The intersection pairing for PL chains, with applications to string topology.

Let M be a compact, oriented PL manifold and let C_*M be its PL chain complex.

The intersection pairing is defined on a subcomplex of $C_*M \otimes C_*M$.

Definition. Let A_* be a chain complex and let B_* be a subcomplex. B_* is *full* if the map $B_* \to A_*$ is a quasi-isomorphism.

Theorem 1. The domain of the intersection pairing is a full subcomplex of $C_*M \otimes C_*M$; similarly for the iterated intersection pairing.

Consequences of Theorem 1.

Notations. Let d be the dimension of M.

If A_* is a graded abelian group, let $\Sigma^{-d} A_*$ denote the graded abelian group which in degree i is A_{d+i}.

Similarly for chain complexes.

Theorem 2. $\Sigma^{-d} C_* M$ is canonically quasi-isomorphic to an E_∞ chain algebra.

Theorem 3. (i) The Eilenberg-Moore spectral sequence whose abutment is $\Sigma^{-d}H_*(LM)$ is a spectral sequence of Batalin-Vilkovisky algebras.

(ii) The operations at E^∞ agree with the Chas-Sullivan operations.

Notations. Let $\mathcal{F}$ denote the framed little 2-disks operad.

Let S_* denote the singular chain functor.

Theorem 4. The Chas-Sullivan operations on $\Sigma^{-d}H_*LM$ are induced by a chain-level action of an operad quasi-isomorphic to $S_*\mathcal{F}$ on a chain complex quasi-isomorphic to $\Sigma^{-d}S_*LM$.

The action in Theorem 4 is given by explicit formulas.

Lefschetz's Definition of the chain-level intersection pairing: TAMS 1926.

Let $C = \sum m_i \sigma_i$ and $D = \sum n_i \tau_i$ be PL chains in M.

The intersection pairing $C \cdot D$ is only defined when C and D are in general position.

Definition. C and D are in *general position* if

$$\dim(C \cap D) \leq \dim(C) + \dim(D) - d.$$

When C and D are in general position, Lefschetz defines

$$C \cdot D = \sum \pm m_i n_j \, \sigma_i \cap \tau_j.$$

The sign depends on the orientations. The terms with dimension $< \dim(C) + \dim(D) - d$ are counted as zero.

The boundary formula.

The expected relation between $\cdot$ and ∂ is

$$\partial(C \cdot D) = \partial C \cdot D \pm C \cdot \partial D.$$

To prove this, Lefschetz restricts the domain by assuming that *all* pairs (σ_i, τ_j), $(\partial\sigma_i, \tau_j)$ and $(\sigma_i, \partial\tau_j)$ on the two sides of the equation are in general position.

This condition allows him to work with one pair of simplices at a time.

Note: If C and D are chains on the *same triangulation*, then (with minor exceptions) Lefschetz's condition forces $C \cdot D = 0$.

By subdividing one can always find a triangulation K for which both C and D are chains on K.

Thus the domain of Lefschetz's intersection pairing is not compatible with subdivision: if $C \cdot D \neq 0$ then (with minor exceptions) it will always be possible to subdivide so that $C \cdot D$ is undefined.

The PL chain complex of M.

The PL chain complex C_*M was first defined by Goresky and MacPherson in 1980.

Given a triangulation K of M, let s_*K denote the simplicial chain complex of K.

The triangulations of M form a partially ordered set with respect to subdivision.

If L is a subdivision of K there is a canonical map $s_*K \to s_*L$.

Definition. C_*M is $\operatorname*{colim}_K s_*K$.

Lefschetz's definition of $C \cdot D$ does not give a partially defined product on C_*M.

The Goresky-MacPherson definition of $C \cdot D$ (Topology 1980)

It is well known that the intersection pairing in homology is a combination of the cup product and Poincaré duality.

Goresky and MacPherson observe that elements of C_*M can be interpreted as relative homology classes (by analogy with the fact that cellular chains of a CW complex are relative homology classes).

They they construct the chain-level intersection pairing by combining the relative cup product and relative Poincaré duality.

The Goresky-MacPherson intersection pairing is defined when all three of the pairs (C, D), $(\partial C, D)$ and $(C, \partial D)$ are in general position.

This condition is compatible with subdivision, so their definition gives an operation whose domain is a subset of $C_*M \times C_*M$.

One could attempt to extend to a partially defined operation on $C_*M \otimes C_*M$ by taking

$$\sum C_i \otimes D_i$$

to

$$\sum C_i \cdot D_i,$$

but it would be difficult to prove this is well-defined and to say exactly what its domain is.

My definition of the chain-level intersection pairing.

The first step is to construct chain-level "Umkehr" maps.

Let

$$f : N \to P$$

be a 1-1 PL map between compact, oriented PL manifolds.

Let $C_*^f P$ be the subcomplex of $C_* P$ consisting of chains C for which both C and ∂C are in general position with respect to $f(N)$.

There is a chain map

$$f_! : C_*^f P \to C_* N$$

which induces the usual Umkehr map on homology.

Next let

$$\varepsilon : C_*M \otimes C_*M \to C_*(M \times M)$$

be the exterior product.

Definition. $G_2 \subset C_*M \otimes C_*M$ is defined to be

$$\varepsilon^{-1} C_*^{\Delta}(M \times M),$$

where

$$\Delta : M \to M \times M$$

is the diagonal.

Definition. The chain-level intersection pairing is the composite

$$G_2 \xrightarrow{\varepsilon} C_*^{\Delta}(M \times M) \xrightarrow{\Delta_!} C_*M.$$

Theorem 1 says that G_2 is a full subcomplex of $C_*M \otimes C_*M$.

The proof of Theorem 1 uses the same basic ingredients as Hudson's proof of the standard general position theorem for PL subspaces.

Background for Theorem 2: partially defined commutative algebras.

Motivation: special Γ-spaces.

Let Γ be the category of based finite sets.

Let $[n] = \{0, \ldots, n\}$, with basepoint 0.

If $1 \leq i \leq n$, let

$$\omega_i : [n] \to [1]$$

take i to 1 and everything else to 0.

Definition. (Segal) Let X be a functor from finite based sets to spaces. For each n let

$$\xi_n : X([n]) \to X([1])^n$$

be the map whose projection on the i-th factor is $X(\omega_i)$. X is a *special* Γ-*space* if each ξ_n is a weak equivalence.

Note: Special Γ-spaces can be "rectified" to infinite loop spaces; that is, there is a functor $\mathcal{R}$ from special Γ-spaces to infinite loop spaces such that $\mathcal{R}(X)$ is weakly equivalent to $X(\{0,1\})$.

In order to give an analogous definition for chain complexes, the existence of the ξ_n must be assumed as part of the structure.

Let Φ be the category of finite sets (including the empty set) and let Ch be the category of chain complexes.

Definition. A Leinster partial commutative DGA is a functor A from Φ to Ch together with quasi-isomorphisms

$$\xi_{S,T} : A(S \coprod T) \to A(S) \otimes A(T)$$

for each S, T and

$$\xi_{\emptyset} : A(\emptyset) \to \mathbb{Z}$$

(where $\mathbb{Z}$ is considered as a chain complex concentrated in degree 0) satisfying certain properties.

Leinster calls these "homotopy algebras."

Proposition. There is a functor $\mathcal{R}$ from Leinster partial commutative DGA's to E_∞ chain algebras such that $\mathcal{R}(A)$ is quasi-isomorphic to $A(\{1\})$.

Proof of Theorem 2.

Fix M.

The definition of G_2 given earlier extends to give a functor G from Φ to Ch.

$G(S)$ is a subcomplex of $(C_*M)^{\otimes|S|}$ and the inclusion is a quasi-isomorphism.

The composite

$$\begin{aligned} G(S \coprod T) &\to (C_*M)^{\otimes|S \coprod T|} \\ &= (C_*M)^{\otimes|S|} \otimes (C_*M)^{\otimes|T|} \end{aligned}$$

factors through $G(S)\otimes G(T)$; this gives the map $\xi_{S,T}$.

Background for Theorem 3: the cocyclic cobar construction.

Starting from M we can create a cosimplicial (in fact a cocyclic) object.

$$\begin{array}{c} \uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \\ M \times M \times M \\ \uparrow\downarrow\uparrow\downarrow\uparrow \\ M \times M \\ \uparrow\downarrow\uparrow \\ M \end{array}$$

The coface maps are diagonal maps (the last coface is a diagonal composed with a cyclic permutation).

The codegeneracy maps are projections.

Applying the PL chain functor C_* to the previous slide gives:

$$
\begin{gathered}
\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \\
C_*(M \times M \times M) \\
\uparrow\downarrow\uparrow\downarrow\uparrow \\
C_*(M \times M) \\
\uparrow\downarrow\uparrow \\
C_*(M)
\end{gathered}
$$

This cocyclic chain complex will be called the cocyclic cobar construction of M and denoted $\mathcal{L}^\bullet_* M$.

Using $\mathcal{L}^{\bullet}_{*}M$ we can make a double (actually a "mixed") chain complex: the horizontal differential is the differential of C_* and the vertical differential is the alternating sum of the coface maps.

The associated spectral sequence is the Eilenberg-Moore spectral sequence.

The loop product in the Eilenberg-Moore spectral sequence.

For each $p, q \geq 0$, let

$$\Delta_{p,q} : M^{p+q+1} \to M^{p+1} \times M^{q+1}$$

be defined by

$$\Delta_{p,q}(x_0, \dots, x_{p+q}) \\ = (x_0, \dots, x_p), (x_0, x_{p+1}, \dots, x_{p+q}).$$

The chain-level Umkehr map $(\Delta_{p,q})_!$ gives a chain map from a full subcomplex of

$$C_*(M^{p+1}) \otimes C_*(M^{q+1})$$

to

$$C_*(M^{p+q+1}).$$

The collection of these these maps induces a product on the Eilenberg-Moore spectral sequence which is compatible with the Chas-Sullivan loop product.

A similar construction gives a Lie bracket in the Eilenberg-Moore spectral sequence which is compatible with the Chas-Sullivan loop bracket.

Background for Theorem 4: cocyclic chain complexes with cup product.

Definition. A *cocyclic chain complex with cup product* is a cocyclic chain complex $X_*^\bullet$ together with maps

$$X^p \otimes X^q \to X^{p+q}$$

satisfying certain identities.

Kaufmann and Tradler (independently) have constructed a chain model for the framed little 2-disks operad. Their chain model acts on the total complex of any cocyclic chain complex with cup product.

The action is given by explicit formulas.

Remark: the Kaufmann-Tradler model is the analog for framed little 2-disks of the McClure-Smith model for the unframed little 2-disks (Kaufmann rediscovered the McClure-Smith model independently).

Proof of Theorem 4.

The partial product on the cocyclic cobar construction $\mathcal{L}_*^\bullet M$ induces a "Leinster partial action" of the Kaufmann-Tradler operad on the totalization of $\mathcal{L}_*^\bullet M$.

This partial action can be rectified to a full action on a quasi-isomorphic chain complex.

When M is simply connected, the totalization of $\mathcal{L}_*^\bullet M$ is quasi-isomorphic to S_*LM, which completes the proof in this case.

When M is not simply connected, there is a generalization of $\mathcal{L}^{\bullet}_{*}M$ (due to Mandell) which is constructed from the $\pi_1(M)$ action on the universal cover of M. The totalization of this object is quasi-isomorphic to S_*LM for all M, and the theory described above generalizes to this setting.