What is a Multi-parameter Renewal Process?

B. Gail Ivanoff — University of Ottawa
Ely Merzbach — Bar Ilan University

Forest Fires and Point Processes
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Motivation:

"The archetypal point processes are the Poisson and renewal processes."

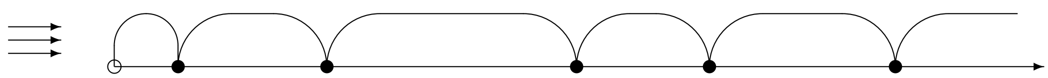
(Daley and Vere-Jones, *An Introduction to the Theory of Point Processes*)

Outline

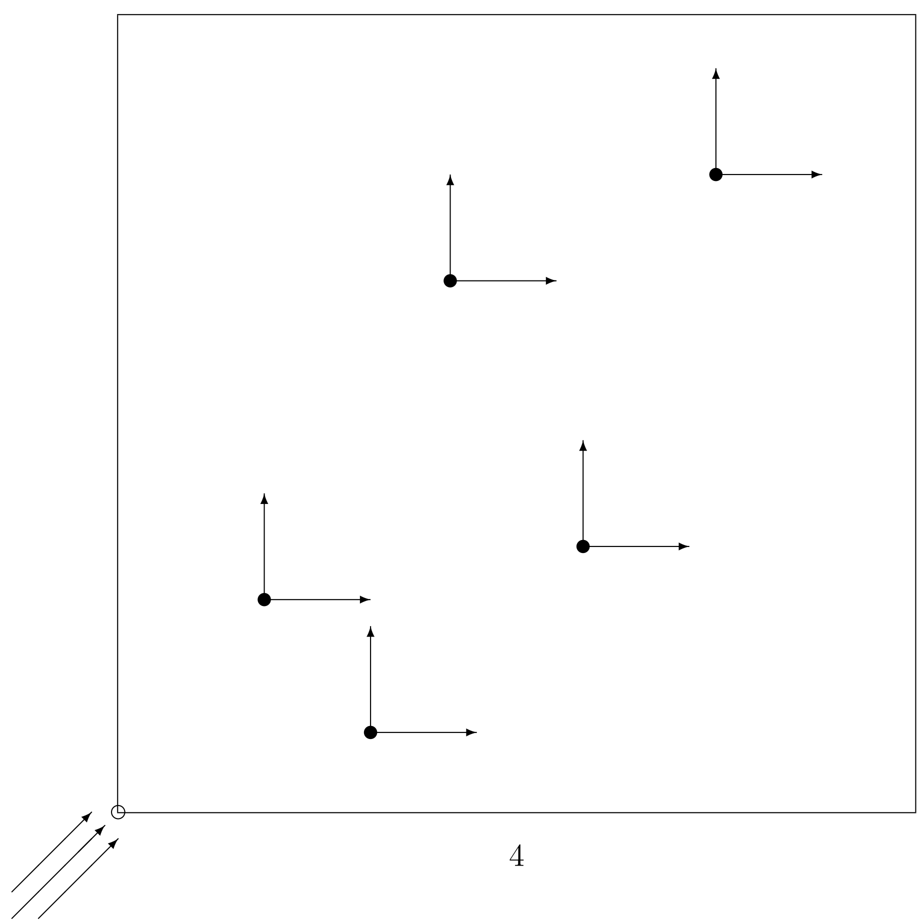
1. The renewal process as a forest fire model
2. Renewal property on $\mathbf{R}_+$
3. The structure of point processes on $\mathbf{R}_+^d$
4. Renewal property on $\mathbf{R}_+^d$
5. Renewal property and the Poisson process
6. Stochastic intensity of the renewal process
7. Open questions

1. The renewal process as a forest fire model

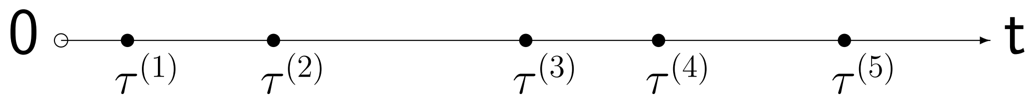
A forest fire under a prevailing wind:
One dimension:



Two dimensions:



2. Renewal property on R_+



Two equivalent ways of looking at the renewal property:

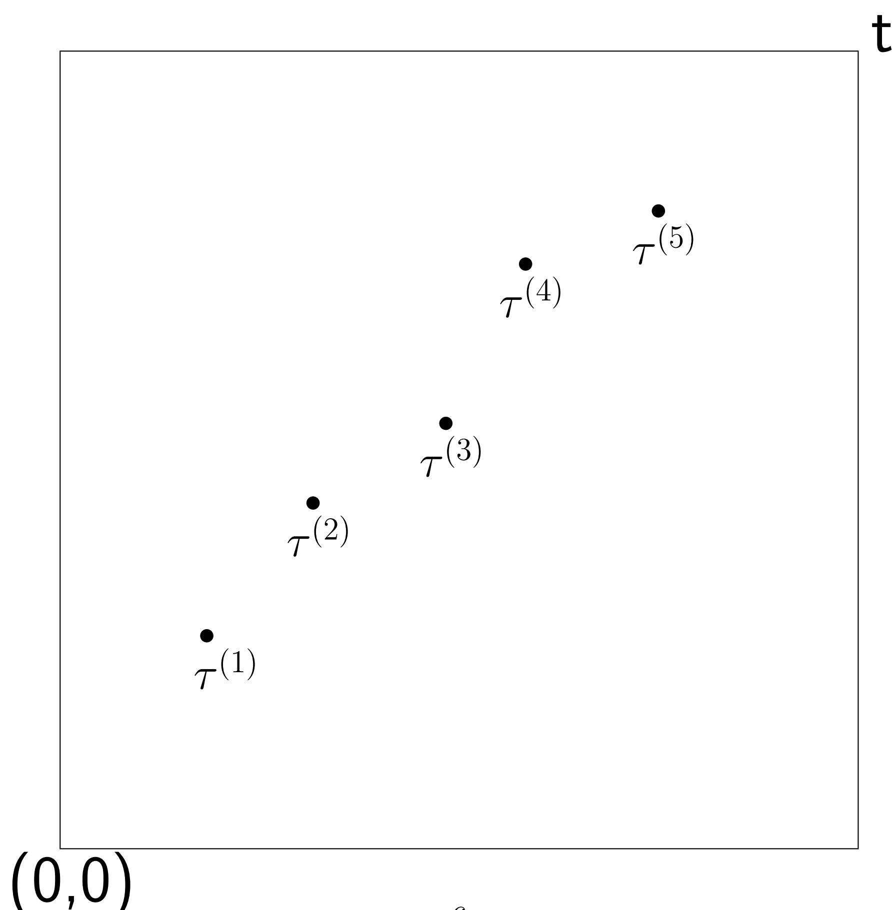
1. $\tau^{(i)} = \sum_{k=1}^{i} X_k$, where $X_1, X_2, ...$ are i.i.d. nonnegative random variables (partial sum process).
2. Given the value of $\tau^{(i-1)}$, $\tau^{(i)}$ is an independent copy of $\tau^{(1)}$ translated by $\tau^{(i-1)}$.

Note that if $N_t = \sum_{k=1}^{\infty} I_{\{\tau^{(k)} \leq t\}}$ is the associated point process, $\tau^{(i)}$ is an $\mathcal{F}^N$ *stopping time*.

Previous extensions of the renewal property to higher dimensions were restricted to *partial sum point processes*:

$$\tau^{(i)} = \sum_{k=1}^{i} X_k$$

where $X_1, X_2, \ldots$ are i.i.d. $\mathbf{R}_+^d$-valued random variables.



This is not a reasonable model for the spread of a forest fire.

A Poisson process is *not* renewal in this sense.

How do we extend the renewal property to higher dimensions to include point processes whose jumps are not totally ordered?

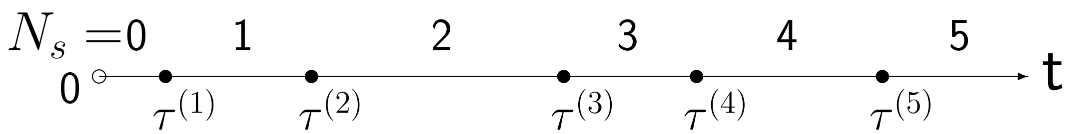
3. The structure of point processes on $\mathbf{R}_+^d$

A point process N on $\mathbf{R}_+^d$ may be characterized either by the locations of its jump points or by the associated process

$$N_t = \#\text{ jump points in } [0, t].$$

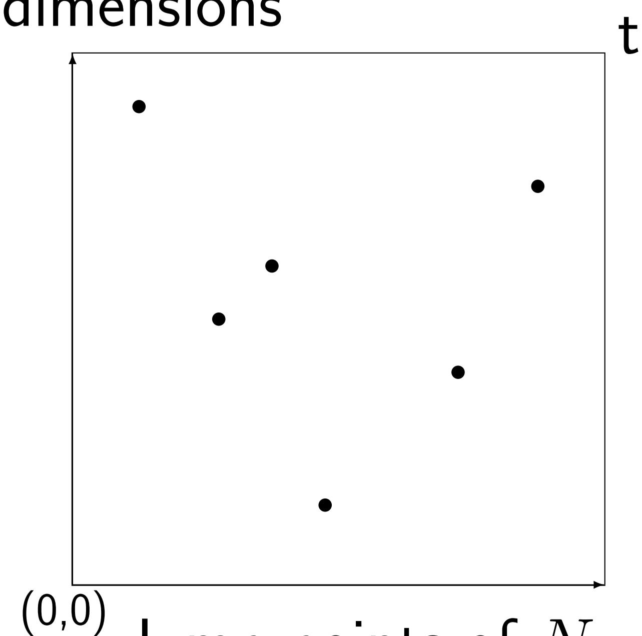
$$(t = (t_1, ..., t_d) \Rightarrow [0, t] = [0, t_1] \times ... \times [0, t_d])$$

One dimension:

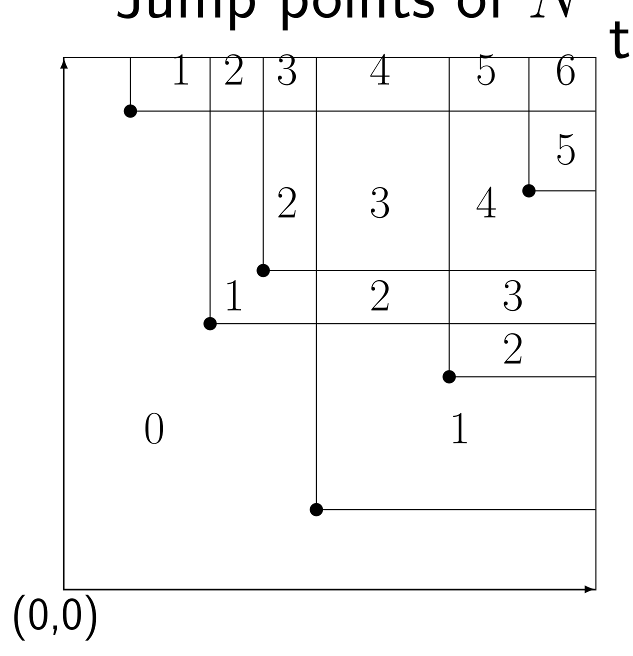


Values of N_s for $0 \leq s \leq t$

Two dimensions



Jump points of N



Values of N_s for $s \in [0, t]$

Note: Jump points of point processes in higher dimensions are *not* stopping times.

In higher dimensions, the appropriate generalization of a stopping point is a stopping *line,* or equivalently a stopping set:

Definition: Let N be a point process on $\mathbf{R}_+^d$ and let $\mathcal{F}^N$ be its associated filtration (i.e. $\mathcal{F}_t^N = \sigma\{N_s : s \in [0,t]\}$, for $t \in \mathbf{R}_+^d$). A closed random set $\xi \subseteq \mathbf{R}_+^d$ is a *stopping set* (wrt $\mathcal{F}^N$) if

$$\{t \in \xi\} \in \mathcal{F}_t^N \ \forall t \in \mathbf{R}_+^d.$$

On $\mathbf{R}_+$, $[0,\tau]$ is a stopping set (wrt $\mathcal{F}^N$) if and only if τ is a stopping time. Let $\xi_i = [0, \tau^{(i)}] = \{s : N_{s-} < i\}$.

On $\mathbf{R}_+^d$, the following are stopping sets:

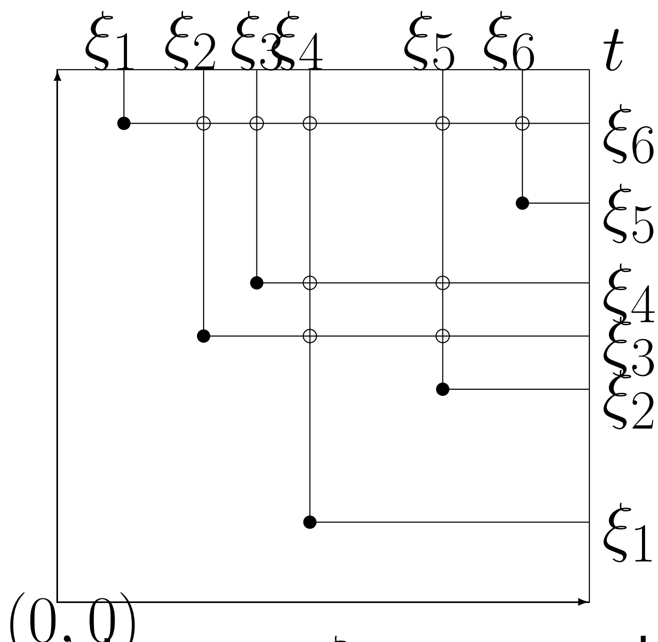
$$\xi_n := \{s : N_{s-} = N_{(0,s)} < n\}.$$

Exposed points of ξ_n:

$$\epsilon(\xi_n) = \min(\overline{\xi_n^c}) = \{\tau_1^{(n)}, \tau_2^{(n)} \ldots\}$$

where

$$\min(B) = \{t \in B : s \not\leq t \forall s \in B, s \neq t\}$$

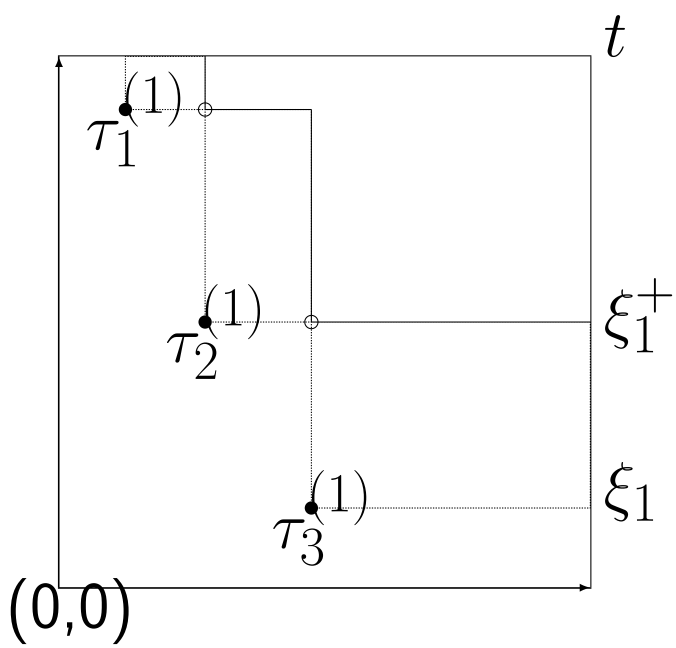


The stopping sets ξ_n generated by N and their exposed points

Note:

1. N determines and is determined by its stopping sets $\xi_n, n \geq 1$.

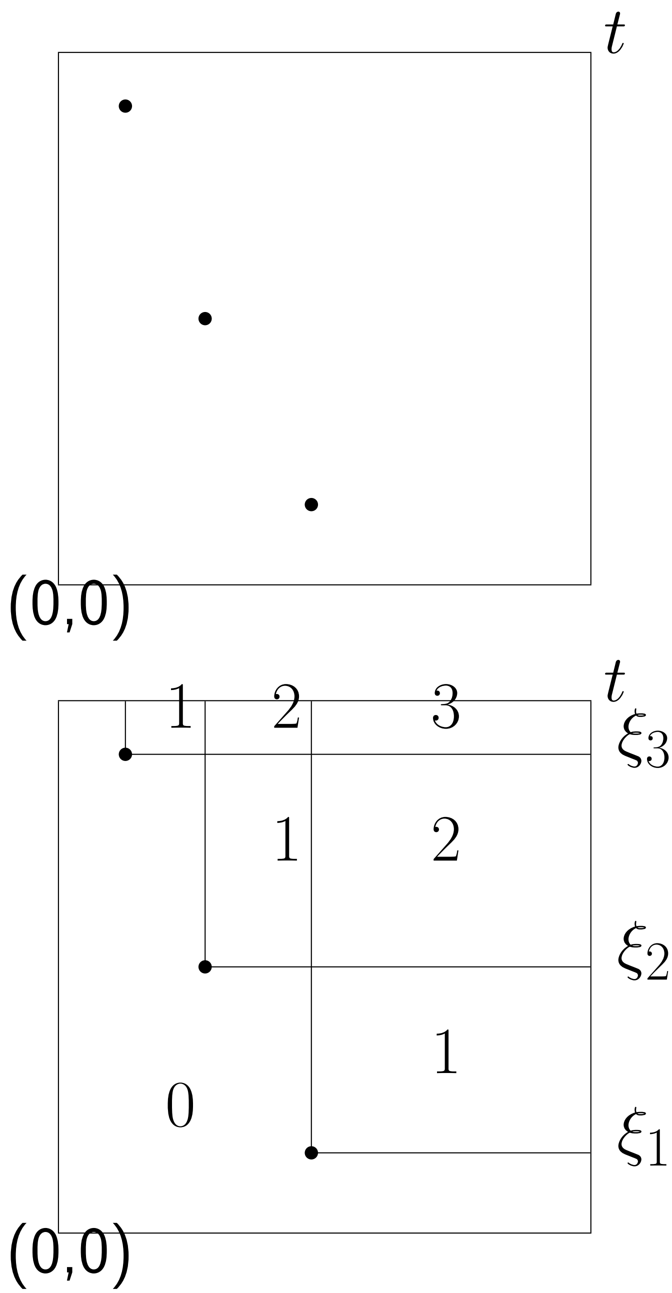
2. For each $\underline{n \geq 1,\ \xi_n \subset \xi_{n+1} \subseteq \xi_n^+}$, where $\xi_n^+ := \{\cup_{k \neq j}(\tau_k^{(n)} \vee \tau_j^{(n)}, \infty)\}^c$. If ξ_n contains only one exposed point, then $\xi_n^+ = \mathbf{R}_+^d$.



The stopping sets ξ_1 and ξ_1^+

The d-dimensional analogue of the single jump process $\mathbf{R}_+$ is the *single line process*:

Definition: A point process M is a single line process if its jump points are all incomparable in the partial order on $\mathbf{R}_+^d$, which is the case if and only if the set of jump points of M is equal to $\epsilon(\xi_1)$. M is known as the single line process associated with ξ_1.



A single line process M and its values

4. Renewal property on $\mathbf{R}_+^2$

The idea: A renewal process on $\mathbf{R}_+$ can be constructed by successively translating i.i.d. copies of the single jump process associated with $\tau^{(1)}$. For a renewal point process N defined on $\mathbf{R}_+^d$ we replace $\tau^{(1)}$ with the single line process associated with ξ_1. Given ξ_n, the renewal times will be the exposed points of ξ_n, and each renewal puts an independent copy of ξ_1 (suitably translated) on a set of the form $(\tau_j^{(n)}, \infty) \cap \xi_n^+$. **Note :**

- These sets are disjoint and incomparable.
- Their union is $\xi_n^+ \setminus \xi_n$.

Definition: A point process N on $\mathbf{R}_+^{\mathbf{d}}$ is renewal if

- The "renewal times" generated by ξ_n are the exposed points $\epsilon(\xi_n) = (\tau_1^{(n)}, \tau_2^{(n)}, \ldots)$.
- Given ξ_n, the process N behaves independently on each of the disjoint sets $(\tau_j^{(n)}, \infty) \cap \xi_n^+$.
- Given ξ_n, the distribution of ξ_{n+1} is equal to the distribution of

$$\xi_n \cup \left(\xi_n^+ \cap \cup_{i=1}^{\infty}(\xi_{1,i} \oplus \tau_i^{(n)})\right)$$

where $\xi_{1,1}, \xi_{1,2}, \ldots$ are i.i.d. copies of ξ_1 and $B \oplus t = \{x + t : t \in B\}$.

Note: This definition includes the partial sum point process.

Simulating a forest fire:

- Choose a suitable model for a single line process M describing the initial spread after ignition. Use this to construct the jump points that define ξ_1.
- Proceed by iteration: once ξ_n has been generated, construct ξ_n^+. On each of the disjoint rectangles of $\xi_n^+ \setminus \xi_n$, put an independent copy of M. The addition of these points defines ξ_{n+1}.

5. Renewal property and the Poisson process

Definition: A point process N on $\mathbf{R}_+^d$ is a Poisson process if there exists a boundedly finite diffuse Borel measure Λ such that for every finite family of disjoint bounded Borel sets $B_1, ..., B_n \subset \mathbf{R}_+^d$ and $k_1, ..., k_n \in \mathbf{N}$, we have

$$P(N_{B_i} = k_i, i = 1, ..., n) = \frac{\Lambda(B_1)^{k_1}}{k_1!} e^{-\Lambda(B_1)} \times ... \times \frac{\Lambda(B_n)^{k_n}}{k_n!} e^{-\Lambda(B_n)}.$$

Λ is called the mean measure of N. If Λ is absolutely continuous with density (λ_t) with respect to Lebesgue measure μ, (λ_t) is called the intensity of N. If $\lambda_t \equiv \lambda$ then N is a *homogeneous* Poisson process.

Definition: Let N be a point process on $\mathbf{R}_+^d$.

- N is *stationary* if the finite dimensional distributions are invariant under translation: i.e. for all $B_1, ..., B_n \in \mathcal{B}$, $n \in \mathbf{N}$ and $t \in T$,
  $$P(N_{B_1} = k_1, ..., N_{B_n} = k_n) = P(N_{B_1 \oplus t} = k_1, ..., N_{B_n \oplus t} = k_n).$$
- N has *independent increments* if for all disjoint $B_1, ..., B_n \in \mathcal{B}$, $n \in \mathbf{N}$, $N_{B_1}, ..., N_{B_n}$ are independent random variables.

Theorem: Let N be a point process on $\mathbf{R}_+^d$. The following are equivalent.

1. N is a homogeneous Poisson process.
2. N is stationary with independent increments.
3. N is a Poisson renewal process.
4. N is renewal with independent increments and no fixed atoms.
5. N satisfies the *waiting time paradox*.

The waiting time paradox on R_+: A point process N on $\mathbf{R}_+$ on satisfies the waiting time paradox if for every $s \leq t \in \mathbf{R}_+$, $N(s, s+t]$ is independent of $\mathcal{F}_s^N$ and

$$N(s, s+t] =_{\mathcal{D}} N(0, t].$$

The waiting time paradox on R_+^d: A point process N on $\mathbf{R}_+^d$ on satisfies the waiting time paradox if for every set of the form $R = \cup_{i=1}^n [0, t_i]$ and all $s, t \in \mathbf{R}_+^d$ such that $s \notin R^o$, $N(s, s+t]$ is independent of $\mathcal{F}_R^N = \sigma\{N_u : u \in R\}$ and

$$N(s, s+t] =_{\mathcal{D}} N(0, t].$$

6. Stochastic intensity of the renewal process

Roughly speaking, the ($\mathcal{F}^N$-) stochastic intensity of a point process N is defined by

$$\lambda_t dt \approx P(N(t, t + dt] = 1 \mid \mathcal{F}_t^N).$$

On $\mathbf{R}_+$, λ determines the law of N, but this is no longer true in higher dimensions.

Consider

$$\mathcal{F}_t^{*N} = \sigma\{N_s : s_1 \leq t_1 \text{ or } s_2 \leq t_2\}$$

The *-stochastic intensity is defined by

$$\lambda_t^* dt \approx P(N(t, t + dt] = 1 \mid \mathcal{F}_t^{*N}).$$

λ^* does not necessarily determine the law of N, but it does if N is Poisson.

*-Intensity of a single line process M on $\mathbf{R}_+^2$

Assumption (F4): Given $\mathcal{F}^M_{(t_1,t_2)}$ the jump points of M on $[0,t_1]\times(t_2,\infty)$ are conditionally independent of the jump points of M on $(t_1,\infty)\times[0,t_2]$.

This is satisfied by the forest fire model.

*-stochastic intensity of M:

$$\lambda_t^{*M} = I(M(B_t)=0)\frac{m(t)}{P(M(B_t)=0)}$$

where m is the density of the mean measure of M and

$$B_t = [0,t] \setminus \{t\}$$

*-Intensity of a renewal process N on $\mathbf{R}_+^2$

N is renewal with associated stopping sets ξ_i and M is the single line process associated with ξ_1.

For $t \in \mathbf{R}_+^2$

$\alpha_t = \max\{\alpha \leq t : \alpha$ a renewal point of $N\}$

*-stochastic intensity of N:

$$\lambda_t^{*N} = \lambda_{t-\alpha_t}^{*M}$$

Conjecture: If M satisfies (F4), then the law of N is determined by its *-stochastic intensity.

Theorem: Let $N^1, N^2, \ldots$ be i.i.d. copies of a renewal process N on $\mathbf{R}_+^2$ and let

$$N_n(t) := \sum_{i=1}^{n} N^i \left(\frac{t}{\sqrt{n}} \right).$$

If M satisfies (F4), then N_n converges weakly in the Skorokhod topology on $\mathbf{R}_+^2$ to a homogeneous Poisson process with intensity $\lambda \equiv m(0)$.

7. Open Questions

- Renewal property on more general state spaces
- Cox processes
- regenerative processes
- compensators
- Limit theorems: renewal theorem, Blackwell's theorem