Noncommutative Geometry Heisenberg calculus and CR geometry

Raphaël Ponge
(Ohio State University)

Abstract

The Fefferman program was launched 25 years ago as a conjecture to express the analysis of the Bergman and Szego kernels of a strictly pseudoconvex domain in purely geometric terms. To date this conjecture is only partially solved, but a more general aim of the Fefferman is to relate the hypoelliptic analysis of the Kohn-Rossi complex to the CR geometric data of the manifold.

What I would like to explain in this talk is how new perspectives in the latter direction can be opened by making use of noncommutative geometry, and in particular of the operator theoretic framework for local index formula of Connes-Moscovici.

I. CR manifolds

Definition. *A CR structure on an orientable manifold M^{2n+1} is given by a rank n complex subbundle $T_{1,0} \subset T_{\mathbb{C}}M$ so that:*

- $T_{1,0}$ is integrable, i.e. $[T_{1,0}, T_{1,0}] \subset T_{1,0}$.

- $T_{1,0} \cap T_{0,1} = \{0\}$, where $T_{0,1} = \overline{T_{1,0}}$.

Equivalently, $H = \Re(T_{1,0} \oplus T_{0,1})$ has the structure of a complex vector bundle of rank n.

Example. Boundary of domain $D \subset \mathbb{C}^n$.

• Horizontal $\bar{\partial}$-complex (Kohn-Rossi):

The decomposition $H \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$ yields a splitting $\Lambda_{\mathbb{C}} H^* = \oplus \Lambda^{p,q}$.

The operator $\bar{\partial}_b : \Gamma(\Lambda^{p,q}) \to \Gamma(\Lambda^{p,q+1})$ is

$$\bar{\partial}_b = \Pi_{p,q+1} \circ d, \quad \Pi_{p,q+1} : \Lambda T^* M \to \Lambda^{p,q}.$$

Since $\bar{\partial}_b^2 = 0$ we get a complex.

The operator,

$$\Box_b = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*,$$

is the Kohn Laplacian.

Fact: $\Box_b$ *is not elliptic, yet can be hypoelliptic for most, but not all, values of* q *(e.g. under Kohn's* $Y(q)$ *condition).*

• <u>Pseudohermitian Structure:</u>

Let θ be a non-vanishing real 1-form annihilating H. The Levi form on $T_{1,0}$ is

$$L_\theta(Z, W) = -i d\theta(\bar{Z}, W).$$

Definition. *1) M is strictly pseudoconvex when θ can be chosen in such way that L_θ is positive definite.*
2) The choice of such a 1-form is called a pseudohermitian structure on M.

Theorem (Tanaka, Webster). *A pseudohermitian structure determines a canonical connection, the Tanaka-Webster connection, which preserves the CR structure. This connection is not torsion-free.*

Open Question (Fefferman's program). *How much can we relate the hypoelliptic analysis of the* $\bar{\partial}_b$*-complex to the pseudohermitian geometric data?*

II. Heisenberg Calculus

Independently invented by Beals-Greiner ('84) and Taylor ('84), extending previous works by Dynin, Folland-Stein, Boutet de Monvel, and others.

• Heisenberg Manifolds:

Definition. *A Heisenberg manifold is a manifold M together with a distinguished hyperplane $H \subset TM$.*

This definition includes CR manifolds, as well as:

- Foliations (already dealt with by Connes-Moscovici);
- Contact manifolds.
- Confoliations of Elyashberg and Thurston.

• Tangent Groupoid of a Heisenberg Manifold:

Lemma. *There exists a 2-form* $\mathcal{L} : H \times H \to TM/H$ *so that*

$$\mathcal{L}_m(X(m), Y(m)) = [X, Y](m) \bmod H_m$$

for sections X, Y *of* H *near* $m \in M$.

Definition. *The tangent Lie group bundle* GM *is obtained by endowing the bundle,*

$$(TM/H) \oplus H$$

with the grading and product such that

$$t.(X_0 + X') = t^2 X_0 + tX', \quad t \in \mathbb{R},$$
$$(X_0 + X').(Y_0 + Y') =$$
$$X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y',$$

for sections X_0, Y_0 *of* TM/H *and* X', Y' *of* H.

Proposition. *We have:*

$$\operatorname{rk} \mathcal{L}_x = 2n \Longleftrightarrow G_xM \simeq \mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n},$$

where $\mathbb{H}^{2n+1}$ *is the* $(2n+1)$*-dimensional Heisenberg manifold.*

This result justifies the terminology Heisenberg manifold.

Remark. This description of G_xM coincides with that in terms of nilpotent approximation of vector fields at x, via the use of Heisenberg coordinates at x (RP '04).

Theorem (RP '04). $M \times M$ *smoothly deforms to* GM*, i.e.*

$$\mathcal{G}_H M := GM \sqcup (M \times M \times (0, \infty)) \mapsto M \times [0, \infty),$$

is a differentiable groupoid.

• Heisenberg Calculus:

The underlying idea is to construct a class of pseudodifferential operators, the Ψ_HDO's, modeled on homogeneous left-invariant ΨDO's on the fibers of the tangent Lie group bundle GM.

Locally the Ψ_HDO's are ΨDO's of type $(\frac{1}{2}, \frac{1}{2})$, but unlike the latter they:

- possess a full symbolic calculus;
- make sense globally on any Heisenberg manifold.

• Description of Ψ_HDO's.

Consider a chart U with a with a frame $X_0,\ldots,X_d$ of TU so that $X_1,\ldots,X_d$ are in H and set $\sigma_j = \text{symb}(\frac{1}{i}X_j)$ and $\sigma=(\sigma_0,\ldots,\sigma_d)$.

Definition. *A Heisenberg symbol $p(x,\xi)$ of order m has an asymptotic expansion,*

$$p \sim \sum_{j\geq 0} p_{m-j}, \quad p_{m-j}(x,\lambda.\xi) = \lambda^{m-j} p_{m-j}(x,\xi),$$

where $\lambda.\xi = (\lambda^2\xi_0, \lambda\xi_1,\ldots,\lambda\xi_d)$.

Definition. *A $\Psi_H DO$ of order m is locally of the form,*

$$P = p(x,-iX) + R, \quad R \textit{ smoothing},$$

$$p(x,-iX)u(x) = (2\pi)^{-(d+1)} \int e^{ix.\xi} p(x,\sigma(x,\xi))\hat{u}(\xi)d\xi,$$

where p is a Heisenberg symbol of order m.

• Principal Symbol and Model Operators:

Definition. $S_m(\mathfrak{g}^*M)$, $m \in \mathbb{C}$, *consists of smooth functions on* $\mathfrak{g}^*M \setminus 0$ *which are homogeneous of degree* m.

Proposition (RP). *The principal symbol of* $P \in \Psi_H^m(M)$ *makes sense globally as an element* $p_m \in S_m(\mathfrak{g}^*M)$.

Definition. *The model operator of* P *at* $x \in M$ *is the left-convolution operator on* G_xM *by* p_m^x, $p_m^x = p_m(x,.)$.

Proposition (Beals-Greiner, Christ et al.). *There is a well defined product,*

$$* : S_{m_1} \times S_{m_2} \to S_{m_1+m_2},$$
$$[(p_{m_1} * p_{m_2})^x]^\vee *_x u = \check{p}_{m_1}^x *_x (\check{p}_{m_1}^x *_x u).$$

Proposition (Beals-Greiner). *For* $j = 1, 2$ *let* $P_j \in \Psi_H^{m_j}(M)$ *have principal symbol* p_{m_j}*. Then* P_1P_2 *has principal symbol* $p_{m_1} * p_{m_2}$ *and so* $(P_1P_2)^x = P_1^x P_2^x \ \forall x \in M$*.*

Theorem (Beals-Greiner). *Let* $P \in \Psi_H^m(M)$ *have principal symbol* p_m*. TFAE:*

(i) p_m *is invertible w.r.t. product* $*$*.*

(ii) $\exists Q \in \Psi_H^{-m}(M)$ *s.t.* $PQ = QP = 1 \bmod \Psi^{-\infty}$*.*

Moreover, if (i) and (ii) hold then P *is hypoelliptic with loss of* $\frac{m}{2}$*-derivatives, i.e. we have Sobolev estimates*

$$\|u\|_{s+m/2} \leq C(\|Pu\|_s + \|u\|_s) \ \forall u \in C_c^\infty(M).$$

• Rockland Condition:

Let $x \in M$. Then to any $\pi \in \widehat{G}_x M$ we can associated an (unbounded) operator π_{P^x} on $\mathcal{H}_\pi$ whose domain contains $C^\infty(\pi)$.

Definition. *P satisfies the Rockland condition at x if π_{P^x} is injective on $C^\infty(\pi)$ for any $\pi \in \widehat{G}_x M$.*

Theorem (Christ et al., RP). *If the Levi form has constant rank and the Rockland condition is satisfied by P and P^t at every point, then p_m is invertible.*

Remark. Above result also applies to Rumin's contact Laplacian.

• Holomorphic Families of Ψ_HDO's

Let $\Omega \subset \mathbb{C}$ be open.

Definition. *A family $(q_z)_{z\in\Omega}$ of symbols is holomorphic when:*

- The order m_z is an analytic function of z;

- (q_z) is a hol. family of smooth functions;

- The bounds of $q \sim \sum_{j\geq 0} q_{z,m_z-j}$ are locally uniform in z.

Definition. *A family (Q_z) of Ψ_HDO's is holomorphic if it is locally of the form*

$$Q_z = q_z(x, -iX) + R_z,$$

where (q_z) is a holomorphic family of symbols and (R_z) is a holomorphic family of smoothing operators.

III. Functional Calculus (heat kernel approach)

- $\Psi_H DO$ representation of the heat kernel:

Let P be a differential operator of even Heisenberg order v such that:

- P is positive, i.e. $\langle Pu, u\rangle \geq 0$;
- P has an invertible principal symbol.

Then e^{-tP} is well defined for $t \geq 0$, is smoothing for $t > 0$.

Moreover, e^{-tP} allows us to invert the heat equation, for the operator

$$Q_0 : C_c^\infty(M \times \mathbb{R}) \to \mathcal{D}'(M \times \mathbb{R}),$$
$$Q_0 u(x,t) = \int_0^\infty e^{-sP} u(x, t-s) ds,$$

satisfies $(P + \partial_t) Q_0 u = Q_0 (P + \partial_t) u = u$.

In terms of distribution kernels the definition of implies Q_0 that it has the *Volterra property* in the sense below.

Definition. $Q : C_c^\infty(M \times \mathbb{R}) \to \mathcal{D}'(M \times \mathbb{R})$ *has the Volterra property if its distribution kernel is of the form* $K_Q(x, y, t - s)$ *where* $K_Q(x, y, t) = 0$ *for* $t < 0$.

Furthermore, letting $k_t(x, y)$ be the heat kernel of P, we have

$$K_{Q_0}(x, y, t) = k_t(x, y) \quad \text{for } t > 0.$$

This equality motivates building a pseudodifferential calculus taking into account the Volterra property in order to study the heat kernel of P.

• Volterra $\Psi_H DO$-calculus:

Developped by Beals-Greiner-Stanton (JDG '84) for deriving the heat kernel asymptotics for the Kohn Laplacian.

Definition. *A Volterra-Heisenberg symbol* $q(x,\xi,\tau)$ *of order* m *has an asymptotic expansion,*

$$q \sim \sum_{j\geq 0} q_{m-j}, \quad q_{m-j}(x,\lambda.\xi,\lambda^v\tau) = \lambda^{m-j} q_{m-j}(x,\xi,\tau),$$

where $q_{m-j}(x,\xi,\tau)$ *has an analytic extension to the region* $\{\Im\tau < 0\}$ *(Paley-Wiener condition).*

Definition. *A Volterra* $\Psi_H DO$ *of order* m *on* $M\times\mathbb{R}$ *has the Volterra property and is locally of the form,*

$$Q = Q(x,-iX,D_t) + R, \quad R \text{ smoothing},$$

where q *is a Volterra-Heisenberg symbol of order* m*.*

Theorem (Beals-Greiner-Stanton). *Assume that $P+\partial_t$ has an invertible principal symbol.*

1) $(P+\partial_t)^{-1}$ is a Volterra-$\Psi_H DO$of order $-v$.

2) As $t \to 0^+$ we have

$$k_t(x,x) \sim t^{-\frac{d+2}{v}} \sum t^{\frac{2j}{v}} a_j(P)(x),$$
$$a_j(P)(x) = \check{q}_{-w-2j}(x,0,1),$$

where the equality on the bottom shows how to compute $a_j(P)(x)$ locally by means of the symbol $q_{-v-2j}(x,\xi,\tau)$ of degree $-v-2j$ of $(P+\partial_t)^{-1}$.

• Complex Powers:

For any $s \in \mathbb{C}$ we can define P^s by standard L^2-functional calculus. Then:

$$P^s = \Gamma(-s)^{-1}\int_0^\infty t^{-s}(1-\Pi_0(P))e^{-tP}\frac{dt}{t}, \quad \Re s < 0,$$
$$P^s = P^k P^{s-k}, \quad k \text{ integer} > \Re s \geq 0.$$

where $\Pi_0(P)$ is the orthogonal projection onto $\ker P$.

Combining this with the $\Psi_H DO$-reprensentation of the heat kernel of P we get:

Theorem (RP). *Assume the principal symbol of $P + \partial_t$ is invertible. Then $(P^s)_{s\in\mathbb{C}}$ is a holomorphic family of $\Psi_H DO$'s such that* ord$P^s = vs$.

Remark. This result is true for the Kohn Laplacian under $Y(q)$-condition.

Remark. This approach has also been used by Melrose-Singer-Varghese ('04) in the context of ΨDO's on Azamaya bundles.

In fact, combining the previous theorem with results of Folland-Stein we obtain:

Theorem (RP). *Assume the Levi form has constant rank and P satisfies the Rockland condition at every point. Then the principal symbol of $P + \partial_t$ is invertible, hence the previous results holds for P.*

Remark. This holds for Rumin's contact Laplacian and allows us to fix a gap in the proof of Julg-Kasparov of the Baum-Connes conjecture for $SU(n, 1)$.

IV. Functional Calculus (resolvent approach)

• $\Psi_H DO$ representation of the resolvent

Let P be a $\Psi_H DO$ of integer order $v > 0$ with an invertible principal symbol.

Definition. $\Theta(P)$ *is the cone which is the complement in* $\mathbb{C} \setminus 0$ *of the union set of spectra of the model operators* P^x, $x \in M$.

Lemma. *Let* $L \subset \Theta(P)$ *be a ray. Then* $\exists$ *an open cone* $\Theta \supset L$ *s.t.* $\forall R > 0$, *letting* $\Lambda = \Theta \cup [D(0,R) \setminus 0]$, $\exists q_{(\lambda)} \in C^\infty \hat{\otimes} \mathrm{Hol}^{-1}(\Lambda)$ *such that:*

(i) Modulo $S^{-\infty} \hat{\otimes} \mathrm{Hol}^{-\infty}(\Lambda)$,

$$(p_m - \lambda) * q_{(\lambda)} = q_{(\lambda)} * (p_m - \lambda) = 1.$$

(ii) $q_{(\lambda)}$ *is almost homogeneous of degree* $-v$, *i.e. for any* $t \in (0,1)$ *we have*

$$q_{(t^v \lambda)}(x, t.\xi) - t^{-v} q_{(\lambda)}(x,\xi) \in S^{-\infty} \hat{\otimes} \mathrm{Hol}^{-\infty}(\Lambda).$$

Definition. $\Psi_H^*(M;\Lambda)$ *consists of operators given by symbols with an asymptotic expansion of parametric almost homogeneous symbols.*

Let $\check{\Theta}(P)$ be the cone obtained from $\Theta(P)$ by removing from it all the rays containing an eigenvalue of P and set

$$\Lambda(P) = \breve{\bar{\Theta}}(P) \cup [D(0, R_0) \setminus 0],$$

where $R_P = \text{dist}(0, \operatorname{Sp} P \setminus 0)$.

Theorem. *1)* $\bar{\Theta}(P)$ *is an open cone.*
2) Any closed cone $\Theta \subset \Theta(P)$ *contains at most finitely many eigenvalues of* P*.*
3) The resolvent $(P-\lambda)^{-1}$ *is in* $\Psi_H(M;\Lambda(P))$*.*

Examples:

1) Selfdajoint case:

$$\Theta(P) \supset \mathbb{C} \setminus \mathbb{R}.$$

2) Selfadjoint sublaplacian and sum of squares $\Delta = -(X_1^2 + \ldots + X_r^2)$:

$$\Theta(P) = \mathbb{C} \setminus [0, \infty).$$

Hence any selfadjoint sublaplacian is bounded from below.

3) Real sublaplacian (e.g. $\Delta = -(X_1^2 + \ldots + X_r^2) + X_0$): $\exists \delta(\Delta) \in (0, \frac{\pi}{2})$ such that

$$\theta(\Delta) \supset \{\frac{\pi}{2} - \delta(\Delta) \leq \arg \lambda \leq \frac{3\pi}{2} + \delta(\Delta)\},$$

Moreover $\delta(\Delta) > 0$ when the Levi form is never zero.

• Complex Powers:

Let $L_\theta = \{\arg \lambda = \theta\}$ be a ray contained in $\Lambda(P)$ and define

$$P_\theta^s = \frac{1}{2i\pi} \int_{\Gamma_\theta} \lambda^s (P-\lambda)^{-1} d\lambda, \quad \text{for } \Re s < 0,$$
$$P_\theta^s = P^k P_\theta^{s-k}, \quad k \text{ integer} > \Re s \geq 0.$$

Proposition. *1) $(P_\theta^s)_{s\in\mathbb{C}}$ is a holomorphic family of Ψ_HDO's such that* $\operatorname{ord} P_\theta^s = ms$.

2) We have

$$P_\theta^{s_1+s_2} = P_\theta^{s_1} P_\theta^{s_2}, \quad P_\theta^0 = (1 - \Pi_0(P)),$$
$$P_\theta^{-k} = P^{-k}, \quad k = 1, 2, \ldots,$$

where P^{-k} denotes the partial inverse of P^k and $\Pi_0(P)$ is the projection onto $E_0(P) = \cup_{j\geq 1} \ker P^j$ and along $E_0(P^*)^\perp$ (note this is also a smoothing operator).

$\boxed{\text{V. Noncommutative Residue Trace}}$

Let (M^{d+1}, H) be a compact Heisenberg manifold.

• Logarithmic Singularity

Proposition. *Let $P \in \Psi_H^m(M)$, $m \in \mathbb{Z}$.*

1) Near the diagonal the kernel $k_P(x,y)$ of P has a behavior of the form

$$k_P(x,y) = \sum_{j=-(m+d+2)}^{-1} a_j(x, \psi_x(y)) \\ - c_P(x) \log \|\psi_x(y)\| + \mathrm{O}(1),$$

where $a_j(x, \lambda.z) = \lambda^j a_j(x,z)$.

2) The coefficient $c_P(x)$ makes sense globally as a density on M.

• Analytic Extension of the Trace:

If $\text{ord} P < -(d+2)$ then P is traceable and we have

$$\operatorname{Tr} P = \int_M k_P(x,x).$$

Theorem. *1) The map $P \to k_P(x,x)$ has a unique analytic continuation $P \to t_P(x)$ to $\Psi_H^{\mathbb{C}\setminus\mathbb{Z}}(M)$.*
2) If $(P_z)_{z\in\mathbb{C}}$ is a holomorphic jauge for $P \in \Psi_H^{\mathbb{Z}}(M)$ then $t_{P_z}(x)$ has at most a simple pole singularity near $z = 0$ such that

$$\operatorname{res}_{z=0} t_{P_z}(x) = -c_P(x).$$

Consider the functionals

$$\mathrm{TR}\, P = \int_M t_P(x) \quad \text{and} \quad \mathrm{Res}\, P = \int_M c_P(x),$$

defined on $\Psi_H^{\mathbb{C}\setminus\mathbb{Z}}(M)$ and $\Psi_H^{\mathbb{Z}}(M)$ respectively.

Theorem. *1) The functional* TR *is the unique analytic continuation of* $P \to \mathrm{Tr}\, P$ *to* $\Psi_H^{\mathbb{C}\setminus\mathbb{Z}}(M)$.
2) We have $\mathrm{TR}[P_1, P_2] = 0$ *whenever* $\mathrm{ord}P_1 + \mathrm{ord}P_2 \notin \mathbb{Z}$.
3) If $(P_z)_{z\in\mathbb{C}}$ *is a holomorphic jauge for* $P \in \Psi_H^{\mathbb{Z}}(M)$ *then* $\mathrm{TR}\, P_z$ *has at most a simple pole singularity near* $z = 0$ *such that*

$$\mathrm{res}_{z=0}\, \mathrm{TR}\, P_z = -\,\mathrm{Res}\, P.$$

• <u>Noncommutative residue</u>

Definition. *The functional* Res *is the noncommutative residue for* Ψ_H*DO's.*

Proposition. *1)* Res *is a local functional, i.e. is given by integration of a density.*

2) Res *is trace, i.e.* $\operatorname{Res}[P_1, P_2] = 0$.

3) We have

$$\operatorname{ord} Q.\operatorname{Res} P = \operatorname{res}_{z=0} \operatorname{TR} P Q_\theta^{-z}$$

for any positive order $\Psi_H DO$ Q *with principal cut* L_θ.

Theorem. *If* M *is connected then* Res *is the unique trace on* $\Psi_H^{\mathbb{Z}}(M)$ *up to constant multiple.*

$\boxed{\text{VI. NCG and area of a CR manifold}}$

• Quantized Calculus (Connes)

Let $\mathcal{H}$ be a Hilbert space. Then the following equivalences hold.

Classical Infinitesimal	Quantized Calculus
Complex Variable	Operator on $\mathcal{H}$
Real Variable	Selfadjoint Operator
Infinitesimal Variable	Compact Operator
Infinites. of order α	Compact Operator s.t. $\mu_k(T) = \mathrm{O}(k^{-\alpha})$
Integral $\int f(x)dx$	Dixmier Trace $⨍ T$

where $\mu_k(T) = (k+1)$'th eigenvalue of $|T| = \sqrt{T^*T}$.

Let (M,H) be a compact Heisenberg manifold.

Theorem (RP). *Let P be a $\Psi_H DO$ on M of negative order $-m$.*

1) P is an infinitesimal operator of order $\frac{m}{\dim M+1}$.

2) If $\text{ord}P = -(\dim M + 1)$*, then*

$$\int\!\!\!\!\!\!- P = \frac{1}{\dim M + 1} \operatorname{Res} P.$$

This allows us to integrate any $\Psi_H DO$, even though it is not an infinitesimal of order ≤ 1, by letting

$$\int\!\!\!\!\!\!- P = \frac{1}{\dim M + 1} \operatorname{Res} P.$$

• Area of a CR manifold:

Let (M^{2n+1}, θ) be a pseudohermitian manifold and let Δ_b be its sublaplacian.

For any $f \in C^\infty(M)$ we have

$$⨍ f \Delta_b^{-(n+1)} = \int_M f(x)(d\theta)^n \wedge \theta.$$

Thus $ds = \sqrt{\Delta_b}$ recaptures the contact volume. This leads us to define

$$\text{Area}_\theta M = ⨍ ds^2.$$

Theorem (RP). *If* $\dim M = 3$ *then*

$$\text{Area}_\theta M = ⨍ ds^2 = \int_M r_M(x) d\theta \wedge \theta,$$

where r_M *denotes the Tanaka-Webster scalar curvature of* M.

$\boxed{\text{VII. Local Index Formulas}}$

• The CM cocycle:

Let $(\mathcal{A}, \mathcal{H}, D)$ be an (even) spectral triple, i.e.:

- $\mathcal{H}$ is a Hilbert space together with a $\mathbb{Z}_2$-grading $\gamma : \mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{H}_- \oplus \mathcal{H}_+$;

- $\mathcal{A}$ is an involutive unital algebra represented in $\mathcal{H}$ and commuting with the $\mathbb{Z}_2$-grading γ;

- D is a selfadjoint unbounded operator on $\mathcal{H}$ with compact resolvent s.t. $\gamma D = -D\gamma$ and $[D, a]$ is bounded $\forall a \in \mathcal{A}$.

The above data yields a well defined index map,

$$\mathrm{ind}_D : K_0(\mathcal{A}) \longrightarrow \mathbb{Z}.$$

We make the following assumptions:

- <u>p-summability:</u> We have

$$\mu_k(D^{-1}) = \mathrm{O}(k^{-1/p}) \quad \text{as } k \to +\infty.$$

- <u>Smoothness:</u> We have $\mathcal{A} \subset \cap_{k\geq 0} \operatorname{dom} \delta^k$, where δ is the derivation $\delta(T) = [|D|, T]$.

- <u>Finite and simple dimension spectrum:</u> Let $\Psi_D^0(\mathcal{A})$ be the algebra generated by γ and the $\delta^k(a)$'s, $a \in \mathcal{A}$. Then the zeta functions,

$$\zeta(P; z) = \operatorname{Tr} P|D|^{-z}, \quad P \in \Psi_D^0(\mathcal{A}),$$

have at most simple pole singularities and the union set of their singularities is discrete.

This allows us to define a trace by letting

$$⨍ P = \operatorname{Res}_{z=0} \operatorname{Tr} P|D|^{-z}, \quad P \in \Psi_D^0(\mathcal{A}).$$

Theorem (Connes-Moscovici '95). *1) Under the above assumptions the following formulas define an even cocycle $\varphi_{\mathrm{CM}} = (\varphi_{2k})$ in the (b, B)-complex of the algebra $\mathcal{A}$.*

- For $k = 0$,

$$\varphi_0(a^0) = \mathrm{res}_{z=0} \operatorname{Tr} a^0 |D|^{-z},$$

- For $k \neq 0$,

$$\varphi_{2k}(a^0, \ldots, a^{2k}) = \sum_{\alpha} c_{k,\alpha} ⨍ \gamma P_{k,\alpha} |D|^{-2(|\alpha|+k)},$$

$$P_{k,\alpha} = a^0 [D, a^1]^{[\alpha_1]} \ldots [D, a^{2k}]^{[\alpha_{2k}]},$$

where the $c_{k,\alpha}$'s are universal rational constants and the symbol $T^{[j]}$ denotes the j'th iterated commutator with D^2.

2) We have:

$$\mathrm{ind}_D[\mathcal{E}] = \langle [\varphi_{\mathrm{CM}}], \mathcal{E} \rangle \quad \forall \mathcal{E} \in K_0(\mathcal{A}).$$

where $\langle ., . \rangle$ is the pairing of cyclic cohomology with K-theory.

Let (M^{d+1}, H) be a compact Heisenberg manifold and let $D : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ be a selfdajoint first order $\Psi_H DO$ such that:

- $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is $\mathbb{Z}_2$-graded with grading γ.

- $D\gamma = -D\gamma$ and the principal symbol of D is invertible.

- The principal symbol of D^2 is contained in the commutant of a subalgebra of $S_H^{\mathbb{Z}}(M, \mathcal{S})$ containing the principal symbols of $[D, a]^{[k]}$ and $a^{[k]}$, $a \in C^\infty(M)$ (e.g. D^2 sublaplacian and D commutes with almost complex structure).

Proposition. $(C^\infty(M), L^2(M,\mathcal{S}), D)$ *is a smooth spectral triple which is* $(\dim M+1)$*-summable and has a simple dimension spectrum contained in*

$$\{k \in \mathbb{Z};\ k \leq \dim M + 1\}.$$

Moreover, the algebra Ψ_D^0 *is contained in* $\Psi_H^0(M,\mathcal{S})$ *and the associated residual trace coincides with the noncommutative residue for* Ψ_H*DO's.*

Let $\mathcal{E}$ be a Hermitian bundle over M equipped with a unitary connection $\nabla^{\mathcal{E}}$. Then

$$\operatorname{ind}_D[\mathcal{E}] = \operatorname{ind} D^+_{\nabla,\mathcal{E}},$$

where $D_{\mathcal{E}}$ is the twisted operator given by the composition

$$\Gamma(\mathcal{S}\otimes\mathcal{E}) \overset{1_{\mathcal{E}}\otimes\nabla}{\rightarrow} \Gamma(\mathcal{S}\otimes T^*M\otimes\mathcal{E}) \overset{\pi_D\otimes 1_{\mathcal{E}}}{\rightarrow} \Gamma(\mathcal{S}\otimes\mathcal{E}),$$
$$\pi_D[(f^0 df^1)\otimes\sigma] = f^0[D,f^1]\sigma.$$

Theorem (RP). *1) There exists an even de Rham current C_D on M such that for any Hermitian data $(\mathcal{E}, \nabla^{\mathcal{E}})$ as above we have*

$$\operatorname{ind} D^{+}_{\nabla,\mathcal{E}} = \langle C_D, \operatorname{Ch} F^{\mathcal{E}} \rangle.$$

where $\operatorname{Ch} F^{\mathcal{E}} = \operatorname{Tr} e^{-F^{\mathcal{E}}}$ is the total Chern form of the curvature $F^{\mathcal{E}}$ of $\nabla^{\mathcal{E}}$.

2) The components C_{2k}, $k = 0, 2, \ldots$, of C_D are given by the following formulas.

- For $k = 0$,

$$\langle C_0, f^0 \rangle = \operatorname{res}_{z=0} \Gamma(z) \operatorname{Tr} f^0 |D|^{-z}.$$

- For $k \neq 0$,

$$\langle C_{2k}, f^0 df^1 \wedge \ldots \wedge df^{2k} \rangle = \sum_{\alpha} c_{k,\alpha} \,⨍\, \gamma P_{k,\alpha} |D|^{-2(|\alpha|+k)},$$

$$P_{k,\alpha} = f^0 [D, f^1]^{[\alpha_1]} \ldots [D, f^{2k}]^{[\alpha_{2k}]},$$

where $⨍$ denotes the noncommutative residue for Ψ_HDO's.

• Example:

Let (M^3, θ) be a pseudohermitian manifold and assume that $T_{1,0}$ has a real structure, i.e. an antilinear involution $Z \to \underline{Z}$.

Extending it to an involution on $\Lambda^{*,*}$ we get a Hodge operator,

$$\underline{*\alpha} \wedge \beta = L_\theta^*(\alpha, \beta) d\theta, \quad \alpha, \beta \in \Lambda^{p,q}.$$

where L_θ^* is the Levi metric on $\Lambda^{*,*}$. We then get a $\mathbb{Z}_2$-grading by letting,

$$\gamma = i^{(p+q)^2+1} * \quad \text{on } \Lambda^{p,q}.$$

Define

$$Q_b = (\bar{\partial}_b^* \bar{\partial}_b - \bar{\partial}_b \bar{\partial}_b^*) - \gamma(\bar{\partial}_b^* \bar{\partial}_b - \bar{\partial}_b \bar{\partial}_b^*)\gamma.$$

Theorem (RP). *1) On each $\Lambda^{p,q}$ the operator Q_b is, up to a sign factor, a sublaplacian with an invertible principal symbol.*

2) The previous assumptions are satisfied by the operator D_b such that

$$Q_b = D_b |D_b|.$$