Nitu Kitchloo

Tits Buildings + Applications.

The idea of studying inf. dim top groups via their Tits buildings goes back to Quillen. This idea was later developed by S. Mitchell but not much recent work seems to use these objects.

There are 2 parts to this talk:

1. Look at examples of (discrete) Tits buildings of coxeter gps.

2. Describe Top. Tits buildings of Affine + Hyperbolic Kac-Moody gps and extend the framework in which recent results of Freed-Hopkins-Teleman hold.

Coxeter Groups (crystallographic)

Def: An $n \times n$ integral matrix $A = (a_{ij})$ is called a generalized Cartan matrix (GCM) if

- $a_{ii} = 2$
- $a_{ij} \leq 0 \quad i \neq j$
- $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$

From this data one defines a Coxeter group (The "Weyl group")

$$W(A) = \langle r_i, i=1,2..n \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle$$

where

m_{ij}	2	3	4	6	∞
$a_{ij}a_{ji}$	0	1	2	3	≥ 4

Given $I \subseteq \{1, 2, \ldots n\}$ define $W_I \subseteq W(A)$ to be the sub. gp generated by $\{r_i, i \in I\}$

Remarks:

1. $W(A)$ may be infinite even if $m_{ij} < \infty$ for all i, j

eg: $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$; $W(A) = \Sigma_3 \ltimes \mathbb{Z}^2$

For the rest of the talk assume that

- $|W(A)| = \infty$, A indecomposable
- $|W_I| < \infty$ if $I \subsetneq \{1, 2, \ldots n\}$

Such GCMs fall into two types:

1. <u>Affine type</u> : Ones where A is positive semi-definate. The $W(A)$s are the Affine Weyl groups associated to Loop groups

2. <u>Hyperbolic Type</u>: Here A has signature $(n-1, 1)$. For $n = 2$ there is an infinite family. For $n > 2$ there are finitely many (in total) (cf: Humphreys)

Let $\mathcal{C}$ = all proper subsets of $\{1,2,\ldots n\}$ (including ϕ)

<u>Def</u>: The Tits building for $W(A)$ is

$$\underline{X} = \operatorname*{hocolim}_{I \in \mathcal{C}} W/W_I$$

$$= (W \times \Delta_{n-1})/\sim \quad ; \ \Delta_{n-1} = n\text{-}1 \text{ simplex with walls indexed by } \mathcal{C}$$

$(u, x) \sim (v, y)$

iff $x = y \in \mathring{\Delta}_I$; $u \equiv v W_I$

Examples:

<u>Rank 2</u>:

Affine: $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

Hyperbolic:

$\begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, $ab > 4$

<u>Rank 3</u>:

Affine: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$

Hyp: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$

Geometric Realization

$W(A)$ may be seen as a group of reflections in a vectorspace t^*, where r_i acts as a reflection about a wall $h_i = 0$, $h_i \in t$

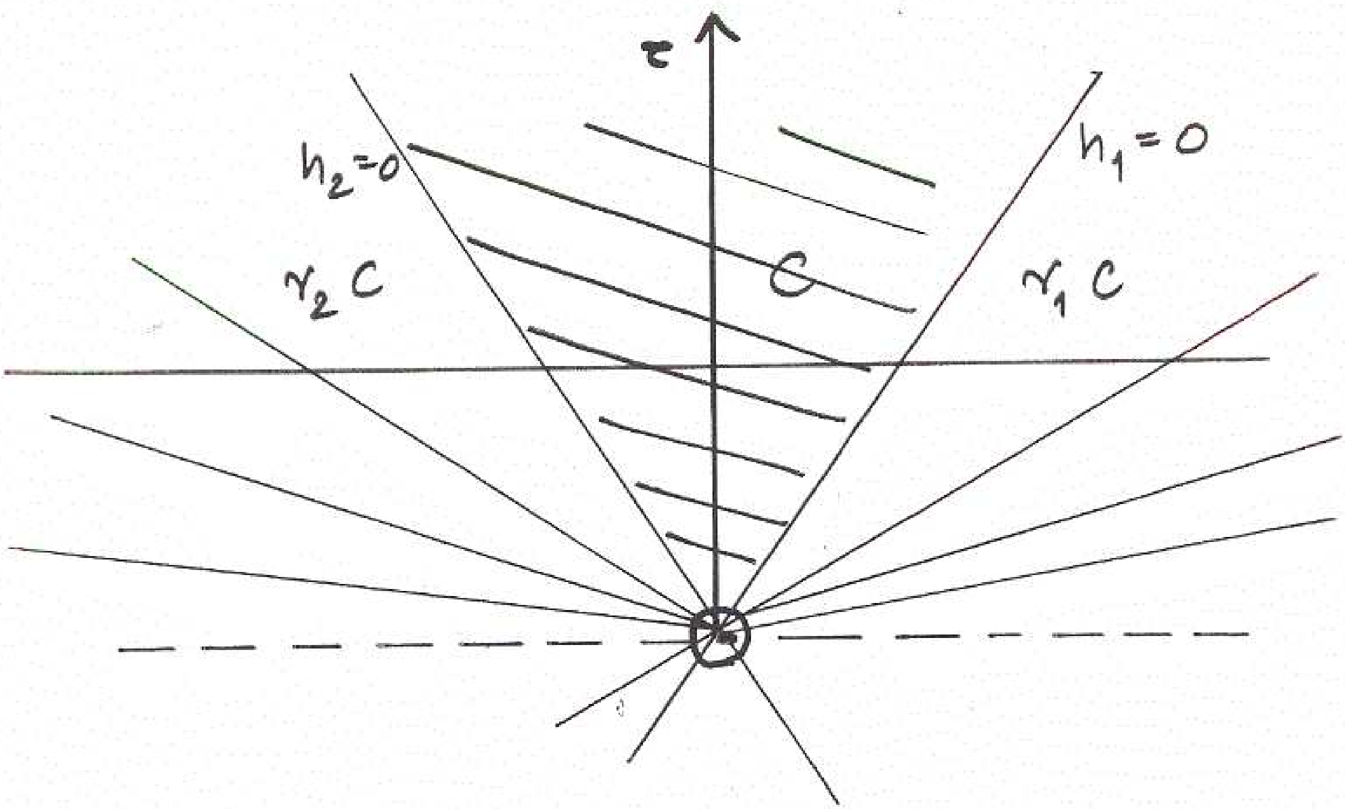
Def: The fundamental Weyl chamber in t^* is

$$C = \{ \alpha \in t^* \mid h_i(\alpha) \geq 0 \quad \forall i \}$$

The Tits cone Y, is the W orbit of C. Moreover, C is the fundamental domain of the W action on Y.
The Tits building is a deformable retract of the interior of Y.

Examples:

1. $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, $W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1 \rangle$



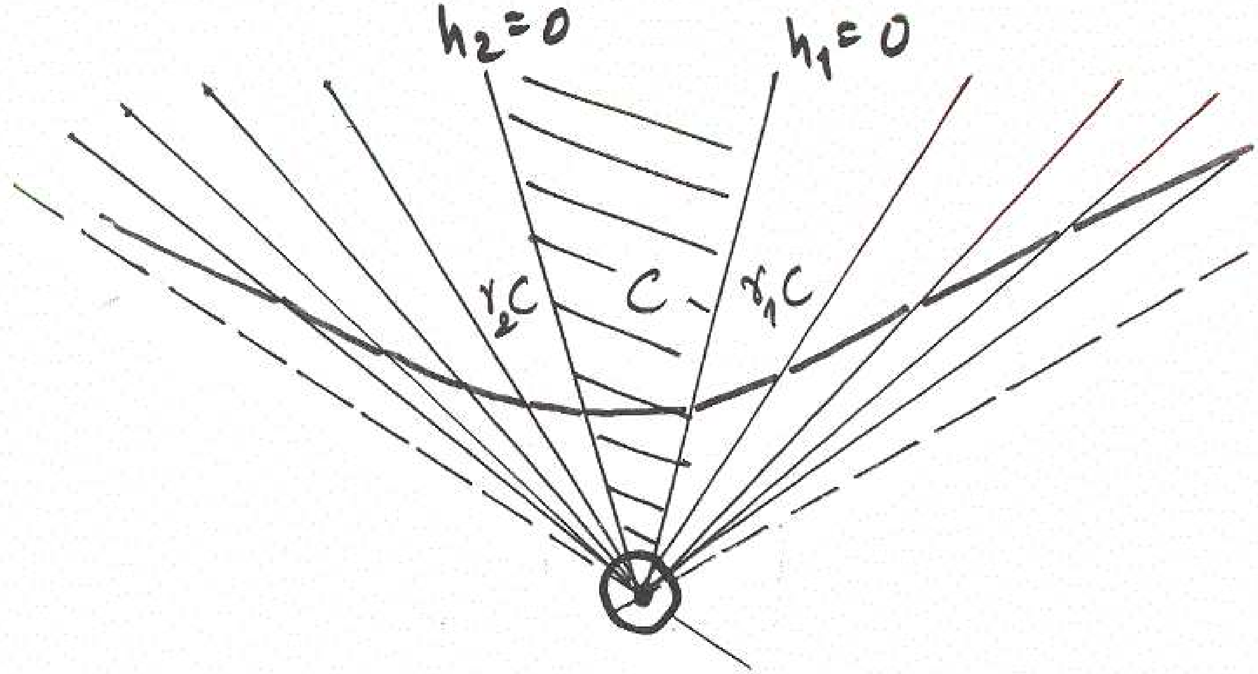
The Tits cone Y is the upper half plane + origin

$$Y = \{\tau > 0\} \cup \{0\}$$ (τ is called the 'level')

The Tits building is the affine line in red.

2. $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$ $ab > 4$

$W = \langle r_1, r_2 \mid r_1^2 = r_2^2 = 1 \rangle$



The Tits cone Y is an open cone with one boundary point (the origin)

Note: The interior of Y <u>cannot</u> be expressed as a linear eq^n using a level.

Let T be a torus with Lie alg $\mathfrak{t}$, then the GCM, A yields an extension

$$1 \longrightarrow T \longrightarrow N(T) \longrightarrow W \longrightarrow 1$$

So we may think of $N(T)$ acting (properly) on $\underline{\mathfrak{X}}$. Infact, $\underline{\mathfrak{X}}$ is the classifying space of proper actions of $N(T)$

<u>Note</u>: A G-space $\underline{E}G$ is said to be the classifying space of proper actions if

$$\underline{E}G^H = \begin{cases} \phi & H \subseteq G \text{ non compact} \\ \simeq * & H \subseteq G \text{ compact} \end{cases}$$

This property uniquely defines $\underline{E}G$ upto G-homotopy.

Our goal is to calculate $N(T)$-equivariant K-theory of $\underline{X}$.

Since $N(T)$ is locally compact acting properly on $\underline{X}$, this K-theory $\mathbb{K}^*_{N(T)}(\underline{X})$ can be defined.

Note that even though T acts trivially on $\underline{X}$, it is <u>not</u> true that

$$\mathbb{K}^*_N(\underline{X}) = \mathbb{K}^*_W(\underline{X}) \otimes R(T)$$

However, consider the <u>W-space</u> $R(T) \times \underline{X}$, then one can show that

$$\mathbb{K}^*_N(\underline{X}) = \mathbb{K}^*_W(R(T) \times \underline{X})^{c.s}$$

(RHS has classes cptly supported on the quotient)

$R(T)$ has two natural W-inv subsets

$$R(T)^{reg}_+ \subseteq R(T)_+ \subseteq R(T) \quad \text{with}$$

$$R(T)_+ = \{\alpha \in R(T) \mid \alpha \in \mathrm{Int}(Y)\}$$

$$R(T)^{reg}_+ = \{\alpha \in R(T)_+ \mid \alpha \notin \text{walls of } Y\}$$

<u>Def</u>: Define summands of $\mathbb{K}^*_{N(T)}(\underline{X})$ by:

$$\mathbb{K}^*_{N(T)}(\underline{X})_+ = \mathbb{K}^*_W(R(T)_+ \times \underline{X})$$

$$\mathbb{K}^*_{N(T)}(\underline{X})^{reg}_+ = \mathbb{K}^*_W(R(T)^{reg}_+ \times \underline{X})$$

<u>Remark</u>: $\mathrm{Int}(Y)$ can be characterized as $\mathrm{Int}(Y) = \{\alpha \in Y \mid |W_\alpha| < \infty\}$

Note:

1. W acts freely on $R(T)_+^{reg}$
2. The compactification of $\mathcal{S}$ is known to be an $(n-1)$ sphere. So we notice that:

$$K_W^{*+n-1}(R(T)_+^{reg} \times \mathcal{S}) = K^{*+n-1}(R(T)_+^{reg}/W \times \mathcal{S})$$
$$= K^*(R(T)_+^{reg}/W)$$

$$\text{so } \mathbb{K}_N^{*+n-1}(\mathcal{S})_+^{reg} = \mathbb{Z} \otimes_{\mathbb{Z}[W]} R(T)_+^{reg}$$

One can also see this using the Bousfield-Kan SS

$$\varprojlim^i \mathbb{K}_{W_I}^*(R(T)_+^{reg}) \Rightarrow \mathbb{K}_W^*(R(T)_+^{reg} \times \mathcal{S})$$

$$\text{These } \varprojlim^i = \begin{cases} \mathbb{Z} \otimes_{\mathbb{Z}[W]} R(T)_+^{reg} & \text{if } i = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Remark:
the following stronger statement is most likely true:

$$\mathbb{K}^{*+n-1}_{N(T)}(\tilde{X})_+ \simeq \mathbb{K}^{*}_{W}(R(T)_+)$$

<u>Reason</u>: $\tilde{X}$ can be identified with hyperbolic space $\mathbb{H}^{n-1}$, with W acting via isometries. It seems likely that $\tilde{X}$ supports a W-inv K-theory orientation.
The above isom is then given by integrating along the projection

$$R(T)_+ \times \tilde{X} \longrightarrow R(T)_+$$

(recall that our classes were compactly supported)

Part II: Work of FHT.

For a (not nece loc cpct) group G acting properly on a space $\mathbb{A}$, FHT define equivariant K-th:

$\mathbb{K}^*_G(\mathbb{A})$.

Brief idea of const: FHT construct a G-equiv universal Hilbert bdle over $\mathbb{A}$ (not trivial unless G cpct). Using equivariant fiberwise Fred. ops on this bdle they define a G-spectrum parametrized over $\mathbb{A}$. Now

$\mathbb{K}^*_G(\mathbb{A}) = \pi_*$ (space of equiv sections)

A natural way to obtain classes in $\mathbb{K}^*_G(\mathbb{A})$ is to construct an equiv Dirac operator parametrized by $\mathbb{A}$.

Kac-Moody gps (Affine + Hyp)

Given a GCM A, there is a top gp $G(A)$ (poss. inf dim). T may be identified with a max torus in $G(A)$ and $N(T)$ with its normalizer. Moreover, if $I \in \mathcal{C}$, then there is a natural max rank (compact Lie) subgroup H_I of G, and the Weyl group of H_I is W_I.

<u>Def</u> The Top. Tits building of $G(A)$ is

$$\mathbb{A} = \operatorname*{hocolim}_{I \in \mathcal{C}} G/H_I = \underbrace{\left(\cancel{G}/T \times \Delta_{n-1}\right)}_{\sim}$$

<u>Remarks:</u>

1. $\mathbb{A}$ is a model for <u>EG</u>.

2. $\mathbb{A}$ may be identified with the G-orbit of $\mathbb{X}$ (or even c) in the (dual of) Lie algebra $\mathfrak{g}$ of G. Moreover $\mathbb{A}$ and $\mathbb{X}$ are related by $\mathbb{X} = \mathbb{A}^T$ (T-fixed pts)

3. if the GCM A is affine corresp. to the loop gp $\tilde{L}K$, then the space $\mathbb{A}$ is the affine space of connections on $S^1 \times K$.

4. We may consider the most general GCM's A, and the corresponding $G(A)$ space $\mathbb{A}$. But unless A is affine or hyp., the stack $(\mathbb{A} : G)$ will not be smooth.

We want to study $\mathbb{K}^*_G(\mathbb{A})$:

Proceed as FHT: consider

$$f : G \times_{N(T)} \mathfrak{X} \longrightarrow \mathbb{A}$$

(recall $\mathfrak{X} = \mathbb{A}^T$)

We have maps:

$$f^* : \mathbb{K}^*_G(\mathbb{A}) \longrightarrow \mathbb{K}^*_G(G \times_N \mathfrak{X}) = \mathbb{K}^*_N(\mathfrak{X})$$

$$f_* : \mathbb{K}^*_N(\mathfrak{X}) \longrightarrow \mathbb{K}^*_G(\mathbb{A})$$

<u>Claim</u>: The foll is a retraction

$$\mathbb{K}^*_G(\mathbb{A}) \xrightarrow{f^*} \mathbb{K}^*_N(\mathfrak{X}) \xrightarrow{f_*} \mathbb{K}^*_G(\mathbb{A})$$

Pf: Locally, this is

$$K^*_{H_I}(\mathfrak{h}^*_I) \longrightarrow K^*_{N_I}(t^*) \longrightarrow K^*_{H_I}(\mathfrak{h}^*_I)$$

which a retraction (cf FHT)

<u>Def</u>: $\mathbb{K}^*_G(\mathbb{A})_+ = \mathbb{K}^*_G(\mathbb{A}) \cap \mathbb{K}^*_N(\mathfrak{X})_+$.

<u>Thm</u>: The image of $\mathbb{K}^*_G(\mathbb{A})_+$ in $\mathbb{K}^*_N(\mathfrak{X})_+$ is exactly $K^*_N(\mathfrak{X})^{reg}_+ \cong \mathbb{Z} \otimes_{\mathbb{Z}[W]} R(T)^{reg}_+$ $\therefore$

<u>Pf</u> As in FHT, idea is to show f_* kills non-reg classes. Recall that locally f_* is pushforward along

$$\begin{array}{ccccc} K^*_{N_I}(t^*) & \longrightarrow & K^*_{H_I}(\mathfrak{h}^*_I) & \to & K^*_G(\mathbb{A}) \\ \wr\!\!\mid & \nearrow_{g_*} & & & \\ K_{N_I}(pt) & & & & \end{array}$$

where $g: G \times_{N_I} pt \longrightarrow G \times_{H_I} \mathfrak{h}^*_I$

Now consider a non regular element in $\mathbb{K}^*_{N_I}(pt) = K^*_{W_I}(R(T)^{nr}_+)$. All such non regular classes are induced up from walls i.e let $\mathcal{U} \subset \mathrm{Int}(Y)$ be some wall, then $K^*_{W_I}(R(T)^{nr}_+)$ is spanned by the images of $K^*_{W_I \cap W_{\mathcal{U}}}(R(T)_+ \cap \mathcal{U})$, as $\mathcal{U}$ ranges over the walls in $\mathrm{Int}(Y)$.

The image of any class in $K^*_{W_I \cap W_{\mathcal{U}}}(R(T)_+ \cap \mathcal{U})$ all the way in $\mathbb{K}^*_G(\mathbb{A})_+$ factors through $K^*_{H_{\mathcal{U}}}(pt)$ where $H_{\mathcal{U}}$ is the centralizer of the wall $\mathcal{U}$. Since the image of $K^*_{W_I \cap W_{\mathcal{U}}}(R(T)_+ \cap \mathcal{U})$ in $K^*_{H_{\mathcal{U}}}(pt)$ is trivial, the image of all non-reg classes is trivial in $K^*_G(\mathbb{A})_+$.

Corollary:

if G is an affine or Hyperbolic Kac-Moody group, then there is an (abstract) additive isom:

$$K_G^{*+n-1}(\mathbb{A})_{\tau} \simeq \mathrm{Rep}_G^{\mathrm{Int}} \longleftrightarrow \mathbb{Z} \underset{\mathbb{Z}[W]}{\otimes} R(T)_+^{reg}$$

Remark: In the affine case FHT establish this isom geometrically as foll: They endow $\mathbb{A}$ with a Clifford module S that supports a family of dirac operators $\mathcal{D}$ parametrized over $\mathbb{A}$. Given $V \in \mathrm{Rep}_G^{\mathrm{Int}}$, they establish the isom by twisting $\mathcal{D}$ by V i.e via an infinite dim Thom isom. In the Hyperbolic case S may be constructed, but $\mathcal{D}$ is not obvious (to me.)