The Twisted k-Homology of Lie Groups

Fields Institute Workshop (
1 Oct 2004
C. Douglas

Twistings

- are bundles$/_X$ of spectra, fibre R

  $\rightsquigarrow$ Twisted R-cohomology of X [htpy classes of sections]

- are classified by $X \to BAut\,R$ [htpy automorphisms]

$$\begin{array}{ccc} X & \longrightarrow & BAut\,R \\ & \searrow & \cup \\ & & BGL_1 R \end{array}$$

[elementary twistings]

[A_∞ ring spectrum]

[htpy units] [trans fncs just mult by htpy unit]

Motivations

From Elliptic Cohomology

- $X \xrightarrow{\tau} BGL_1 H\mathbb{C} \simeq B\mathbb{C}^* \dashrightarrow BS^1 \to \mathbb{Z} \times BU$ [BS^1 generating k] [really $\to \mathbb{Z}\times BO$ for TMF analog]

  $H\mathbb{C}$-twisting $\tau \rightsquigarrow$ k-thy class, "line bundle" $\mathcal{L}(\tau)$

  $H\mathbb{C}^*_\tau X = H\mathbb{C}^*(X; \mathcal{L}(\tau))$

- $X \xrightarrow{\tau} BGL_1 k \simeq T \times S \to T \to TMF$ [T generating TMF]

  $T = k(\mathbb{Z},3) \tilde{\times} k(\mathbb{Z}/2, 1)$

  k-twisting $\tau \rightsquigarrow$ TMF class, "elliptic line bundle" $\mathcal{L}(\tau)$

  eg. $k(\mathbb{Z},2) \to P$

  $\qquad\qquad\quad\downarrow$

  $\qquad\qquad X \to k(\mathbb{Z},3)$

  ~ stack locally iso to Line = {line bndles}

  ~ a 2-line bundle on X $\in$ 2-vect bnds on X

  $K^*_\tau X =$ "$k^*(X; \mathcal{L}(\tau))$"

(Motivations)

From Physics

D-branes $\sim \partial$ conditions for open strings

$\rightsquigarrow$ charge in $K^{\tau}(\text{Spacetime})$ [τ = Neveu-Schwarz H-flux]

$K_{\cdot}^{\tau} G$ = top. model for D-branes in WZWCFT.

Refinement: D-branes $\longrightarrow MSpin_{\cdot}^{c,\tau} G \longrightarrow K_{\cdot}^{\tau} G$

Computing Twisted k-Homology

[Note: $\exists$ equivariant analogs to all techniques & computations]

Background [on Twisted Homology Theories]

Given $\begin{matrix} F \to E \\ \downarrow \\ X \end{matrix}$ bnd of spectra, ie $\begin{matrix} F_i \to E_i \\ \downarrow \\ X \end{matrix}$ & $\Sigma_X E_i \to E_{i+1}$ [E not a spectrum so no confusion]

- $F^n(X) = \operatorname{colim} [X, \Omega^i F_{i+n}] = \operatorname{colim} \Gamma_h(X, X \times \Omega^i F_{i+n})$

$\downarrow$

$E^n(X) = \operatorname{colim} \Gamma_h(X, \Omega_X^i E_{i+n})$, [struc $\Omega_X^i E_{i+n} \to \Omega_X^{i+1} \Sigma_X E_{i+n} \to \Omega_X^{i+1} E_{i+1+n}$]

- $F_n(X) = \operatorname{colim} [S^{i+n}, (X \times F_i)/X]$

$\downarrow$

$E_n(X) = \operatorname{colim} [S^{i+n}, E_i/X]$, [struc $\Sigma(E_i/X) = (\Sigma_X E_i)/X \to E_{i+1}/X$]

Definition [of Twisted k-Theory]

Given $X \xrightarrow{\alpha} k(\mathbb{Z},3)$

$$\begin{matrix} k(\mathbb{Z},2) \to P(\alpha) \\ \downarrow \\ X \end{matrix}$$

Form $\begin{matrix} k \to E_\alpha = P(\alpha) \times_{k(\mathbb{Z},2)} k \\ \downarrow \\ X \end{matrix}$ by "$k(\mathbb{Z},2) \times k \to k$
$(L, V) \longmapsto L \otimes V$" [· via $PU(H) \circlearrowleft \mathrm{Fred}(H)$ · via $k(\mathbb{Z},2) \circlearrowleft MSpin^c$ · note $\dim V = 0$]

Define $K_\alpha^{\cdot}(X) = E_\alpha^{\cdot}(X)$

Alt Def $K_\alpha^{\cdot}(X) = [P(\alpha), k]_{k(\mathbb{Z},2)}$ [Impt later for alg geo def]

(Computing)

Primary Tool [Twisted Rothenberg-Steenrod Spectral Sequence]

Rothenberg-Steenrod SS (Segal):

For $S_\cdot : \Delta^{op} \to \mathcal{C}$, have $E^2 = H_p(E_q(S_\cdot)) \Rightarrow E_{p+q}(|S_\cdot|)$

Twisted k-Hom RSSS

is RSSS for $\mathcal{K} = \begin{cases} \cdot\ (X; E),\ E \text{ a } k\text{-bnd on } X \\ \cdot \text{ fibrewise h.eq.} \end{cases}$

[Initially suggested by Hopkins]

Example

Given $\tau = k \in H^2(\Omega G; \mathbb{Z}) \leftrightarrow \begin{matrix} L^{-k} \\ \downarrow \\ \Omega G \end{matrix}$

Form $B_\tau \Omega G = B_\cdot(*, \Omega G, *_\tau) : \Delta^{op} \to \mathcal{K}$

where $\Omega G \times k \to k$ defines $*$ [as module over ΩG]

$\Omega G \times k \xrightarrow[\tau \times id]{} k(\mathbb{Z},2) \times k \to k$ defines $*_\tau$

Note $\cdot\ \mathcal{K}_*(\Omega G \to *_\tau) = (k.\Omega G \to (k.*)_\tau)$

$c \longmapsto \langle L^k, c \rangle$

[$\mathcal{K}$ denotes hom w/ coefs in the spectrum bundle]

$\cdot\ |B_\tau \Omega G| = (G; E_\tau)$

[Identify E^2 term & E^∞ term]

Finally $E^2_{p,q} = \mathrm{Tor}^{k.\Omega G}_{p,q}(k.*, (k.*)_\tau) \Longrightarrow k^\tau_{p+q} G$

Example: G_2

① k-Homology of Loop Group

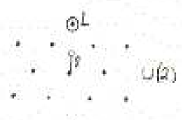
$k.\Omega G_2 = \mathbb{Z}[a,b,x]/(a(a+3)-2b)$

[HEMSS, AHSS, solve mult ext $H G_2 \to H.\Omega G_2 \to k.\Omega G_2$]

② Representatives for k-Hom Classes

$V_3 = G_2/U(2) \xrightarrow[(Bott)]{i} \Omega G_2$

$\cup \qquad g \longmapsto g \ell g^{-1} \ell^{-1}$



$V_2 = SU(3)/U(2)$

$\cup$

$V_1 = SU(2)/U(1)$

$a = i_*[V_1] - 1$

$b = i_*[V_2] - 1$

$x = i_*[V_3] - 1$

[Prove a, b, x gen via AHSS of $G_2/U(2)$ & ΩG_2]

[Alternate generators: P-Dual to classes on V_3]

(Computing)

(Example: G_2)

③ The Twisting Map $K.\Omega G \to (K.*)_\tau$

Given $V = G/H \xrightarrow{i} \Omega G$

Compute $\langle L^k, i_*[V] \rangle = \langle i^* L^k, [V] \rangle$

$$= \langle ch(i^*L^k) \cup Td(V), [V]_H \rangle$$

$$= \sum (-1)^i \operatorname{rnk} H^i(V; i^*L^k) \qquad \text{(HRR)}$$

$$= \dim \mathrm{Irrep}_G(i^*L^k) \qquad (\text{Holom. induc.}, i^*L^k \text{ dom.})$$

Conclude for G_2

$$a \mapsto (k+1) - 1 \qquad =: c_1$$

$$b \mapsto \binom{k+2}{2} - 1 \qquad =: c_2$$

$$c \mapsto \frac{(k+1)(k+2)(2k+3)(3k+4)(3k+5)}{120} - 1 \qquad =: c_3$$

④ The Tor Calculation

$$\mathrm{Tor}^{K.\Omega G_2}(\mathbb{Z}, \mathbb{Z}_\tau) = H(\mathbb{Z}\langle T_1, T_2, T_3 \rangle \{S_1\}; dT_i = c_i, dS_1 = (c_1+3)T_1 - 2T_2)$$

⌜Judiciously chosen⌝ ⌞Tate res.⌟

$$= \mathbb{Z}/\gcd(c_1, c_2, c_3) \langle y \rangle$$

iterated filtration spectral sequences

Example: Spin$(2n+1)$

$$K.\Omega \mathrm{Spin}(2n+1) = \frac{\mathbb{Z}[\sigma_1, \dots, \sigma_{n-1}, 2\sigma_n, 2\sigma_{n+1} + \sigma_n, \dots, 2\sigma_{2n-1} + \sigma_{2n-2}]}{(\rho_1, \dots, \rho_{n-1})} \qquad \text{(Clark)}$$

$$\rho_k = \sigma_k^2 + \sum_{i=0}^{k-1} (-1)^{k-i} \sigma_i \sum_{j=k}^{2k-i-1} \binom{k-i-1}{j-k} (2\sigma_{j+1} + \sigma_j)$$

Tor calc!

⌜Need good Tate & IFSS⌟

Omitted

(Computing)

Theorem: For G cmpt, conn, s.c., simple, rank n,

$$K_\bullet^{\tau(k)} G \cong \Lambda[x_1, \dots, x_{n-1}] \otimes \mathbb{Z}/c(G,k)$$

⌜Can give $c(G,k)$ explicitly⌟

Pf: • E^2 $\begin{array}{|cccc} * & * & * & \cdots \\ \circ & \circ & \circ & \\ * & * & * & \cdots \\ \circ & \circ & \circ & \end{array}$ gen in Tor_1 & Tor_2 [as algebra] $\Rightarrow$ no diff.

• no diff $\rightsquigarrow$ resolves E_7 & E_8 Tor calc $\Rightarrow \mathrm{Tor} = \Lambda \otimes \mathbb{Z}/c$. □

⌜Prove no additive, no mult extensions⌟

Twisted Spin^c Bordism

Def: $MSpin_i^{c,\tau}(X) = \pi_i\left(\left(P(\tau) \times_{K(\mathbb{Z},2)} MSpin^c\right)/X\right)$

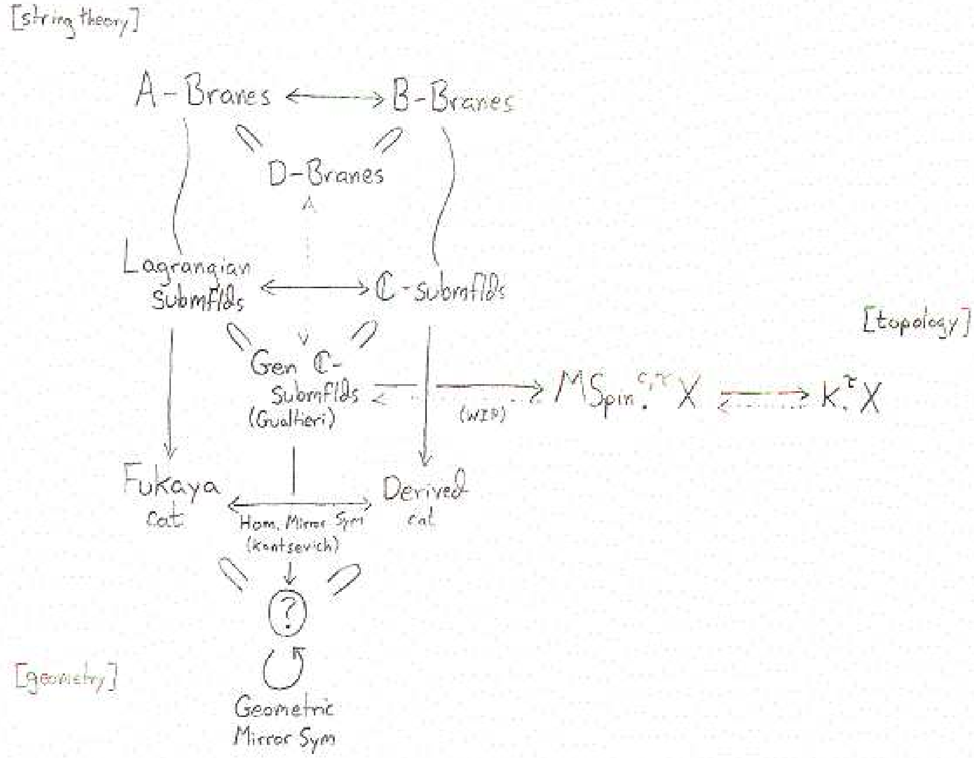
Index: $MSpin_\bullet^{c,\tau}(X) \to K_\bullet^\tau(X)$

Model: $MSpin_\bullet^{c,\tau}(X) = \left\{ \begin{array}{l} \text{bordism of ortd mflds } M \\ \text{w/ } i: M \to X,\ c \in C^2(M) \\ \text{s.t. } \delta c = \beta w_2(\nu(M)) - i^*\tau \end{array} \right.$

Note: Struc of TRSSS $\rightsquigarrow$ [suggests some] tw Spin^c mfld reps for tw K classes

(Eg: tw Spin^c 5-mfld for ext gen of $K_\bullet^\tau SU(3)$)

Mirror Symmetry



⌜Twisting enters in general picture, also has been recognized on Lagrangian side⌟

⌜History of realization of GCS ↔ tw $Spin^c$ connection⌟

⌜Said D-branes $\longrightarrow MSpin^{c,\tau}$, mathematically GCS $\longrightarrow MSpin^{c,\tau}$⌟

⌜Of course not all TKT classes will have GCS reps⌟

⌜Derived cat of coh. sheaves.
Derived Fukaya cat. = more than Lagrangians