-1})x^{\alpha}$$

for a slowly varying function L and some $\alpha > 0$.


x_2
(X_{11}, X_{12})
(X_{21}, X_{22})
(X_{41}, X_{42})
(X_{31}, X_{32})
x_1


x_2
(X_{11}, X_{12})
$h(X_1, X_2)$
(X_{21}, X_{22})
(X_{41}, X_{42})
(X_{31}, X_{32})
x_1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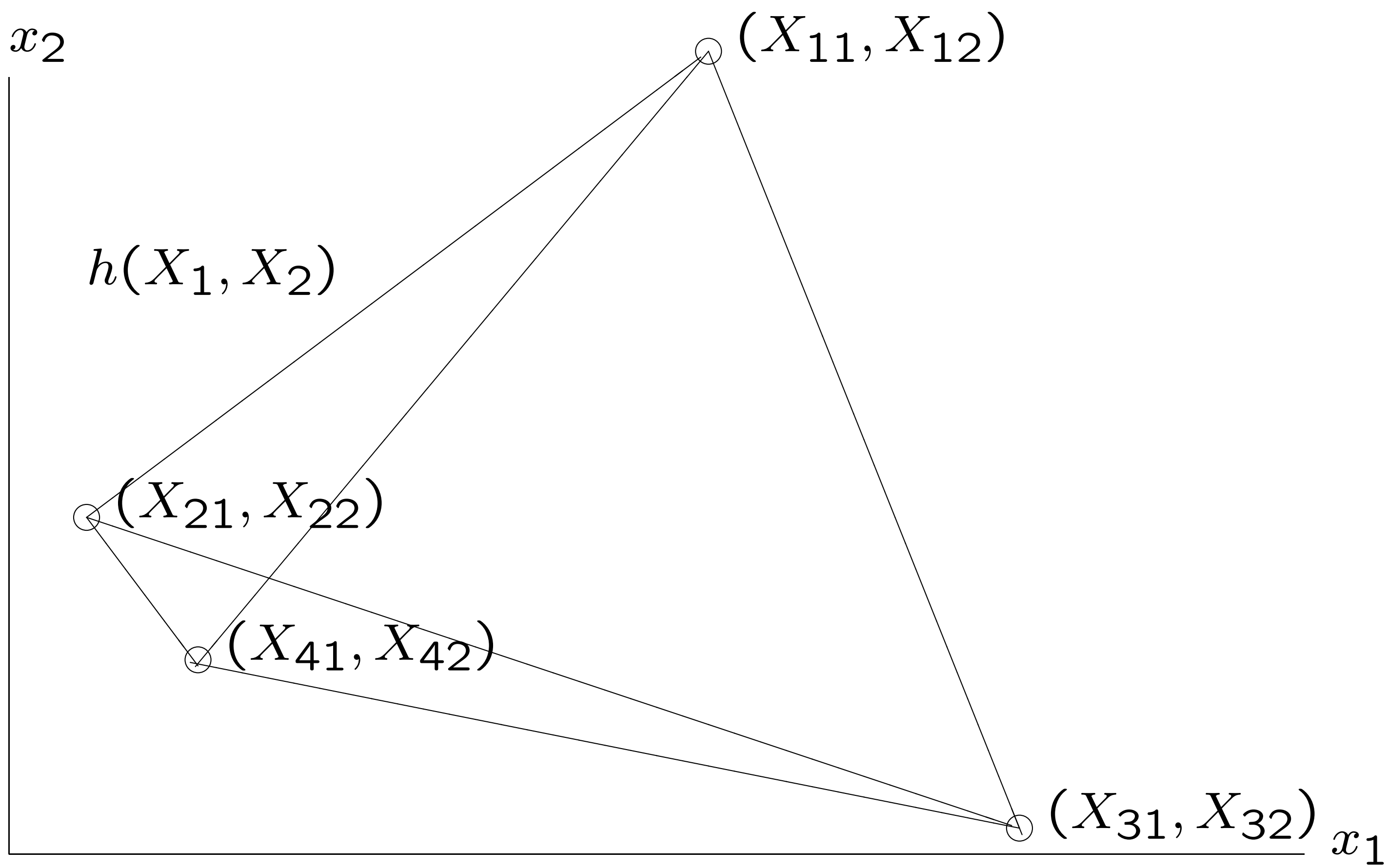


The $h(X_i, X_j)$ are not independent.

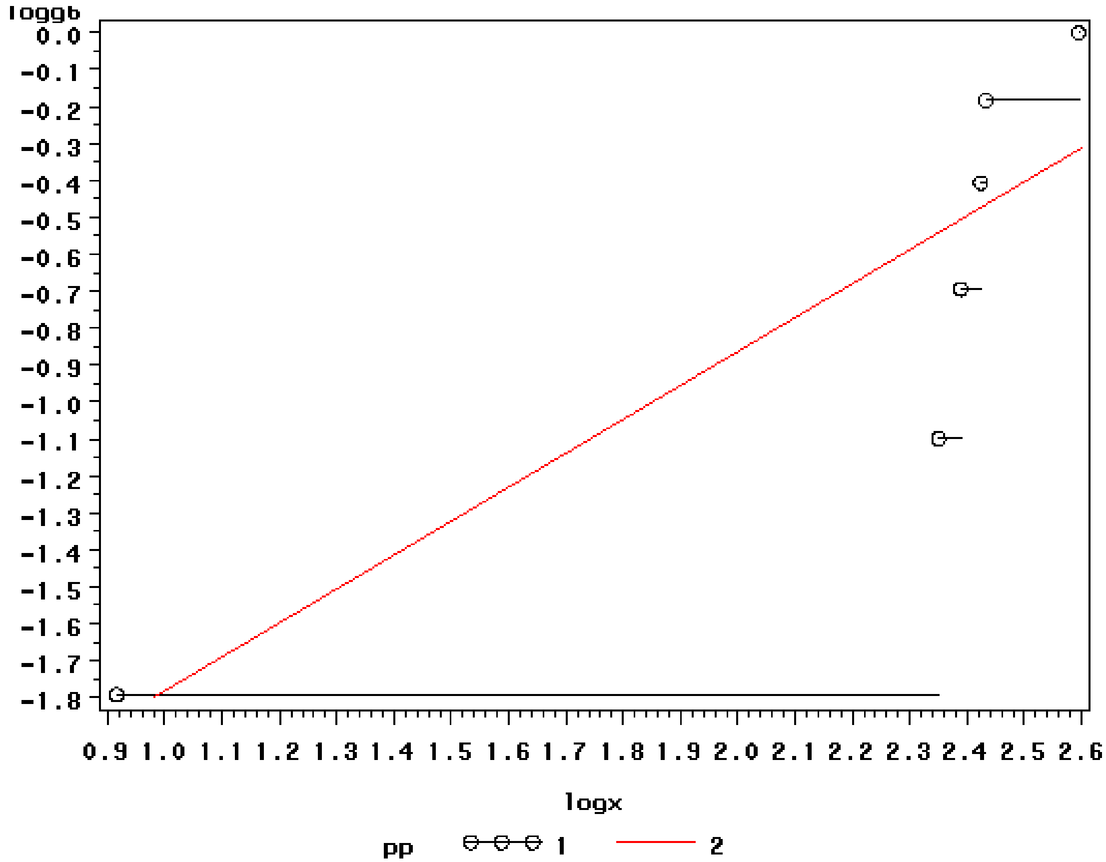
If we believe

$$P[h(X_1, X_2) \leq x] \; \sim x^{\alpha}$$

then we can write down the empirical distribution function, G_k, of the $h(X_i, X_j)$, try to fit it by x^{α} (or $\log G_k(x)$ by $\alpha \log x$), and so estimate α, i.e. as $x \downarrow 0$,

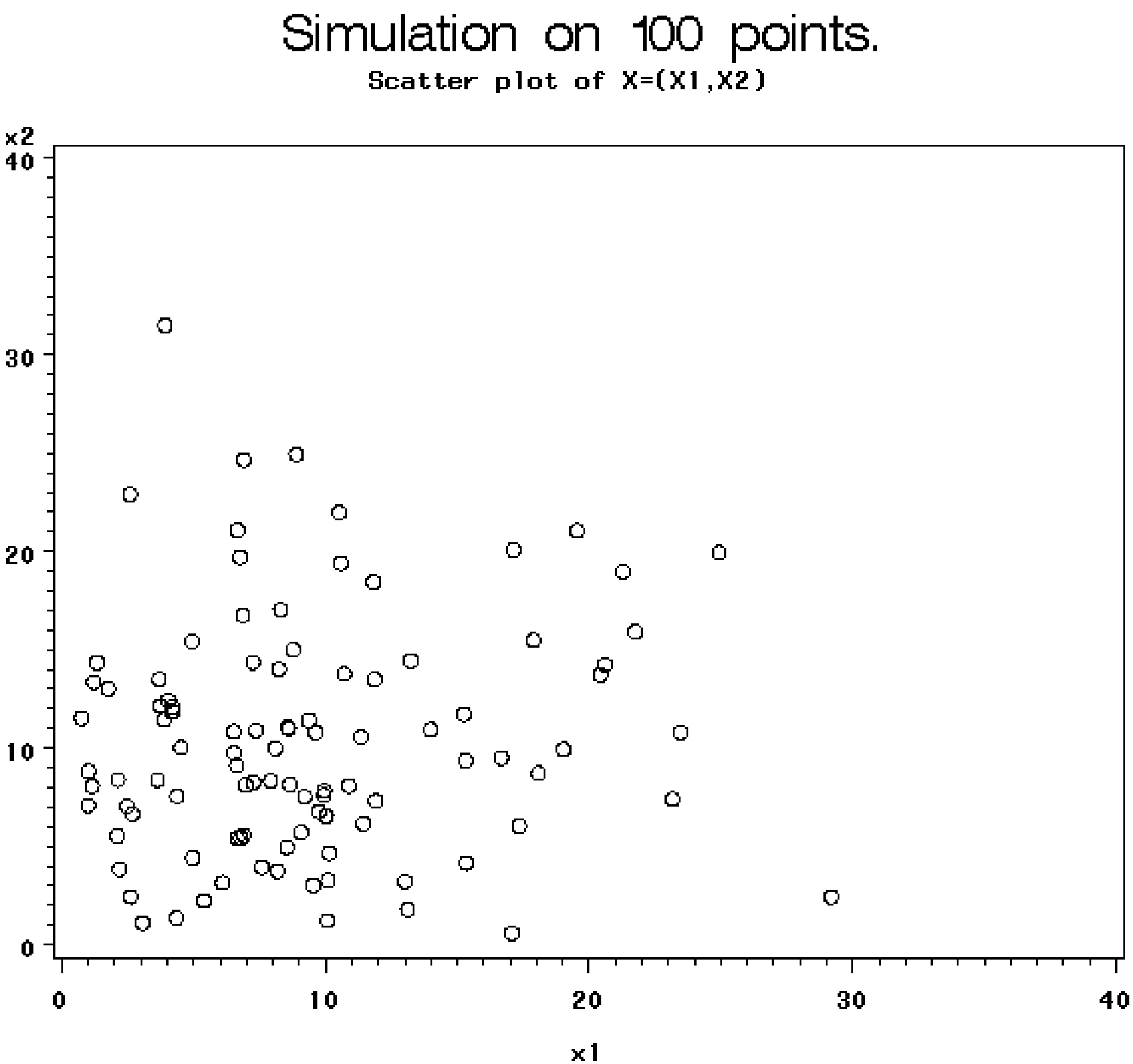
$$\log G(x) = \log P[h(X_1, X_2) \leq x] \; \sim \log c + \alpha \log x$$

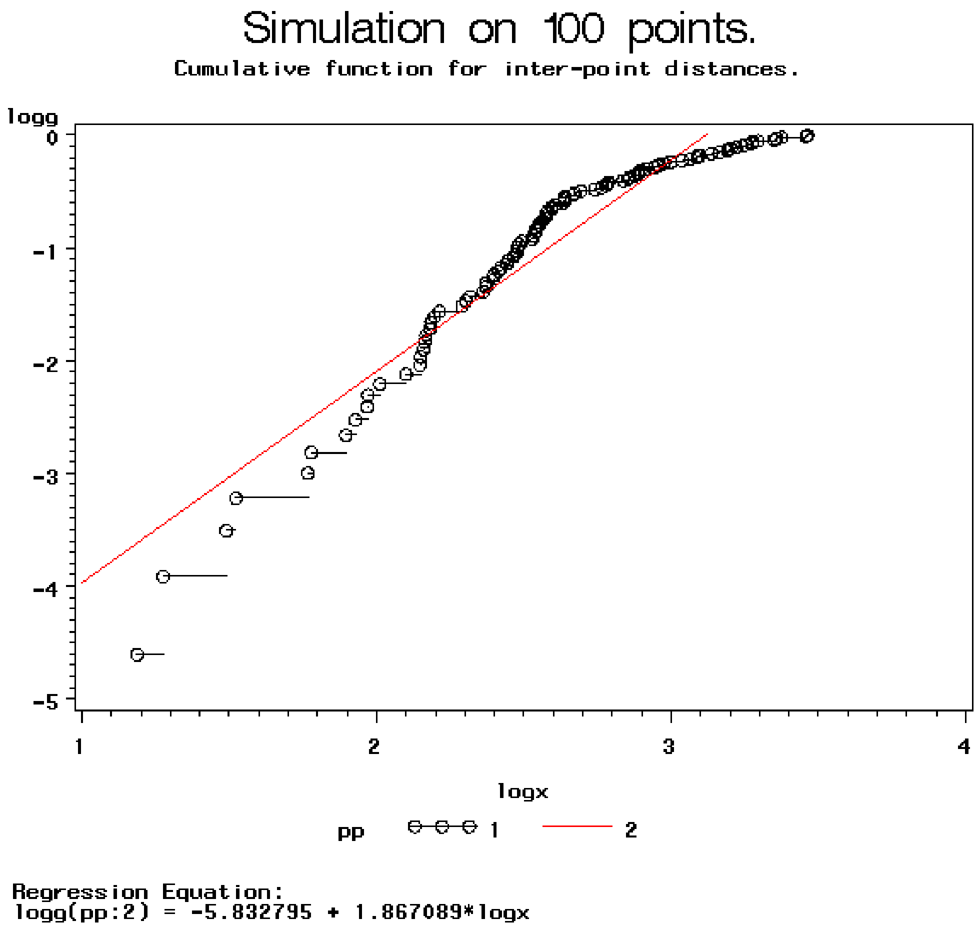
This is not too impressive on the 4-point data set:



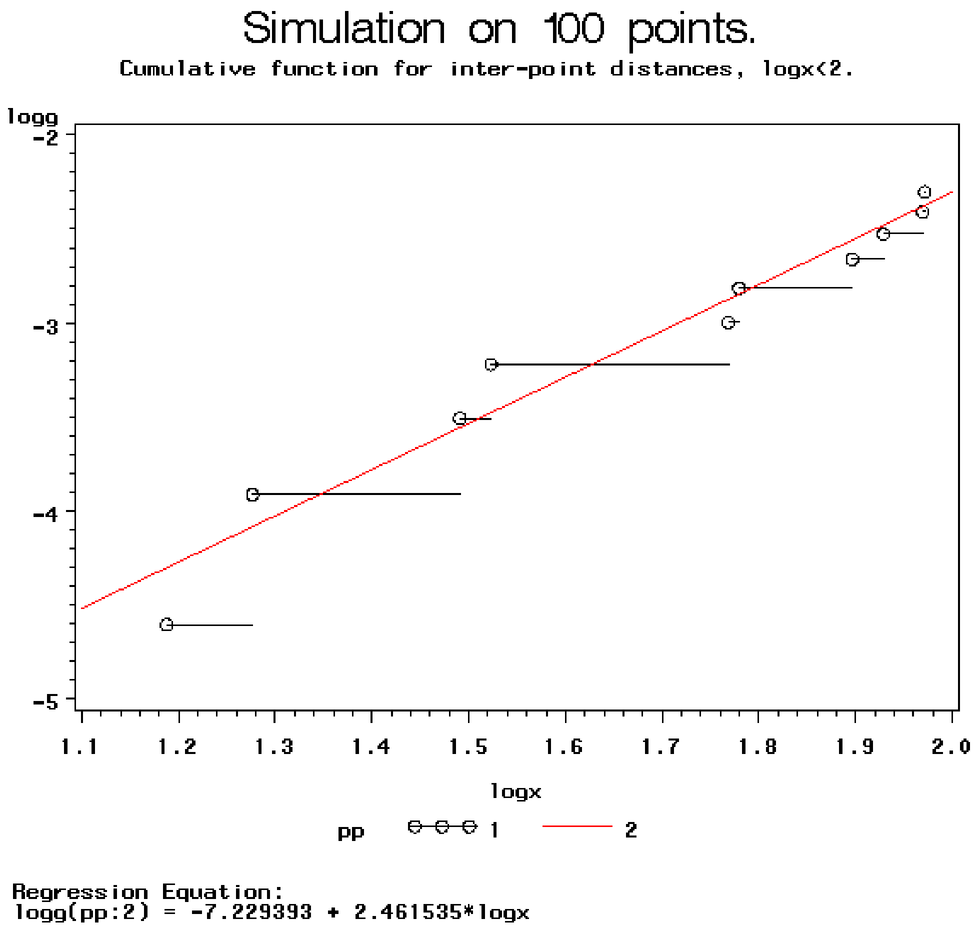
Regression Equation:
logg6(pp:2) = -2.702746 + 0.91877*logx

2.5
10.5
10.9
11.3
11.4
13.4


Simulation on 100 points.
Scatter plot of X=(X1,X2)
x2
40
30
20
10
0
0
10
20
30
40
x1

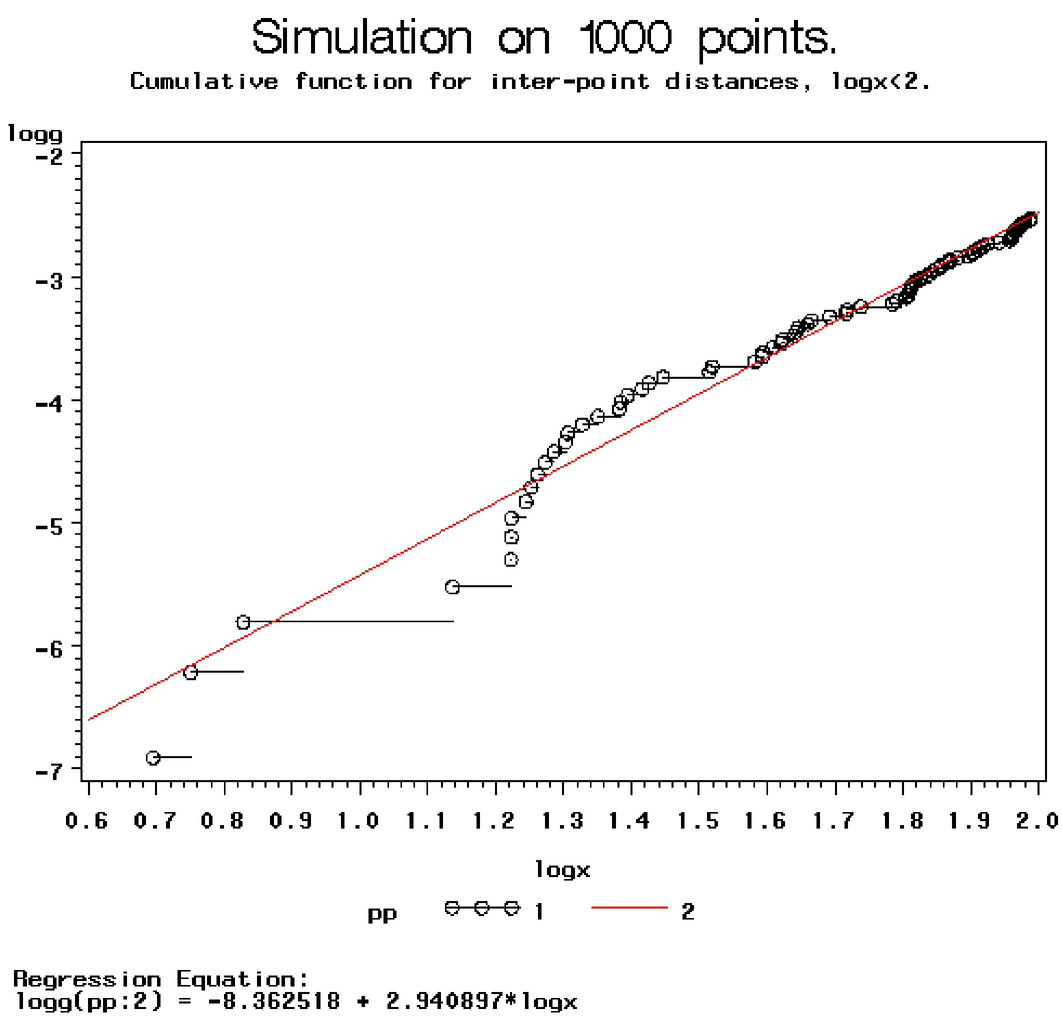


The straight line fit to all the $h(x_i, x_j)$.



The straight line fit to a lower tail of the $h(x_i, x_j)$. Good here but ...

... sometimes this is not as convincing.



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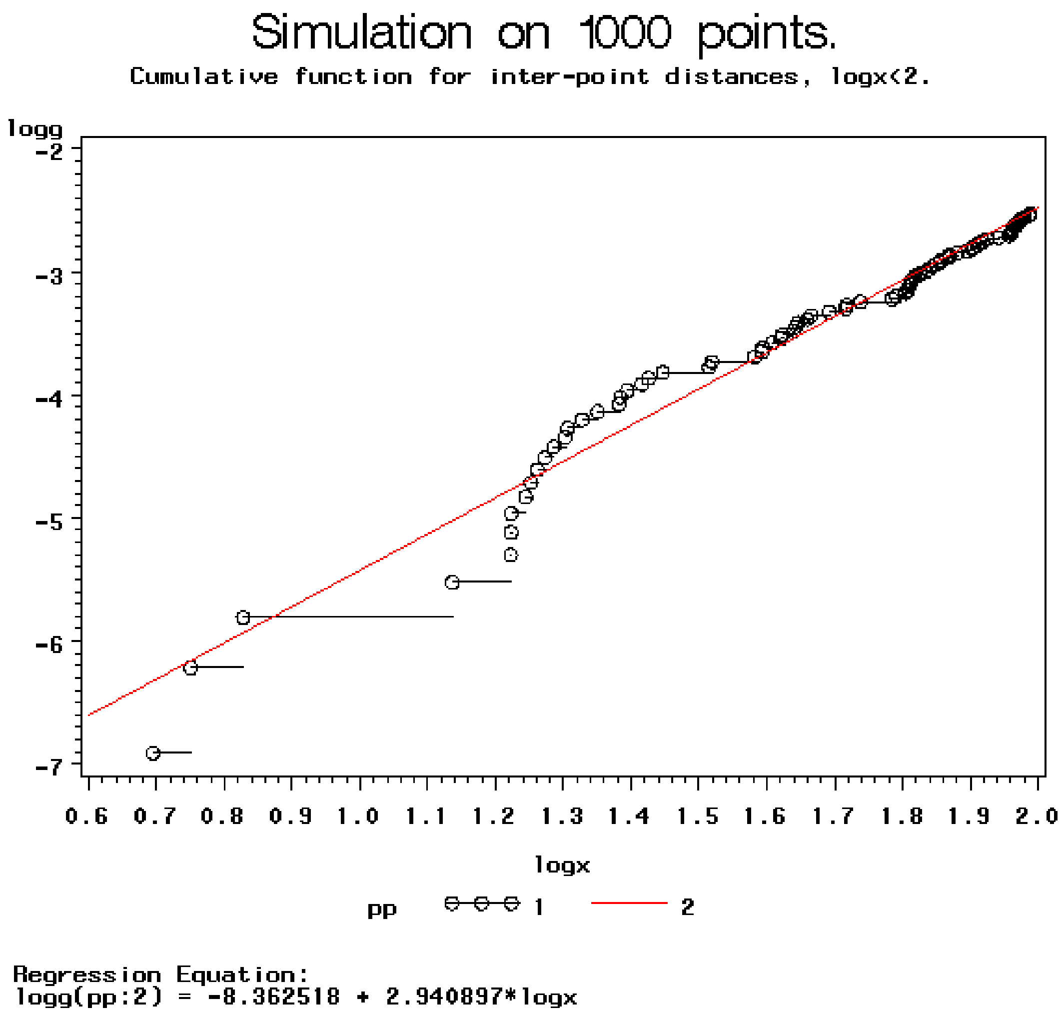


Asymptotics?

Can we say anything about the limiting properties of such an estimator?

3. A limit theorem for minimal inter-point distances.

1. X_i iid, h symmetric non-negative kernel. For example, take $h(x,y) = |x-y|^\gamma$.

2. Regular variation condition: For $\alpha > 0$, as $x \to 0$

$$P[h(X_1, X_2) \le x] = L(x^{-1})x^\alpha.$$

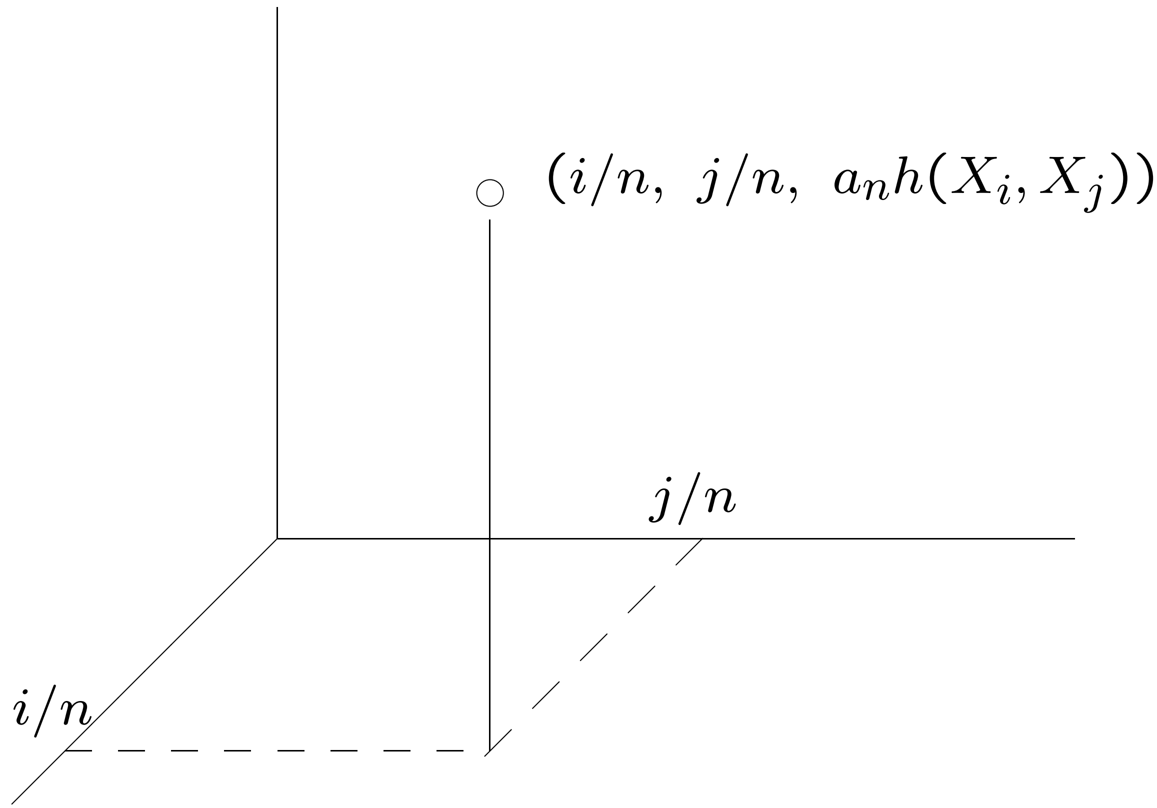
3. As $n \to \infty$, for all $x > 0$,

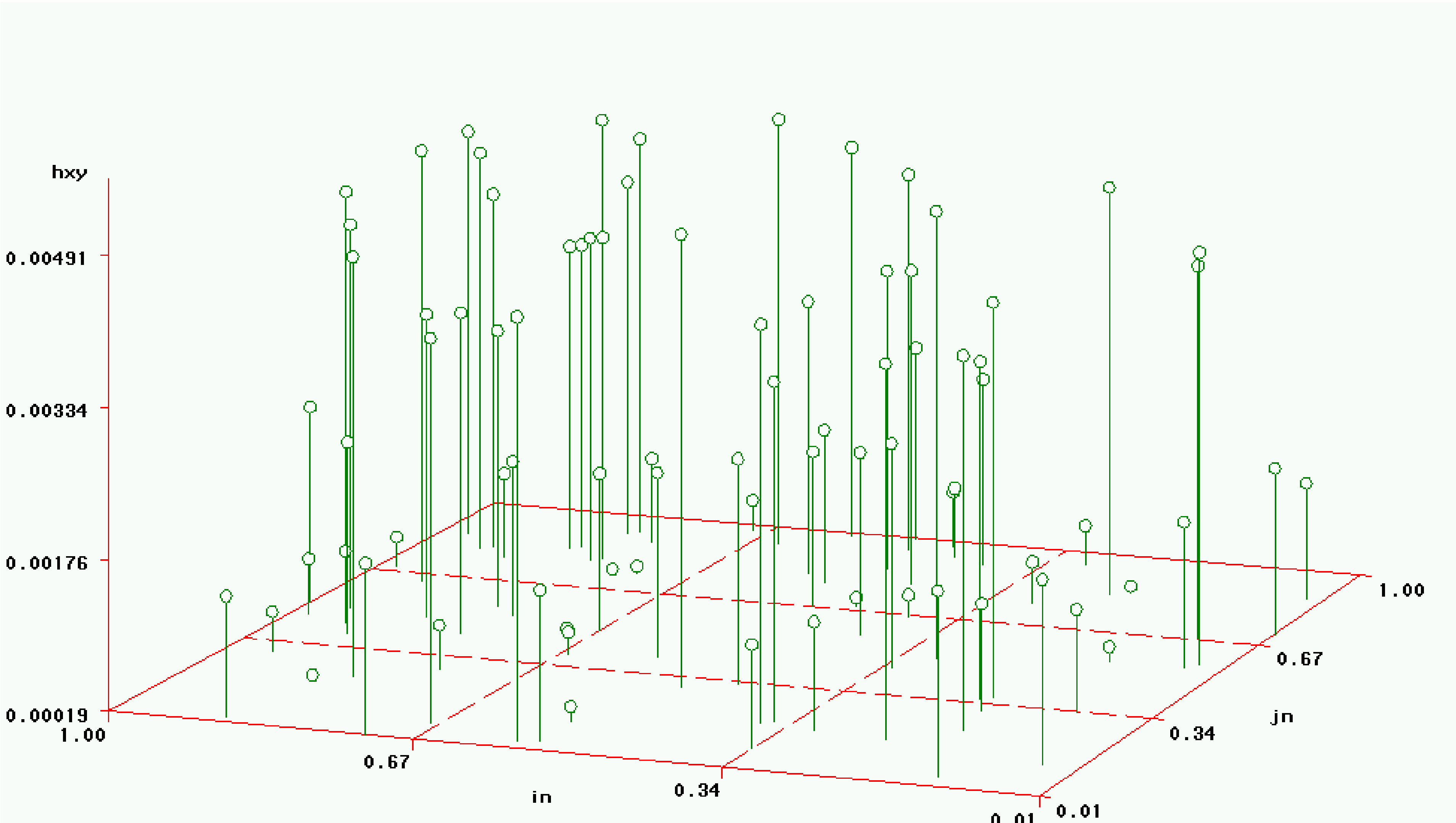
$$n^3 P\left[a_n h(X_1, X_2) \le x,\ a_n h(X_1, X_3) \le x\right] \to 0$$

In analogy with extremal processes we can define the point process

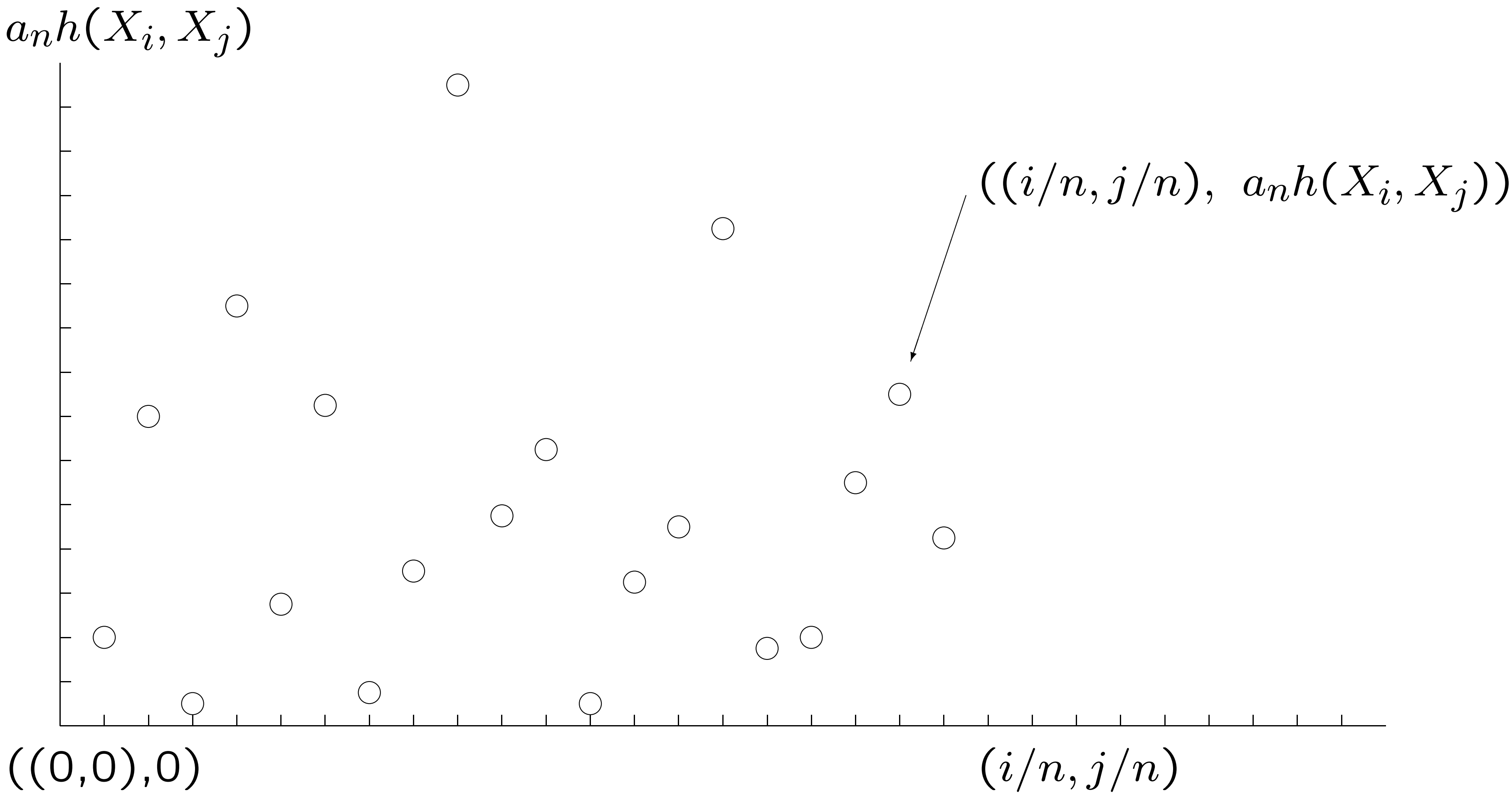
$$N_n(A\times B) = \#\{i < j \in \{1\ldots n\} : \Big((i/n,\ j/n), a_n h(X_i, X_j)\Big) \in A\times B\}$$

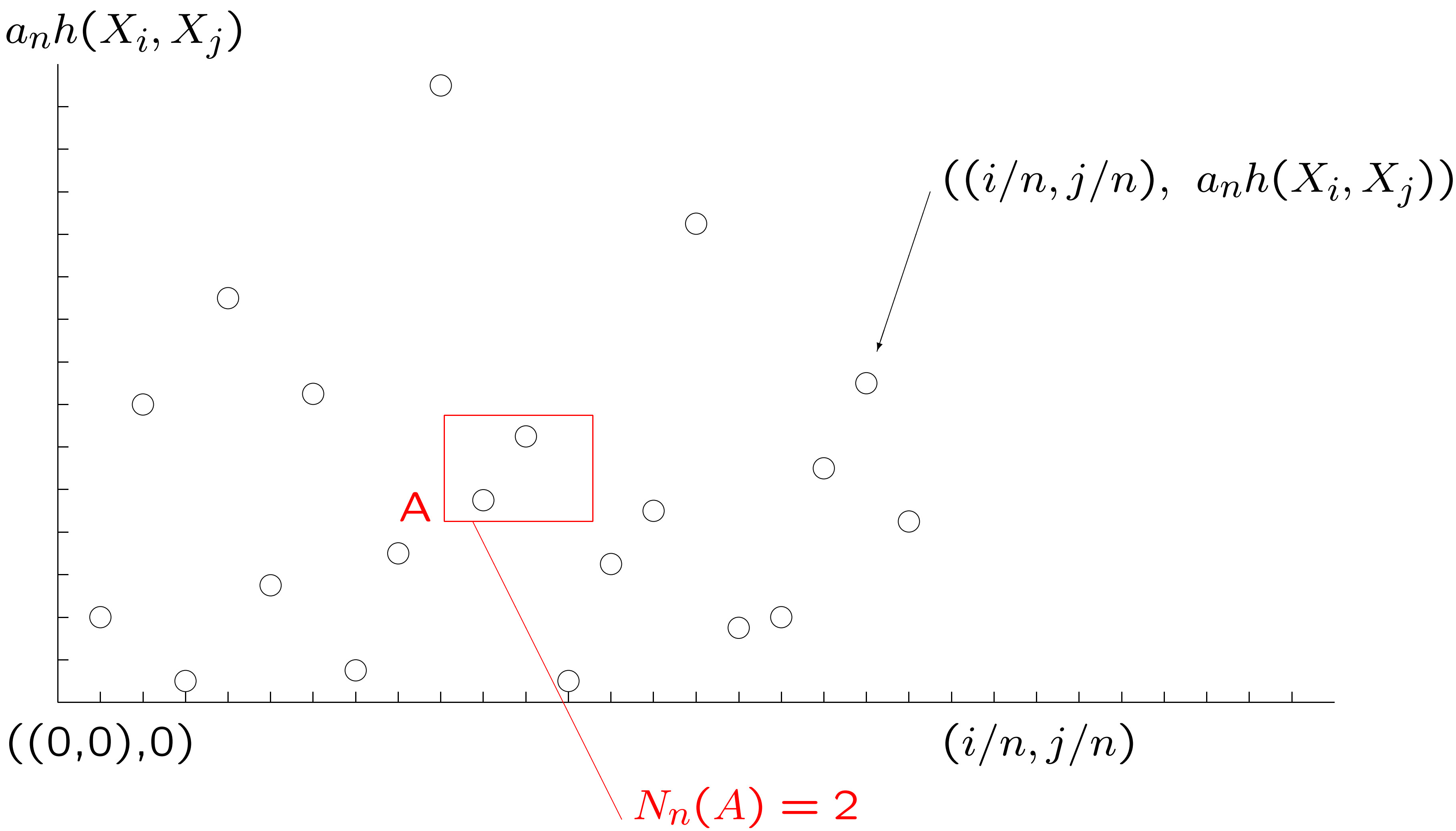
and look for its weak limit.


$(i/n,\ j/n,\ a_n h(X_i, X_j))$
j/n
i/n

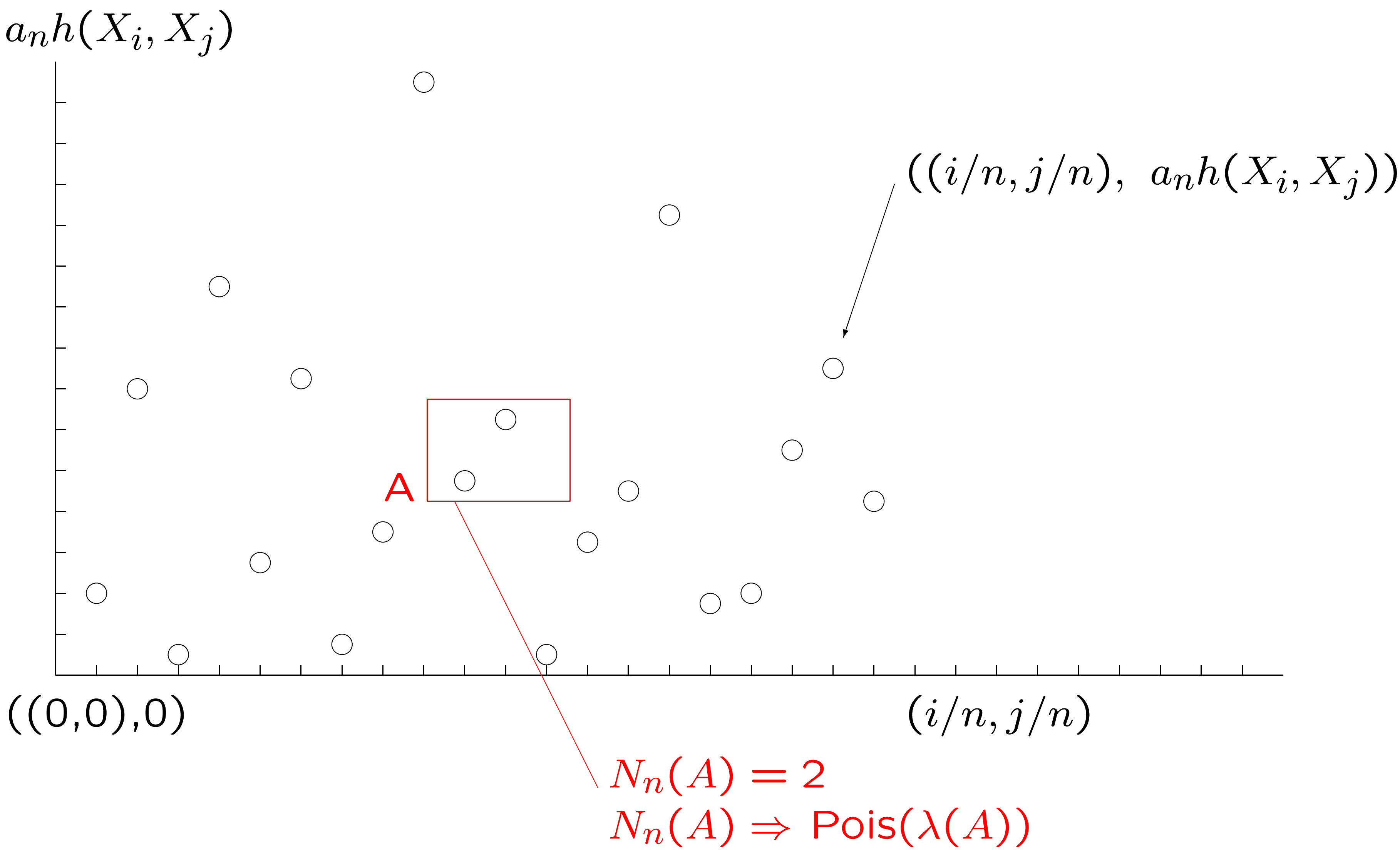


$h(X_i, X_j) = |X_i - X_j| < 0.05$ for uniform X_i.


$a_n h(X_i, X_j)$
$((i/n, j/n),\ a_n h(X_i, X_j))$
((0,0),0)
$(i/n, j/n)$



$a_n h(X_i, X_j)$
$((i/n, j/n),\ a_n h(X_i, X_j))$
A
((0,0),0)
$(i/n, j/n)$
$N_n(A) = 2$


$a_n h(X_i, X_j)$
$((i/n, j/n),\ a_n h(X_i, X_j))$
A
((0,0),0)
$(i/n, j/n)$
$N_n(A) = 2$
$N_n(A) \Rightarrow \text{Pois}(\lambda(A))$

Theorem. [*]*Assume* $P[h(X_1, X_2) \le x] = L(x^{-1})x^{\alpha}$ *and other conditions. Then*

$$N_n \Rightarrow N$$

where N *is a point process on*

$$\{(x, y):\ 0 < x \le 1,\ 0 < y < x\} \times \Re^{+}$$

with intensity

$$\eta\left((a_1, b_1] \times (a_2, b_2] \times (a_3, b_3]\right) = 2(b_1 - a_1)(b_2 - a_2)(b_3^{\alpha} - a_3^{\alpha})$$

[*]*Poisson limits for* U*–statistics*, AD, Herold Dehling, Thomas Mikosch, Olimjon Sharipov, *Stoch. Proc. Appl.* **99**, 137-157, (2002).

How does this help in estimating α?

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There are several point estimates from several contexts.

- Takens' estimator for the correlation dimension.

- The spatial K-function near zero.

- Hill estimator.

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- The spatial K-function near zero.
- Hill estimator.

We can see how well each of these potential estimators performs.

Takens' estimator for the correlation dimension

As an alternative to the Grassberger-Procaccia estimator, Takens * introduced a dimension estimator motivated by the maximum likelihood principle. Assume

$$P(\|\mathbf{X}_1 - \mathbf{X}_2\| \le x) = x^{\alpha}\,, \quad \text{for } 0 \le x \le \delta.$$

Takens' estimator modified with $\delta_n = \delta / a_n$.

$$\widehat{\alpha}_T = \left[\frac{- \sum_{i=2}^{n} \sum_{j=1}^{i-1} \log(\|\mathbf{X}_i - \mathbf{X}_j\| / \delta_n) I_{[0,\delta_n]}(\|\mathbf{X}_i - \mathbf{X}_j\|)}{\sum_{i=2}^{n} \sum_{j=1}^{i-1} I_{[0,\delta_n]}(\|\mathbf{X}_i - \mathbf{X}_j\|)} \right]^{-1}.$$

*Takens, F. (1985) In *Lecture Notes in Math.* **1125**, pp. 99–106

By a continuous mapping argument, exploiting a representation in terms of gamma variables, and by simple facts, we identify the limit distribution for the modified Takens' estimator:
$\widehat{\alpha}_T^{-1}$ has asymptotic expectation

$$\alpha^{-1}$$

and variance

$$\alpha^{-2}\left[P(N(\delta^{\alpha})=0)+E[N(\delta^{\alpha})I_{\{N(\delta^{\alpha}\})>0}]^{-1}\right].$$

As $\delta \to 0$, the variance is of the order $\alpha^{-2}[1+\delta^{\alpha}]$. Note that it does not shrink to 0.

Poisson convergence of the K-function

In the spatial analysis of point patterns the K-function is used as a measure of spatial dependence*. A sample version of it is given by the U-statistic

$$K_n(\delta) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} I_{[0,\delta]}(a_n \|\mathbf{X}_i - \mathbf{X}_j\|) \,.$$

Thus we have the kernel

$$h(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \,,$$

and so we may conclude that $K_n(\delta) = N_n(\mathbf{E}_1 \times [0, \delta])$ converges in distribution to a Poisson random variable with mean δ^α.

*Cressie (1993) *Statistics for Spatial Data.* Wiley, New York.

More generally, the K_n-processes converge in distribution in $M_p(\mathbf{R}_+)$ to a Poisson process K with mean measure $\alpha\, x^{\alpha-1}\, dx$:

$$K_n(\cdot) = N_n(\mathbf{E}_1 \times \cdot) = \sum_{i=2}^{n} \sum_{j=1}^{i-1} \varepsilon_{a_n|\mathbf{X}_i - \mathbf{X}_j|}(\cdot) \stackrel{d}{\rightarrow} K(\cdot)\,. \qquad (1)$$

Writing $K(\delta) = K([0,\delta])$, it follows that there is distributional convergence near zero,

$$(K_n(\delta))_{\delta \geq 0} \stackrel{d}{\rightarrow} (K(\delta))_{\delta \geq 0}\,.$$

The continuous mapping theorem for $\mathbf{D}[\Delta_0, \Delta_1]$ with $0 < \Delta_0 < \Delta_1 < \infty$ mapping to $\mathbf{C}[0, b]$ yields convergence of squared error between the data and a linear fit to logarithms;

$$\begin{aligned} B_n &= \left(\int_{\Delta_0}^{\Delta_1} (\log^+ K_n(\delta) - (\beta_0 + \beta \log \delta))^2 \, d\delta\right)_{\beta \in [0,b]} \\ &\xrightarrow{d} B = \left(\int_{\Delta_0}^{\Delta_1} (\log^+ K(\delta) - (\beta_0 + \beta \log \delta))^2 \, d\delta\right)_{\beta \in [0,b]}, \end{aligned}$$

in $\mathbf{C}[0, b]$.

Another application of the continuous mapping shows that the LS minimizer $\beta(n)$ of B_n on $[0, b]$ converges to the minimizer $\widehat{\beta}$ of B on $[0, b]$ (where $\log^+ x = \log(\max(1, x))$):

$$\begin{aligned}
\beta(n) &= \frac{\int_{\Delta_0}^{\Delta_1} (\log\delta - \overline{\log\delta})(\log^+ K_n(\delta) - \overline{\log^+ K_n(\delta)})\, d\delta}{\int_{\Delta_0}^{\Delta_1} (\log\delta - \overline{\log\delta})^2\, d\delta} \\
\xrightarrow{d} \widehat{\beta} &= \alpha + \text{ an expression in } (\alpha, K, \Delta_0, \Delta_1). \qquad (2)
\end{aligned}$$

So the best linear fit to extremes in the K-function is an asymptotically biased estimator of α.

Hill estimation of α.

Write

$$h_{(1)} \leq \cdots \leq h_{(n(n-1)/2)}$$

for the order statistics of the sample $h(\mathbf{X}_i, \mathbf{X}_j)$, $i = 2, \ldots, n, j = 1, \ldots, i-1$. A classical estimator of α in the univariate case is *Hill's estimator** given by

$$\widehat{\alpha}_{n,m} = -\left(\frac{1}{m} \sum_{i=1}^{m} \log(h_{(i)}/h_{(m)})\right)^{-1}$$

for $m \geq 1$;

Theorem. *Under regular variation conditions, if $m = m_n \to \infty$ and $\sqrt{m_n}/n \to 0$, then Hill's estimator is consistent, i.e. $\widehat{\alpha}_{n,m} \xrightarrow{P} \alpha$.*

*Hill, B.M. (1975)*Ann. Statist.* **3**, 1163–1174.

- Takens' estimator.

$$\alpha_T = \left[\frac{-\sum_{i=2}^{n}\sum_{j=1}^{i-1} \log(\|\mathbf{X}_i - \mathbf{X}_j\|/\delta_n) I_{[0,\delta_n]}(\|\mathbf{X}_i - \mathbf{X}_j\|)}{\sum_{i=2}^{n}\sum_{j=1}^{i-1} I_{[0,\delta_n]}(\|\mathbf{X}_i - \mathbf{X}_j\|)}\right]^{-1}.$$

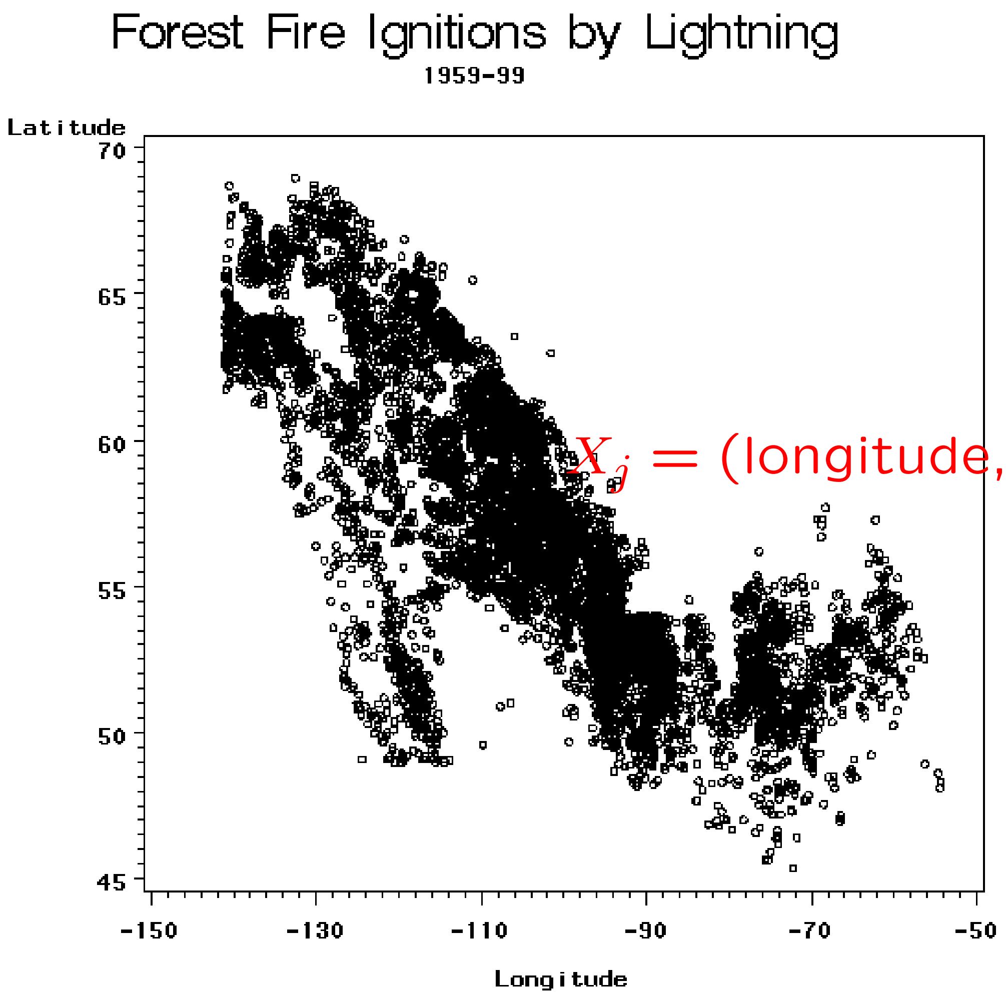
- The spatial K-function near zero.

$$\beta(n) = \frac{\int_{\Delta_0}^{\Delta_1} (\log\delta - \overline{\log\delta})(\log^+ K_n(\delta) - \overline{\log^+ K_n(\delta)})\, d\delta}{\int_{\Delta_0}^{\Delta_1} (\log\delta - \overline{\log\delta})^2\, d\delta}$$

- Hill's estimator.

$$\alpha_{n,m} = -\left(\frac{1}{m}\sum_{i=1}^{m} \log(h_{(i)}/h_{(m)})\right)^{-1}$$

4. Application to data.



$X_j =$ (longitude,latitude,detection date)

To illustrate the estimators, we can compute the estimators of α on this data.

We assume the space-time data on ignitions to be iid observations from a single density.

$$h((x,y,t)_1,(x,y,t)_2) = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(t_1-t_2)^2}$$

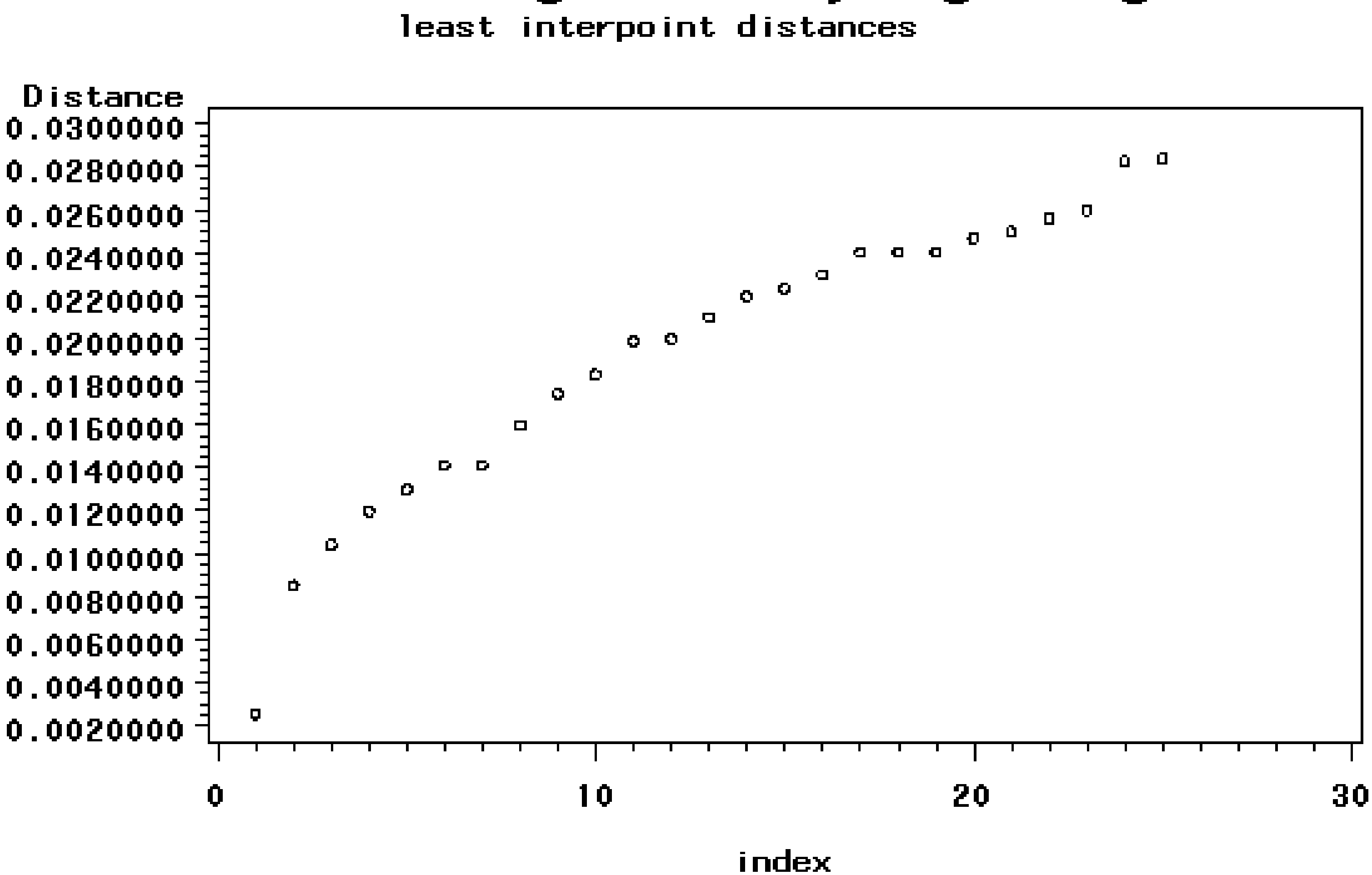
Here x is longitude, y is latitude, and t is time in years$*$1000.

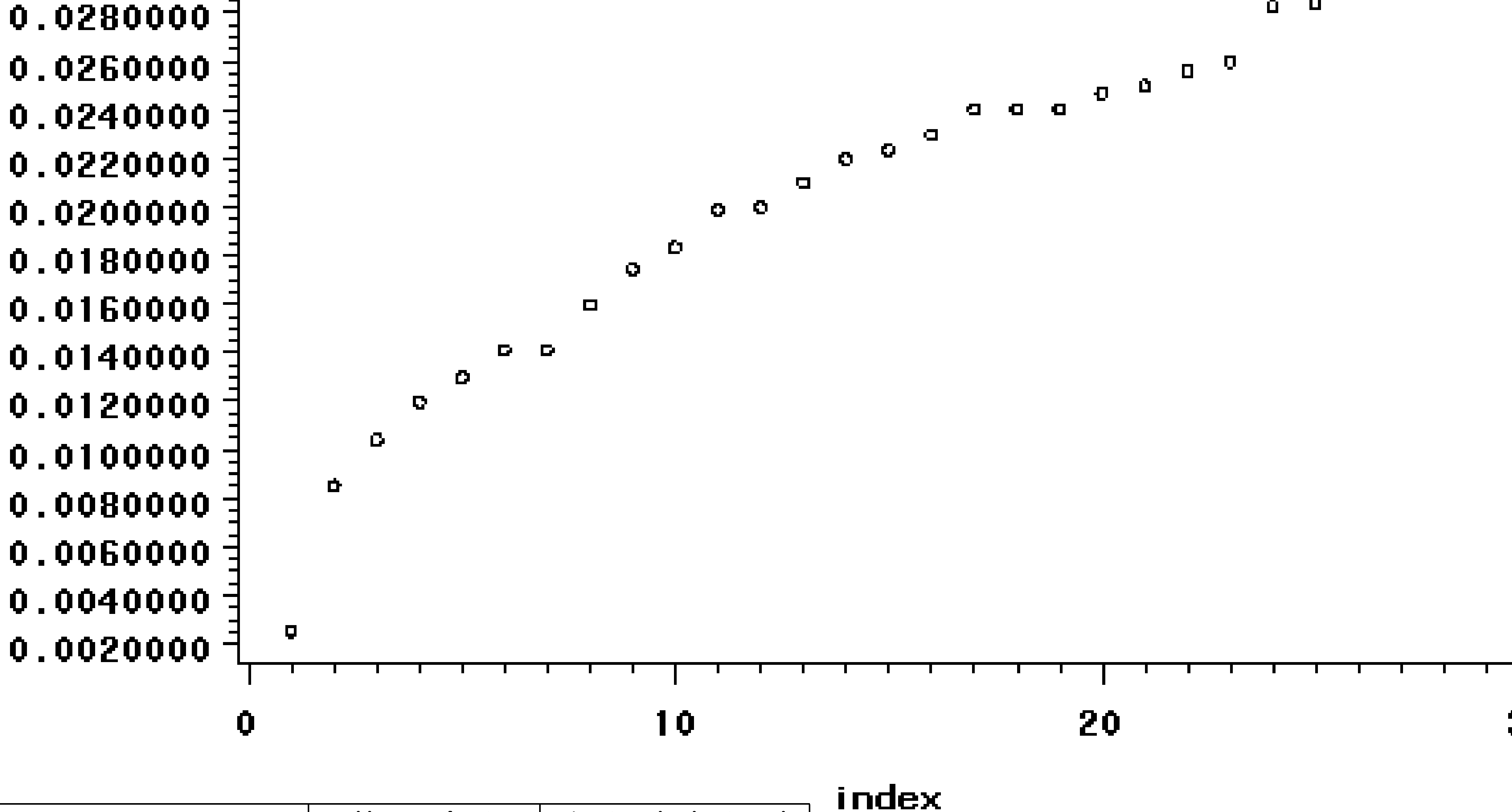
As the index is estimated on just a few small inter-point distances; the estimate will effectively be determined by the most intense "areas".

Inter-point distances of 0 were deleted (38 out of 8050).

Conditional on using the same number of extreme values, Takens' and Hill's estimators yield the same values. The Takens estimator employs essentially a fixed number of minimal inter-point values, the Hill estimator a slowly increasing number.

Forest Fire Ignitions by Lightning





	all points	1st deleted
Takens $\delta = 0.03$	1.90	2.32
K function	1.54	2.71
Hill $m = 25$	2.11	2.61

Here we seem to have (approximately)

$$P(\|\mathbf{X}_1 - \mathbf{X}_2\| \leq x) \sim x^2$$

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$$P(\|\mathbf{X}_1 - \mathbf{X}_2\| \le x) \sim x^2$$

$$1/\left[\widehat{\alpha}^{-1} + 2\sqrt{\widehat{\alpha}^{-2}(1+\delta^{\alpha})}\right] = 0.63$$

Here we seem to have (approximately)

$$P(\|\mathbf{X}_1 - \mathbf{X}_2\| \leq x) \sim x^2$$

Simulating 3-dimensional standard normal . . .

Takens/Hill	K fn	
2.5	2.9	
3.5	2.4	
3.8	2.9	
4.1	4.7	
3.6	3.5	
3.50	3.28	averages

Here we seem to have (approximately)

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Simulating 3-dimensional standard normal . . .

Takens/Hill	K fn	
2.5	2.9	
3.5	2.4	
3.8	2.9	
4.1	4.7	
3.6	3.5	
3.50	3.28	averages
3.00		exact

Here we seem to have (approximately)

$$P(\|\mathbf{X}_1 - \mathbf{X}_2\| \leq x) \sim x^2$$

Some from 2-dimensional normal . . .

Takens/Hill	K fn	
2.0	1.4	
3.2	2.4	
3.5	5.0	
1.7	1.4	
2.1	2.1	
2.50	2.16	averages
2.00		exact

It seems that $P(\|\mathbf{X}_1 - \mathbf{X}_2\| \leq x)$ for the ignition data is more consistent with $\alpha = 2$ than $\alpha = 3$.

The best approximate CI we have is $[0.63, \infty]$.

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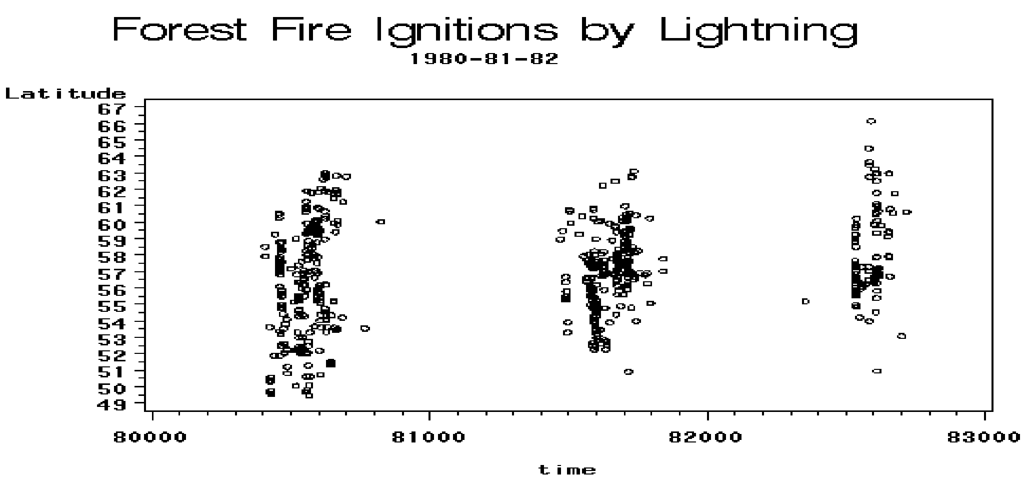
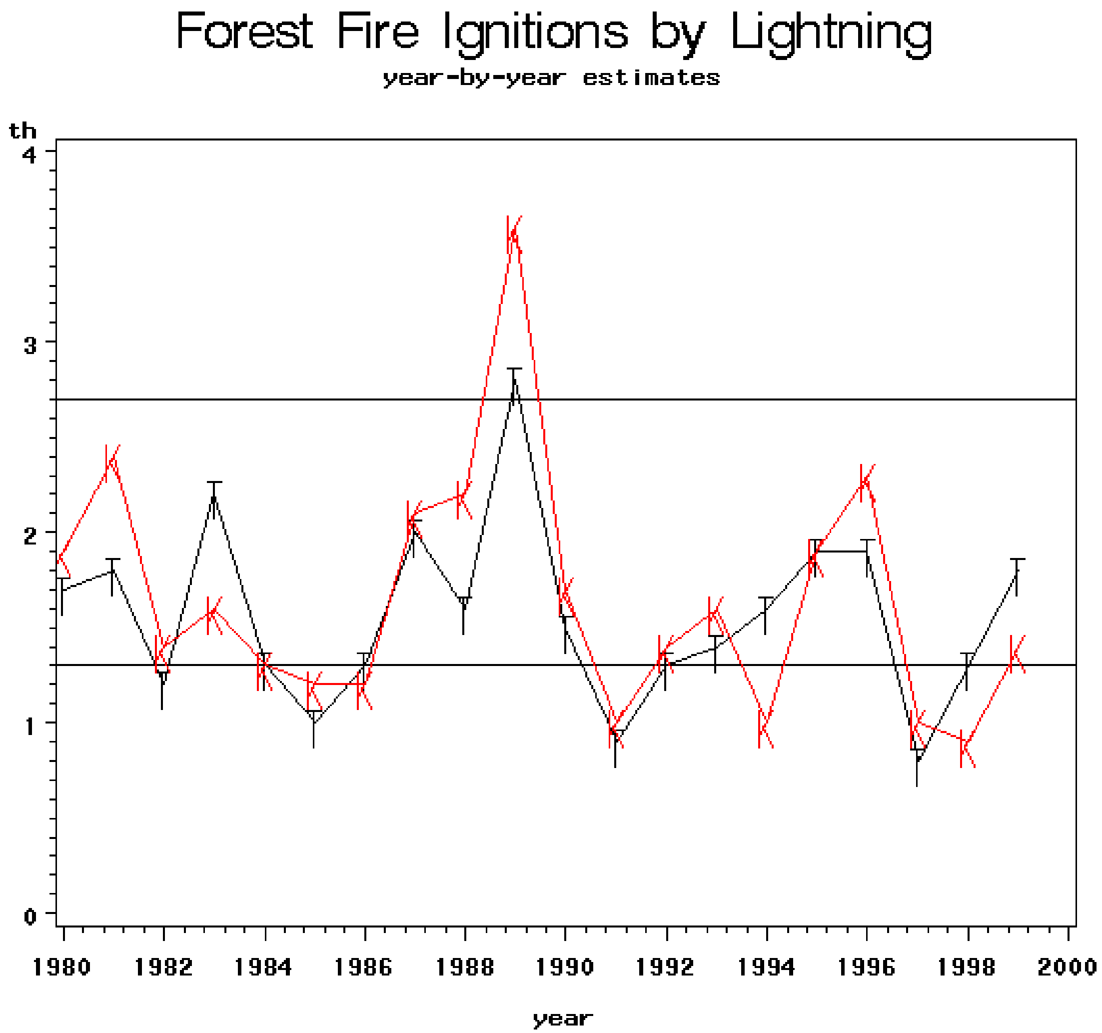
The best approximate CI we have is $[0.63, \infty]$.

If we treat each year (summer) as an independent sample, we can compute an estimate for each year, and then compute an approximate CI based on the independent estimates.





$\overline{\alpha} = 1.57$

$[1.34,\ 1.79]$

How to interpret $\overline{\alpha} = 1.57$?

This observation seems to indicate that (in the heart of most intense lightning storms) that the process of ignitions seems to behave more like a random process in dimension 1.5 rather than a random process in dimension 3. This could arise, for example, if an ignition spawned "daughter" ignitions (either by "spotting" or by clustering of the underlying lightning strikes) only along a branching path downwind of the initial site.

In a more practical vein, this value and $P[h(X_1, X_2) \leq x] \simeq x^{\alpha}$ can be employed to estimate the chance of a second ignition in close proximity to the first.

Thanks to the organizers for a stimulating conference.