Dirac Operators on Loop Spaces

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Abstract

In this paper we describe how to construct a Dirac operator on the loop space of a string manifold. The key step is to construct an inner product on the cotangent bundle of the loop space. There is then a Hilbert bundle which is the fibrewise completion of the cotangent bundle. This bundle is used to construct the spin bundle so that the Clifford multiplication map extends to the domain of a connection allowing one to define the Dirac operator.

1 Introduction

In this paper we describe a way to construct a Dirac operator on the loop space of a string manifold. The key to this construction is to shift the focus from the tangent bundle of the loop space to the cotangent bundle.

To explain the relevance of this remark and to describe a road map for our construction, we review the finite dimensional situation. Let M be a closed, oriented, Riemannian, spin manifold of even dimension d. The structure group of M thus reduces to SO_d and lifts to Spin_d. A *spin structure* on M is a choice of such lift. Let $Q \to M$ be a spin structure on M; so Q is a principal Spin_d-bundle. The Lie algebras of SO_d and of Spin_d are the same so any SO_d-connection on M canonically lifts to a Spin_d-connection. In particular the Levi-Civita connection on M lifts to a connection on Q called the *spin connection*.

The group Spin_d acts on $\mathbb{R}^d$ via the covering map $\mathrm{Spin}_d \to SO_d$ and the tangent bundle of M can be described as the bundle $TM = Q \times_{\mathrm{Spin}_d} \mathbb{R}^d$. It also has a unitary $2^{d/2}$ representation Δ called the *spin representation* which decomposes as the sum of two irreducible representations, $\Delta = \Delta^+ \oplus \Delta^-$. There is a Spin_d-equivariant bilinear map $\pi : \mathbb{R}^d \times \Delta \to \Delta$ with the property that it interchanges the factors Δ^+ and Δ^-. This extends to a linear map $\pi : \mathbb{C}^d \otimes \Delta^+ \to \Delta^-$. This map is called *Clifford multiplication*.

Transferring this to M, we obtain bundles $S^\pm := Q \times_{\mathrm{Spin}_d} \Delta^\pm$ together with a fibrewise linear map $\pi : T_{\mathbb{C}}M \otimes S^+ \to S^-$ which extends to a linear map $\pi : \Gamma(T_{\mathbb{C}}M \otimes S^+) \to \Gamma(S^-)$. The spin connection on Q defines a covariant differential operator ∇ on S^+. The Dirac operator, $\partial\!\!\!/$, is defined as the composition of ∇ with π.

In defining $\partial\!\!\!/$ we have used two identities from finite dimensional linear algebra. The range of ∇ is $\Gamma(\mathcal{L}(T_{\mathbb{C}}M, S^+))$, the space of sections of the bundle of fibrewise (complex) linear maps from $T_{\mathbb{C}}M$ to S^+; the domain of π is $\Gamma(T_{\mathbb{C}}M \otimes S^+)$. To compose ∇ with π we need to equate $\mathcal{L}(T_{\mathbb{C}}M, S^+)$ with $T_{\mathbb{C}}M \otimes S^+$. To do this, we observe first that for finite dimensional vector spaces, $\mathcal{L}(V, W) \cong V^* \otimes W$; and second that the metric on M identifies TM with T^*M (and hence their complexifications). Thus $\partial\!\!\!/$ should be thought of as the map:

$$\partial\!\!\!/ : \Gamma(S^+) \xrightarrow{\nabla} \Gamma(\mathcal{L}(T_{\mathbb{C}}M, S^+)) \cong \Gamma(T^*_{\mathbb{C}}M \otimes S^+) \cong \Gamma(T_{\mathbb{C}}M \otimes S^+) \xrightarrow{\pi} \Gamma(S^-).$$

In infinite dimensions neither of the identities $\mathcal{L}(V, W) \cong V^* \otimes W$ nor $V \cong V^*$ is in general valid. If one tries to replicate the argument in the case of a loop space one finds that the range of the covariant differential operator, $\Gamma(\mathcal{L}(T_{\mathbb{C}}LM, S^+))$, is very much larger than the domain of the Clifford multiplication map, $\Gamma(T_{\mathbb{C}}LM \otimes S^+)$. Even though Clifford multiplication extends to a completion of the tensor product, there is still a difference.

A spin representation can be constructed for any vector space V with a continuous inner product, $\langle \cdot, \cdot \rangle$. It consists of a Hilbert space, $\mathbb{H}$, either finite or infinite dimensional depending on the dimension of V, and there is a continuous map $\pi : V \to \mathcal{L}(\mathbb{H})$. This can be considered as a separately continuous bilinear map, $V \times \mathbb{H} \to \mathbb{H}$, which extends to a linear map with domain the tensor product $\pi : V_{\mathbb{C}} \otimes \mathbb{H} \to \mathbb{H}$.

Therefore, suppose that X is a manifold such that the cotangent bundle can be completed to a (finite or infinite dimensional) Hilbert bundle. Applying the spin construction to the cotangent bundle rather than to the tangent bundle produces bundles S^+, S^- with a fibrewise linear map $T^*_{\mathbb{C}}X \otimes S^+ \to S^-$. In the finite dimensional case, $X = M$, the identification of TM and T^*M via the metric leads to an identification of the spin bundles and thus we have only altered matters cosmetically (or cosmetrically). In infinite dimensions, $X = LM$, the domain of the new Clifford multiplication map is significantly larger than the original. We shall show in proposition 2.2 that it extends to the projective completion of the tensor product, $T^*_{\mathbb{C}}LM \widetilde{\otimes} S^+$, which is naturally isomorphic to $\mathcal{L}(T_{\mathbb{C}}LM, S^+)$.

Thus by taking as our starting point the cotangent bundle with an inner product, we solve both issues and can construct a Dirac operator. It is the composition:

$$\partial\!\!\!/ : \Gamma(S^+) \xrightarrow{\nabla} \Gamma(\mathcal{L}(T_{\mathbb{C}}LM, S^+)) \xrightarrow{\pi} \Gamma(S^-).$$

It transpires that the completion of the cotangent bundle is naturally isomorphic to the completion of the tangent bundle and thus the spin bundle defined by the cotangent bundle is equivalent to that defined by the tangent bundle. This proves useful in that the spin bundle associated to the tangent bundle has a natural connection in terms of structure on the original manifold.

This paper is organised as follows: in section 2 we review the construction of the spin representation in infinite dimensions. In section 3 we derive the obstruction to a spin structure on a loop space and define a string structure. In section 4 we examine the finer structure of the tangent bundle of a loop space and construct the inner product on the cotangent bundle. In section 5 we give the details of the construction of the Dirac operator. Finally, in the appendix we examine the set of inner products on the space of distributions. These results are not directly used in the construction, but are a related interesting side issue.

2 The Spin Representation

In this section we shall review the essential details of the construction of the spin representation in infinite dimensions, also referred to as the Fock representation. This is gleaned mostly from [PR94] with the application to loop spaces coming from [PS86].

Let V be an infinite dimensional real vector space with a continuous inner product, $(\cdot,\cdot)$. Let J be a choice of unitary structure on V; that is, J is an orthogonal transformation on V such that $J^2 = -1$. Let V_J denote V with this complex structure and let $\langle \cdot, \cdot \rangle$ be the hermitian inner product on V_J defined by $\langle u, v\rangle = (u, v) + i\,(u, Jv)$.

Let $\mathbb{H}_J$ be the Hilbert space completion of $\Lambda^\bullet V_J$, the exterior power of V_J, with respect to the inner product:

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_l \rangle = \begin{cases} 0 & l \neq k \\ \det(\langle u_i, v_j \rangle) & l = k. \end{cases}$$

We use the notation $\mathcal{L}(\mathbb{H}_J)$ for $\mathcal{L}(\mathbb{H}_J, \mathbb{H}_J)$, the Banach space of (complex) continuous linear maps from $\mathbb{H}_J$ to itself. Define operators $c : V \to \mathcal{L}(\mathbb{H}_J)$ and $a : V \to \mathcal{L}(\mathbb{H}_J)$ by:

$$\begin{aligned} c(v) u_1 \wedge \cdots \wedge u_k &= v \wedge u_1 \wedge \cdots \wedge u_k \\ a(v) u_1 \wedge \cdots \wedge u_k &= \sum_{j=1}^{k} (-1)^{j-1} \langle u_i, v \rangle u_1 \wedge \cdots \wedge \widehat{u_i} \wedge \cdots \wedge u_k. \end{aligned}$$

Let $\pi : V \to \mathcal{L}(\mathbb{H}_J)$ be the operator $c + a$.

Proposition 2.1 *The operator c is complex linear and a is conjugate linear, regarding V as V_J, and they satisfy the* canonical anti-commutation relations*:*

$$\begin{aligned} \{c(u), a(v)\} &= \langle u, v \rangle \\ \{c(u), c(v)\} = \{a(u), a(v)\} &= 0 \end{aligned}$$

where for operators X, Y, $\{X, Y\} = XY + YX$.

Hence π is real linear and satisfies $\pi(v)^2 = (v, v)\, I$.

Proposition 2.2 *Suppose that V is a complete nuclear reflexive space. The map $\pi : V \to \mathcal{L}(\mathbb{H}_J)$ defines a continuous linear map $\pi : \mathcal{L}(V^*, \mathbb{H}_J) \to \mathbb{H}_J$. This map satisfies: $\pi(x)h = \pi(x \otimes h)$ where $x \otimes h : V^* \to \mathbb{H}_J$ is the map $f \to f(x)h$.*

Proof. Let H denote the Hilbert space completion of V with respect to the inner product topology defined by $(\cdot,\cdot)$. From [PR94, ch 2.4], we know that $\pi : V \to \mathcal{L}(\mathbb{H}_J)$ extends to an isometric inclusion $\pi : H \to \mathcal{L}(\mathbb{H}_J)$. The map $H \times \mathbb{H}_J \to \mathbb{H}_J$, $(x,\xi) \to \pi(x)\xi$, is therefore continuous. From [Sch71, ch III, §6], it extends to a continuous linear map with domain the projective tensor product $H \widetilde{\otimes} \mathbb{H}_J$.

The inclusion $V \to H$ induces a continuous linear map $V \widetilde{\otimes} \mathbb{H}_J \to H \widetilde{\otimes} \mathbb{H}_J$. From [Sch71, ch IV, §9.4], as V is a complete nuclear space then the space $V \widetilde{\otimes} \mathbb{H}_J$ is isomorphic to $\mathcal{L}_e(V_\tau^*, \mathbb{H}_J)$; where this denotes the space of linear maps from V^* to $\mathbb{H}_J$. The topology on V^* is the Mackay topology and the topology on the space of maps is that of uniform convergence on equicontinuous sets.

From [Sch71, ch IV, §5] we deduce that as V is reflexive, the Mackay topology on the dual agrees with the strong topology. Also as V is reflexive, it is barrelled and so equicontinuous sets in V^* are the same as bounded sets. Hence $\mathcal{L}_e(V_\tau^*, \mathbb{H}_J) = \mathcal{L}(V^*, \mathbb{H}_J)$. □

The *implementation question* is the following: let $O(V)$ be the orthogonal group of V. For which $g \in O(V)$ is there some $U_g \in U(\mathbb{H}_J)$ such that $\pi(gv) = U_g \pi(v) U_g^{-1}$? It is answered by:

Theorem 2.3 ([PR94, ch 3]) *For $g \in O(V)$ there is some $U_g \in U(\mathbb{H}_J)$ such that $\pi(gv) = U_g \pi(v) U_g^{-1}$ if and only if $[g, J]$ is a Hilbert-Schmidt operator. Moreover, if U_g and U'_g both implement g then $U_g = \lambda U'_g$ for some $\lambda \in S^1$.*

An operator $T : H_1 \to H_2$ between Hilbert spaces is said to be *Hilbert-Schmidt* if for some, and hence every, orthogonal basis $\{e_i\}$ of H_1 then $(\|Te_i\|)$ is square summable. The subgroup of $O(V)$ consisting of g such that $[g, J]$ is Hilbert-Schmidt is written $O_J(V)$ in [PR94].

The theory of spin representations is closely related to that of polarisations. There are various equivalent definitions of a polarisation, we choose the one that is closest to the theory of unitary structures. The theory of polarisations and the relationship with loop groups is the subject of [PS86]. The following definitions are equivalent to those from [PS86, ch 6] although we have used notation similar to that of [PR94] for better comparison with the theory of unitary structures.

Definition 2.4 *Let H be a complex Hilbert space. A* polarising operator *on H is an operator $J \in \mathcal{L}(H)$ such that $J^2 + I$ is trace class and $J \pm iI$ are not finite rank.*

A polarisation *on H is an equivalence class of polarising operators under the relation $J_1 \sim J_2$ if and only if $J_1 - J_2$ is Hilbert-Schmidt.*

Let $\mathcal{J}$ be a polarisation on H. The restricted general linear group *of H with respect to $\mathcal{J}$, $\mathrm{Gl}_{\mathcal{J}}(H)$, is defined as the subgroup of $\mathrm{Gl}(H)$ consisting of those A for which $[A, J]$ is Hilbert-Schmidt for one, and hence all, $J \in \mathcal{J}$.*

In [PS86], the notation used is $\mathrm{Gl}_{\mathrm{res}}(H)$. The notation $\mathrm{Gl}_{\mathcal{J}}(H)$ emphasises the dependence on the polarisation $\mathcal{J}$. The operator used in the above definition is slightly different from the operator J used in [PS86, ch 6]. To get from the one to the other, multiply by $-i$.

Clearly a polarising operator J defines a polarisation by taking the equivalence class of J. Thus a unitary structure J on a real Hilbert space H gives rise to a polarisation on the complexification $H_{\mathbb{C}}$ by taking the equivalence class of J, extended to the complexification by linearity. With respect to this polarisation, it is evident that $O_J(H) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap O(H)$.

There are three equivalent definitions of a unitary structure given in [PR94, ch 2.1]. Using these correspondences, a careful examination of [PS86, ch 12] reveals that the standard unitary structure on $L^2(S^1, \mathbb{R}^{2n})$ is defined in the following way: Let $\{e_k\}$ be the standard basis for $\mathbb{R}^{2n}$. Let $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the complex structure $J_0 e_{2k} = e_{2k-1}$, $J_0 e_{2k-1} = -e_{2k}$. The unitary structure on $L^2(S^1, \mathbb{R}^{2n})$ is defined by the operator J which satisfies:

$$\begin{aligned} J(v \cos k\theta) &= v \sin k\theta \\ J(v \sin k\theta) &= -v \cos k\theta \\ J(v) &= J_0(v). \end{aligned}$$

Here we identify $\mathbb{R}^{2n}$ with the subspace of constant loops in $L^2(S^1, \mathbb{R}^{2n})$.

The standard polarisation operator J on $L^2(S^1, \mathbb{C}^m)$ satisfies the identity:

$$J(vz^k) = -(-1)^{\mathrm{sign}(k)} i v z^k.$$

Proposition 2.5 *The standard polarisation on $L^2(S^1, \mathbb{C}^{2n})$ is that defined by the standard unitary structure on $L^2(S^1, \mathbb{R}^{2n})$. If m is odd, the standard polarisation on $L^2(S^1, \mathbb{C}^m)$ does not contain a unitary structure for $L^2(S^1, \mathbb{R}^m)$.*

Proof. To distinguish the operators, let $J_{\mathbb{R}}$ denote the unitary structure on $L^2(S^1, \mathbb{R}^{2n})$ and also its extension to $L^2(S^1, \mathbb{C}^{2n})$. Let $J_{\mathbb{C}}$ be the polarising operator on $L^2(S^1, \mathbb{C}^m)$. The first part of the proposition follows from the observation that $J_{\mathbb{R}}$ and $J_{\mathbb{C}}$ agree on the subspace of $L^2(S^1, \mathbb{C}^{2n})$ consisting of loops orthogonal to the constant loops. This has finite codimension and so $J_{\mathbb{C}} - J_{\mathbb{R}}$ is finite rank. Thus $J_{\mathbb{R}}$ and $J_{\mathbb{C}}$ define the same polarisation on $L^2(S^1, \mathbb{C}^{2n})$.

Let m be odd. Let $\{e_k\}$ be the standard basis for $\mathbb{R}^m$. Let $J_0 : \mathbb{R}^m \to \mathbb{R}^m$ be the map $J_0 e_{2k} = e_{2k-1}$, $J_0 e_{2k-1} = -e_{2k}$, $J_0 e_m = 0$. Let $J_{\mathbb{R}}$ be the map on $L^2(S^1, \mathbb{R}^m)$ defined using J_0 as for the even dimensional case. This restricts to a unitary structure on the subspace $\langle e_m \rangle^{\perp}$. As before, $J_{\mathbb{R}}$ and $J_{\mathbb{C}}$ agree on the subspace of loops orthogonal to the constant loops and thus define the same polarisation on $L^2(S^1, \mathbb{C}^m)$.

Let K be a unitary structure on $L^2(S^1, \mathbb{R}^m)$. The space $L^2(S^1, \mathbb{C}^m)$ decomposes orthogonally according to the eigenspaces of $J_{\mathbb{R}}$ and of K. Corresponding to $J_{\mathbb{R}}$ we have $L^2(S^1, \mathbb{C}^m) = V_+ \oplus V_- \oplus \mathbb{C}$ as $\pm i$-eigenspaces and the 0-eigenspace. Corresponding to K we have $L^2(S^1, \mathbb{C}^m) = W_+ \oplus W_-$. Let Σ denote the operation of complex conjugation on $L^2(S^1, \mathbb{C}^m)$. Then $\Sigma W_{\pm} = W_{\mp}$, $\Sigma V_{\pm} = V_{\mp}$, and $\Sigma \mathbb{C} = \mathbb{C}$.

The identity map decomposes as the matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} : V_+ \oplus V_- \oplus \mathbb{C} \to W_+ \oplus W_-.$$

Here $a : V_+ \to W_+$ is the inclusion of V_+ followed by the projection onto W_+, and similarly for the other entries. Since the identity map commutes with complex conjugation, $d = \Sigma b \Sigma$, $e = \Sigma a \Sigma$, and $f = \Sigma c \Sigma$.

Now assume that $J_{\mathbb{R}} - K$ is Hilbert-Schmidt.

The operator $b : V_- \to W_+$ can be written as $\frac{1}{4}(I - iK)(I + iJ_{\mathbb{R}})P$ where $P : V_+ \oplus V_- \oplus \mathbb{C} \to V_+ \oplus V_-$ is the orthogonal projection. This expands to $\frac{1}{4}(I + KJ_{\mathbb{R}} + i(J_{\mathbb{R}} - K))P$. As $K^2 = -I$, $I + KJ_{\mathbb{R}} = K(J_{\mathbb{R}} - K)$ and therefore b is Hilbert-Schmidt. Similarly, d is Hilbert-Schmidt. Since c and f have domain $\mathbb{C}$, they are finite rank. Thus the operator $a + e$ differs from the identity by a compact operator so is Fredholm of index zero.

Since a and e start from orthogonal subspaces and end in orthogonal subspaces, the fact that $a + e$ is Fredholm implies that both a and e are also Fredholm. The identity $e = \Sigma a \Sigma$ then implies that $\operatorname{Index} a = \operatorname{Index} e$. The matrix form of $a + e$ is:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & e & 0 \end{bmatrix}$$

from which it is evident that the index of $a + e$ is $\operatorname{Index} a + \operatorname{Index} e + 1$. This is incompatible with $\operatorname{Index} a = \operatorname{Index} e$ and so we deduce that $J_{\mathbb{R}} - K$ cannot be Hilbert-Schmidt. Hence there is no unitary structure for $L^2(S^1, \mathbb{R}^m)$ in the standard polarisation of $L^2(S^1, \mathbb{C}^m)$. □

For the record, we note the following properties of the groups associated to the standard polarisation on $L^2(S^1, \mathbb{C}^{2n})$ and the standard unitary structure on $L^2(S^1, \mathbb{R}^{2n})$.

Lemma 2.6 *Let $H = L^2(S^1, \mathbb{R}^{2n})$ and let J be the standard unitary structure on H. Let $H_{\mathbb{C}} = L^2(S^1, \mathbb{C}^{2n})$ be the complexification and $\mathcal{J}$ the standard polarisation on $H_{\mathbb{C}}$.*

1. *$O_J(H) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap O(H)$;*
2. *let $U_{\mathcal{J}}(H_{\mathbb{C}}) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap U(H_{\mathbb{C}})$, then $U_{\mathcal{J}}(H_{\mathbb{C}}) \to \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}})$ is a deformation retract;*
3. *let $\mathrm{Gl}_J(H) = \mathrm{Gl}_{\mathcal{J}}(H_{\mathbb{C}}) \cap \mathrm{Gl}(H)$, then $O_J(H) \to \mathrm{Gl}_J(H)$ is a deformation retract; and*
4. *$U_{\mathcal{J}}(H_{\mathbb{C}}) \simeq \Omega U$, $O_J(H) \simeq \Omega O$.*

In [PS86, ch 6], it is shown that the natural action of LU_{2n} on $H_{\mathbb{C}}$ defines an inclusion $LU_{2n} \to U_{\mathcal{J}}(H_{\mathbb{C}})$. Since $LO_{2n} = LU_{2n} \cap O(H)$ and $L\,\mathrm{Gl}_{2n}(\mathbb{R}) = L\,\mathrm{Gl}_{2n}(\mathbb{C}) \cap \mathrm{Gl}(H)$, it follows that the natural actions of LO_{2n} and $L\,\mathrm{Gl}_{2n}(\mathbb{R})$ on H define inclusions $LO_{2n} \to O_J(H)$ and $L\,\mathrm{Gl}_{2n}(\mathbb{R}) \to \mathrm{Gl}_J(H)$.

The action of $O_J(H)$ on $\mathbb{H}_J$ is projective. That is, there is a central S^1-extension of $O_J(H)$, usually written $\mathrm{Pin}_J(H)$ (the identity component being $\mathrm{Spin}_J(H)$), which acts unitarily on $\mathbb{H}_J$. This central extension is classified by a generator of $H^2(O_J(H), \mathbb{Z})$, which is isomorphic to $\mathbb{Z}$.

Examining LO_{2n}, we see that it has four components. The identity component is the semi-direct product $SO_{2n} \times \Omega \operatorname{Spin}_{2n}$ which has double cover $L \operatorname{Spin}_{2n}$. The central extension of $O_J(H)$ pulls back to a central S^1-extension of $L \operatorname{Spin}_{2n}$ written $\widetilde{L} \operatorname{Spin}_{2n}$. This is classified by a generator of $H^2(L \operatorname{Spin}_{2n}, \mathbb{Z})$, which is also isomorphic to $\mathbb{Z}$. Note also that the transgression map $\tau : H^\bullet(\operatorname{Spin}_{2n}, \mathbb{Z}) \to H^{\bullet-1}(L \operatorname{Spin}_{2n}, \mathbb{Z})$ is an isomorphism from degree 3 to degree 2.

We observe that $\mathbb{H}_J$ decomposes as $\mathbb{H}_J^+ \oplus \mathbb{H}_J^-$ corresponding to the decomposition of ΛH_J as $\Lambda^{\mathrm{ev}} H_J \oplus \Lambda^{\mathrm{odd}} H_J$. The identity component of $\operatorname{Pin}_J(H)$, whence also $\widetilde{L} \operatorname{Spin}_{2n}$, preserves this decomposition.

Finally, the circle action on $L^2(S^1, \mathbb{R}^{2n})$ lies in $O_J(H)$ and has a canonical lift to $\operatorname{Pin}_J(H)$. This defines a circle action on $\mathbb{H}_J$. The circle action on $L \operatorname{Spin}_{2n}$ therefore lifts to $\widetilde{L} \operatorname{Spin}_{2n}$ and the action of $\widetilde{L} \operatorname{Spin}_{2n}$ on $\mathbb{H}_J$ is circle equivariant.

3 String Manifolds and Spin Connections

In this section we explain how a string structure on a manifold defines a connection on the spin bundle of the loop space. Let M be an oriented, Riemannian manifold of even dimension d. Let $P \to M$ be the principal SO_d-bundle determined by the metric and the orientation. Let $\omega : TP \to \mathfrak{so}_d$ be the Levi-Civita connection on M.

The group Spin_d is the connected double cover of SO_d, universal if $d > 2$. A *spin structure* on M is a principal Spin_d-bundle $Q \to M$ such that Q is a double covering of P and the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spin}_d \times Q & \longrightarrow & Q \\ \big\downarrow & & \big\downarrow \\ SO_d \times P & \longrightarrow & P. \end{array}$$

The manifold M admits a spin structure if and only if $w_2(M) = 0$; the set of isomorphism classes of spin structures is in bijective correspondence with $H^1(M; \mathbb{Z}_2)$.

In order that the loop space, LM, admit a spin structure the structure group of LM must lift from $L \operatorname{Spin}_d$ to $\widetilde{L} \operatorname{Spin}_d$. We would also like this to be S^1-equivariant. The $L \operatorname{Spin}_d$-principal bundle on LM is LQ. Thus we are asking for an S^1-bundle, equivalently a line bundle, over LQ with certain properties. The primary property is that on fibres it must pull-back to the fibration $S^1 \to \widetilde{L} \operatorname{Spin}_d \to L \operatorname{Spin}_d$.

As explained in [Bry93, ch VI], line bundles on loop spaces are closely related to gerbes on the original manifold. In particular, the central extension $\widetilde{L} \operatorname{Spin}_d$ of $L \operatorname{Spin}_d$ corresponds to the gerbe of Spin_d classified by the generator of $H^3(\operatorname{Spin}_d; \mathbb{Z})$ (recall that as a simply connected, simple Lie group, there is a canonical isomorphism of $H^3(\operatorname{Spin}_d; \mathbb{Z})$ with $\mathbb{Z}$ and hence a canonical generator). Rather than asking for a line bundle over LQ we therefore ask for a gerbe over Q. This has the considerable advantage that the line bundle defined by the gerbe will be $\operatorname{Diff}^+(S^1)$-equivariant.

We have to answer the following question: what is the obstruction to constructing a gerbe on Q which on fibres pulls-back to the fundamental gerbe on Spin_d? We can rephrase this question in cohomological terms where it becomes: when can we find an element $a \in H^3(Q;\mathbb{Z})$ such that if $i : \mathrm{Spin}_d \to Q$ is the inclusion of a fibre then i^*a is the generator of $H^3(\mathrm{Spin}_d;\mathbb{Z})$?

To answer this we examine the Serre spectral sequence of the fibration $\mathrm{Spin}_d \to Q \to M$. The first part of the E_2-term is:

3	$H^3(\mathrm{Spin}_d;\mathbb{Z})$				
2	0	0	0		
1	0	0	0	0	
0	$H^0(M;\mathbb{Z})$	$H^1(M;\mathbb{Z})$	$H^2(M;\mathbb{Z})$	$H^3(M;\mathbb{Z})$	$H^4(M;\mathbb{Z})$
	0	1	2	3	4

This contains all the possible contributions to $H^3(Q;\mathbb{Z})$. The only part that might not persist to the E_∞-term is $H^3(\mathrm{Spin}_d;\mathbb{Z})$ in the $(0,3)$ position. This persists until the E_4-term where the differential is $d_4 : H^3(\mathrm{Spin}_d;\mathbb{Z}) \to H^4(M;\mathbb{Z})$. Let $\lambda \in H^4(M;\mathbb{Z})$ denote the image of the canonical generator of $H^3(\mathrm{Spin}_d;\mathbb{Z})$ under d_4. If $\lambda = 0$ then $H^3(Q;\mathbb{Z}) \cong H^3(M;\mathbb{Z}) \oplus H^3(\mathrm{Spin}_d;\mathbb{Z})$ and the inclusion of a fibre induces the projection $H^3(M;\mathbb{Z}) \oplus H^3(\mathrm{Spin}_d;\mathbb{Z}) \to H^3(\mathrm{Spin}_d;\mathbb{Z})$. If $\lambda \neq 0$ then $H^3(Q;\mathbb{Z}) = H^3(M;\mathbb{Z})$ and the inclusion of a fibre is the zero map on H^3. The class λ is known to satisfy $2\lambda = p_1(M)$ which has led to it being written as $p_1(M)/2$. This notation is somewhat misleading as λ depends on the choice of spin structure on M.

Definition 3.1 *A manifold M is a* string *manifold if it is an oriented, Riemannian, spin manifold such that $\lambda = 0$ together with a choice of* string structure*. That is, a choice of gerbe, $\mathcal{G}$, over the spin structure $Q \to M$ which on fibres is the fundamental gerbe on* Spin_d.

Once we have a string structure, there is a natural notion of a string connection.

Definition 3.2 *A* string connection *on a string manifold with string manifold with string structure $\mathcal{G}$ consists of the Levi-Civita connection on Q and a* Spin_d*-equivariant connective structure on the gerbe $\mathcal{G}$.*

Theorem 3.3 *A string connection on M defines a* $\mathrm{Diff}^+(S^1)$*-equivariant spin connection on LM.*

Compare this result with that of [Man02].

Proof. The Levi-Civita connection on M is a map $\omega : TP \to \mathfrak{so}_d$. As $\mathrm{Spin}_d \to SO_d$ is a covering map, it is a local diffeomorphism and so $\mathfrak{spin}_d = \mathfrak{so}_d$. Thus the Levi-Civita connection lifts to a connection on Q via $\omega' : TQ \to TP \xrightarrow{\omega} \mathfrak{so}_d = \mathfrak{spin}_d$. The loop of this is a $\mathrm{Diff}^+(S^1)$-equivariant map $L\omega' : TLQ \to L\mathfrak{spin}_d$. This is also a connection.

The gerbe with its connective structure defines a $\mathrm{Diff}^+(S^1)$-equivariant S^1-bundle $\widetilde{L}Q \to LQ$ with a connection $\alpha : T\widetilde{L}Q \to \mathbb{R}$. As the gerbe on M

pulls back to the fundamental gerbe on fibres, so also $\widetilde{L}Q \to LQ$ pulls back to $\widetilde{L}\,\mathrm{Spin}_d \to L\,\mathrm{Spin}_d$ on fibres. Also, the connection is $L\,\mathrm{Spin}_d$-equivariant. Hence $L\omega' \oplus \alpha : T\widetilde{L}Q \to L\mathfrak{spin}_d \oplus \mathbb{R}$ is a connection on $\widetilde{L}Q$. □

4 The Finer Structure of the Tangent Bundle

In this section we examine the tangent bundle of the loop space of a manifold. In [PS86] it is shown that the various loop groups of polynomial loops ($L_{\mathrm{pol}}U_n$), loops which extend to an annulus of radii r and r^{-1} $(L_r U_n)^1$, analytic loops ($L_\omega U_n$), and smooth loops (LU_n) are homotopic and therefore, say, a principal LU_n-bundle should have an associated principal, say, $L_{\mathrm{pol}}U_n$-bundle. Similarly, for a vector bundle with fibre $L\mathbb{C}^n$ and structure group LU_n there should be a subbundle with fibre $L_{\mathrm{pol}}\mathbb{C}^n$. In this section we shall give a construction of this subbundle for the loop bundle $LE \to LM$ defined by looping a vector bundle $E \to M$.

This construction owes its inception to J. Morava, [Mor01]. That paper was subsequently withdrawn as the main result was found to be stronger than the construction allows, see for example [CS04]. The aim of [Mor01] was to construct a finite dimensional subbundle of the tangent bundle of an almost complex loop space. The aim here is weaker as we seek to construct an infinite dimensional subbundle which is fibrewise dense. Comparing the result of this construction with the work in [CS04] and [Sta] reveals that this is, in a loose sense, the smallest subbundle of the tangent bundle that can always be defined.

Let M be a simply connected manifold, $\pi : E \to M$ a complex vector bundle of finite dimension n. Choose an inner product, $\langle \cdot, \cdot \rangle$, on E and let ∇ be a covariant differential operator on E compatible with the inner product.

We shall find it useful to regard loops on M as periodic paths. That is, a smooth loop in M is a smooth map $\gamma : \mathbb{R} \to M$ such that $\gamma(t+1) = \gamma(t)$ for all $t \in \mathbb{R}$. Similarly, a point in $L_\gamma E$ is a smooth map $\alpha : \mathbb{R} \to E$ with $\pi \circ \alpha = \gamma$ and such that $\alpha(t+1) = \alpha(t)$. Thus $L_\gamma E$ is identified with the subspace of $\Gamma(\gamma^* E)$ of those sections X satisfying $X(t+1) = X(t)$, noting that for any section Y, $Y(t+1)$ and $Y(t)$ must at least be in the same fibre of E.

For any smooth path $\beta : \mathbb{R} \to M$, the covariant differential operator ∇ defines a covariant differential operator $D_\beta : \Gamma(\beta^* E) \to \Gamma(\beta^* E)$. This has the following properties:

Lemma 4.1 *1. For $v \in E_{\beta(0)}$, there is a unique section $X_v \in \Gamma(\beta^* E)$ such that $X_v(0) = v$ and $D_\beta X_v = 0$. The assignment $v \to X_v$ is linear.*

2. For any $X, Y \in \Gamma(\beta^* E)$,

$$\frac{d\langle X, Y \rangle}{dt} = \langle D_\beta X, Y \rangle + \langle X, D_\beta Y \rangle.$$

3. The assignment $\beta \to D_\beta$ is smooth in β.

[1] We shall find it notationally convenient not to specify which of r or r^{-1} is the inner or outer radius. Thus $L_r U_n$ and $L_{r^{-1}} U_n$ refer to the same object.

Lemma 4.2 *Let $\gamma : \mathbb{R} \to M$ be periodic. If $X \in \gamma^* E$ satisfies $X(t+1) = X(t)$ for all t then $(D_\gamma X)(t+1) = (D_\gamma X)(t)$ for all t. Hence D_γ defines a fibrewise operator $LE \to LE$. This operator is skew-adjoint.*

Proof. As γ is periodic, the map $t \to t+1$ defines a shift map $\sigma : \Gamma(\gamma^* E) \to \Gamma(\gamma^* E)$. As D_γ is defined from data on E, $\sigma D_\gamma = D_\gamma \sigma$. Therefore if $\sigma X = X$, $\sigma D_\gamma X = D_\gamma X$.

That D_γ is skew-adjoint follows from the formula for the inner product on $L_\gamma E$ together with property 2 above:

$$\begin{aligned}\int_0^1 \langle D_\gamma X, Y \rangle + \langle X, D_\gamma Y \rangle dt = \int_0^1 \frac{d\langle X, Y\rangle}{dt} dt \\ = \langle X(1), Y(1) \rangle - \langle X(0), Y(0) \rangle.\end{aligned}$$

This is zero because X and Y are periodic. □

With D_γ defined, we can state the main theorem of this section. Let $L_{\mathrm{pol}}\mathbb{C}^n \subseteq L\mathbb{C}^n$ denote the linear span of the vectors $\{vz^k : v \in \mathbb{C}^n, k \in \mathbb{Z}\}$. This is a dense subspace of $L\mathbb{C}^n$.

Theorem 4.3 *Let $L_{pol}E \subseteq LE$ be the subset such that $L_{pol}E \cap L_\gamma E$ is the span of the eigenvectors of D_γ. Then $L_{pol}E$ is a vector bundle over LM with fibre $L_{pol}\mathbb{C}^n$ and structure group $L_{pol}U_n$. The natural inclusion $L_{pol}E \to LE$ has fibrewise dense image and on fibres corresponds to the inclusion $L_{pol}\mathbb{C}^n \to L\mathbb{C}^n$.*

Proof. Let $\gamma : \mathbb{R} \to M$ be a periodic path, i.e. a point in LM. Let $E_t = E_{\gamma(t)}$ be the fibre of $\gamma^* E$ above $t \in \mathbb{R}$. As γ is periodic, $E_{t+1} = E_t$. The holonomy map of γ is the map $H_\gamma : E_0 \to E_0$ defined by $v \to X_v(1)$, where X_v is defined as in lemma 4.1. Property 2 implies that this map is unitary. From the uniqueness of the map $v \to X_v$, we deduce that $X_v(t+1) = X_{H_\gamma v}(t)$.

Let u be an eigenvalue of H_γ with eigenvalue $\lambda \in S^1$. Let $k \in \mathbb{R}$ be such that $e^{-ik} = \lambda$. Let $Z_{u,k}(t) = e^{itk} X_u(t) \in \Gamma(\gamma^* E)$. We have $Z_{u,k}(t+1) = e^{ik} e^{itk} X_{H_\gamma u}(t)$, whence as $v \to X_v$ is linear and $H_\gamma u = \lambda u$, $Z_{u,k}(t+1) = e^{ik} \lambda e^{itk} X_u(t)$ and hence $Z_{u,k} \in L_\gamma E$. Since $D_\gamma X_u = 0$, $D_\gamma Z_{u,k} = ik Z_{u,k}$ and hence $Z_{u,k}$ is an eigenvalue of D_γ with eigenvalue ik.

Conversely, suppose that Z is an eigenvector of D_γ with eigenvalue λ. From the equation:

$$\frac{d\langle Z, X_u \rangle}{dt} = \langle D_\gamma Z, X_u \rangle + \langle Z, D_\gamma X_u \rangle = \lambda \langle Z, X_u \rangle$$

we deduce that $Z(t) = e^{\lambda t} X_v(t)$ where $v = Z(0)$. The periodicity of Z implies that $v = e^{\lambda} H_\gamma v$ and hence v is an eigenvector of H_γ with eigenvalue $e^{-\lambda}$.

Let $z : \mathbb{R} \to \mathbb{C}$ be the map $z(t) = e^{2\pi i t}$. It is simple to see that the action of multiplication by z takes $Z_{u,k}$ to $Z_{u,k+2\pi}$.

Let $L_{\mathrm{pol}} E_\gamma \subseteq L_\gamma E$ be the span of the eigenvectors of D_γ.

The operator H_γ is unitary, hence diagonalisable. The eigenvalues of H_γ are a finite subset of the circle. A choice of point e^{-is}, $2\pi > s \geq 0$, not in this subset defines a choice of numbers $s > k_1 > \ldots > k_l > s - 2\pi$ such that $\{e^{-ik_1}, \ldots, e^{-ik_l}\}$ are the distinct eigenvalues of H_γ. Let $\psi : \mathbb{C}^n \to E_0$ be a linear isomorphism.

Define the map $\Psi : L_{\text{pol}}\mathbb{C}^n \to L_{\text{pol}}E_\gamma$ as follows: it is sufficient to define $\Psi(vz^m)$ for $v \in \mathbb{C}^n$ and $m \in \mathbb{Z}$. The vector $\psi(v) \in E_0$ has a unique orthogonal decomposition $\psi(v) = u_1 + \cdots + u_l$ into eigenvectors of H_γ with distinct eigenvalues. Define $\Psi(vz^m) = z^m Z_{u_1,k_1} + \cdots + z^m Z_{u_l,k_l}$. This map is a linear isomorphism.

To show that the spaces $L_{\text{pol}}E_\gamma$ fit together to make a vector bundle over LM, it is sufficient to note that the two choices defining the isomorphism Ψ can be locally chosen in a smooth way: a local choice of trivialisation $\psi : \mathbb{C}^n \to E_0$ can be chosen smoothly as $E_0 \to LM$ is a vector bundle, and the choice of point on the circle can also be chosen smoothly – in fact, because the set of eigenvalues of H_γ is a finite set which varies smoothly with γ then there is a neighbourhood of γ in which these eigenvalues do not meet a particular fixed point on the circle.

The structure bundle of this vector bundle has a similar construction. Let $P \to M$ be the principal U_n-bundle associated to E. The structure bundle of LE is LP which, as with LM, we think of as the space of periodic paths in P.

A point in $L_\gamma P$ is a periodic smooth choice of frame for each E_t. Thus if $\alpha \in L_\gamma P$, $\alpha(t)$ is a basis for E_t. Let $\{k_1, \ldots, k_l\}$ and $\psi : \mathbb{C}^n \to E_0$ be as above. Define $\alpha(t) = (\Psi(e_1)(t), \ldots, \Psi(e_n)(t))$. The structure bundle, $L_{\text{pol}}P$, of $L_{\text{pol}}E$ has fibre at γ given by $\alpha \cdot L_{\text{pol}}U_n$. □

Once we have found a subbundle with fibre $L_{\text{pol}}\mathbb{C}^n$, it is a simple matter to adapt this construction to "thicken" this bundle and produce subbundles with fibre $L_r\mathbb{C}^n$ and with fibre $L_\omega\mathbb{C}^n$. Abstractly, these can be defined by taking $L_\alpha E = L_{\text{pol}}E \times_{L_{\text{pol}}U_n} L_\alpha\mathbb{C}^n$. If E were the complexification of a real vector bundle W then these constructions would respect the underlying real structure of $L_\alpha E$ and so define corresponding subbundles of LW.

Using duality, this construction produces completions of $(LE)^*$. Abstractly, $(L_\alpha E)^* = L_{\text{pol}}P \times_{L_{\text{pol}}U_n} (L_\alpha\mathbb{C}^n)^*$. Therefore, one way by which we could find a Hilbert completion of $(LE)^*$ would be to fix a Hilbert completion H of $(L\mathbb{C}^n)^*$ such that the action of $L_{\text{pol}}U_n$ on $(L\mathbb{C}^n)^*$ extends to a smooth action on H. Then define the completion of $(LE)^*$ to be $L_{\text{pol}}P \times_{L_{\text{pol}}U_n} H$. The disadvantage of this approach is that the action of $L_{\text{pol}}U_n$ on any completion of $L\mathbb{C}^n$ is not unitary so this construction does not give a natural inner product on the fibres. One can be defined using a partition of unity, but there is a better approach.

Lemma 4.4 *Let $\gamma \in LM$. Let $\cos D_\gamma : L_{pol}E_\gamma \to L_{pol}E_\gamma$ be the operator:*

$$\cos D_\gamma := \sum_{k \geq 0} \frac{(-1)^k}{(2k)!} D_\gamma^{2k}$$

This operator is a well-defined linear isomorphism. If Z is an eigenvector of D_γ with eigenvalue λ then $(\cos D_\gamma)Z = (\cos\lambda)Z$.

Proof. The statement about the action of $\cos D_\gamma$ on the eigenvectors of D_γ is immediate from the definition. As the eigenvalues of D_γ are purely imaginary, the eigenvalues of $\cos D_\gamma$ are real, positive, and greater than or equal to 1. Hence as $L_{\mathrm{pol}}E_\gamma$ is defined as the span of the eigenvalues of D_γ, $\cos D_\gamma$ is a well-defined linear bijection. The topology on $L_{\mathrm{pol}}E_\gamma$ is the inductive limit topology from the family of finite dimensional subspaces defined by taking the span of a finite family of eigenvalues of D_γ. Since $\cos D_\gamma$ preserves this family, it preserves the inductive topology and is thus a topological homeomorphism. Hence it is a linear isomorphism. □

Definition 4.5 *Let $L^2_e E$ be the vector bundle over LM such that $L^2_e E_\gamma$ is the subspace of LE consisting of those paths X for which $\langle(\cos D_\gamma)X, (\cos D_\gamma)X\rangle$ is defined. Let $\langle\langle\cdot,\cdot\rangle\rangle$ be the inner product on $L^2_e E$ defined by $\langle\langle X, Y\rangle\rangle = \langle(\cos D_\gamma)X, (\cos D_\gamma)Y\rangle$.*

Lemma 4.6 *Let $L^2_e\mathbb{C}^n$ be the space of loops in $\mathbb{C}^n$ which extend over the annulus of radii (e^{-1}, e) and have square-integrable boundaries. Then $L^2_e E \cong L_{pol}U_n \times_{L_{pol}U_n} L^2_e\mathbb{C}^n$.*

Proof. Let $\Psi : L_{\mathrm{pol}}\mathbb{C}^n \to L_{\mathrm{pol}}E_\gamma$ be the inclusion of a fibre as constructed in the proof of theorem 4.3. We need to show that the L^2_e-inner product on $L_{\mathrm{pol}}\mathbb{C}^n$ is equivalent to the pull back of $\langle\langle\cdot,\cdot\rangle\rangle$ via Ψ. If we choose a unitary basis of E_0 consisting of eigenvectors of the holonomy operator, we can express both $L_{\mathrm{pol}}\mathbb{C}^n$ (via $\psi : \mathbb{C}^n \to E_0$) and $L_{\mathrm{pol}}E_\gamma$ as the orthogonal direct sum of n-subspaces each isomorphic to $L_{\mathrm{pol}}\mathbb{C}$. The map Ψ respects this decomposition.

The standard basis for $L_{\mathrm{pol}}\mathbb{C}$ is $\{z^m : m \in \mathbb{Z}\}$. This basis is orthogonal for the L^2_e-inner product and for the pull back of $\langle\langle\cdot,\cdot\rangle\rangle$, but orthonormal for neither. The norm of z^m with respect to each inner product is given by:

$$\langle z^m, z^m\rangle_e = e^{|m|}, \qquad \langle\langle z^m, z^m\rangle\rangle = \cosh(k+m)$$

for some fixed $k \in (-2\pi, 2\pi)$. Since both the sequences $(e^{|m|}/\cosh(k+m))$ and $(\cosh(k+m)/e^{|m|})$ are bounded, the identity map on $L_{\mathrm{pol}}\mathbb{C}$ extends to an isomorphism between the Hilbert space completions. □

Proposition 4.7 *The inner product $\langle\langle\cdot,\cdot\rangle\rangle$ is equivariant under the circle action. The map $X \to (\cos D_\gamma)X$ is an equivariant isometry $L^2_e E \to L^2 E$, the standard Hilbert bundle completion of LE. The dual of $L^2_e E$ is a Hilbert bundle completion of $(LE)^*$ with a fibrewise equivariant inner product.*

Using the Riesz identification of a Hilbert space with its dual, we have the following Escher-like diagram of inclusions:

$$\begin{array}{ccc} L^2_e E & \longrightarrow & LE \\ \uparrow & & \downarrow \\ (LE)^* & \longleftarrow & L^2 E \end{array}$$

though, unlike Escher's *Ascending and Descending*, traversing a loop in this diagram does not leave one where one started. The triples $((LE)^*, L^2_e E, LE)$ and $(LE, L^2 E, (LE)^*)$ are known as *rigged spaces*.

5 The Construction of the Dirac Operator

The construction of the Dirac operator now proceeds without hindrance. There are two equivalent approaches, differing in whether one wishes to emphasise the rôle of the tangent bundle or of the cotangent bundle. Let M be a finite dimensional, simply connected, string manifold with a choice of string structure and string connection. The loop space LM thus has a spin structure with spin connection. The Levi-Civita connection on the tangent bundle of M defines the finer structure on the tangent bundle of LM as in section 4.

To emphasise the rôle of the tangent bundle, we take the spin bundles associated to the tangent bundle with their spin connection. Proposition 2.2 assures us that the domain of the Clifford multiplication map π extends to $\Gamma(\mathcal{L}(T^*_{\mathbb{C}} LM, S^+))$. The inner product from section 4 defines a linear injection $b : T^*_{\mathbb{C}} LM \to T_{\mathbb{C}} LM$ (factoring through $L^2_e TM$). The map $A \to A \circ b$ defines a linear injection $\mathcal{L}(T_{\mathbb{C}} LM, S^+) \to \mathcal{L}(T^*_{\mathbb{C}} LM, S^+)$.

Definition 5.1 (Version 1) *The Dirac operator on the loop space LM is the operator $\not{\partial} := \pi \circ b \circ \nabla$:*

$$\not{\partial} : \Gamma(S^+) \xrightarrow{\nabla} \Gamma(\mathcal{L}(T_{\mathbb{C}} LM, S^+)) \xrightarrow{b} \Gamma(\mathcal{L}(T^*_{\mathbb{C}} LM, S^+)) \xrightarrow{\pi} \Gamma(S^-).$$

To emphasise the rôle of the cotangent bundle, we use the isomorphism $L^2_e TM \to L^2 TM$ to pull-back the spin connection from $L^2 TM$ to the dual of $L^2_e TM$. We then construct the spin bundles of LM directly from the dual of $L^2_e TM$. As this is the fibrewise completion of $T^* LM$, the Clifford multiplication map extends to $\mathcal{L}(TLM, S^+)$.

Definition 5.2 (Version 2) *The Dirac operator on the loop space LM is the operator $\not{\partial} := \pi \circ \nabla$:*

$$\not{\partial} : \Gamma(S^+) \xrightarrow{\nabla} \Gamma(\mathcal{L}(T_{\mathbb{C}} LM, S^+)) \xrightarrow{\pi} \Gamma(S^-).$$

The two constructions are equivalent in that the isomorphism of (the dual) of $L^2_e TM$ with $L^2 TM$ induces an isomorphism of spin bundles and an isomorphism (by construction) of the connections. Thus the two definitions differ solely in emphasis.

A Inner Products on the Space of Distributions

In this appendix we examine inner products on $L\mathbb{R}^{n*}$. The goal is to classify the inner products on $L\mathbb{R}^{n*}$ which have the following properties: the inner product is invariant under the circle action, the involution of reversing loops is orthogonal, and the operations of multiplication by $\cos\theta$ and $\sin\theta$ are continuous.

We shall actually work with $\mathcal{S}^*$, the dual of the space of rapidly decreasing, complex-valued, $\mathbb{Z}$-indexed sequences. As a sequence space, this is particularly simple to describe and therefore to work with. Taking Fourier coefficients defines an isomorphism $L\mathbb{C} \to \mathcal{S}$ which allows us to transfer information from $\mathcal{S}^*$ to $L\mathbb{C}^*$. Using the description of $L\mathbb{C}^n$ as $L\mathbb{C} \otimes \mathbb{C}^n$, we can extend the description to the dual of $L\mathbb{C}^n$, and thence to the dual of the underlying real space $L\mathbb{R}^n$.

As a preliminary, we shall show that there is no "natural" inner product on $L\mathbb{C}^*$. That is, if $L\mathbb{C}^\times = L(\mathbb{C}^\times)$ denotes the space of never-zero smooth loops in $\mathbb{C}$ then there is no inner product on $L\mathbb{C}^*$ such that the group $L\mathbb{C}^\times$ acts continuously with respect to the inner product topology. This is in stark contrast to the situation for $L\mathbb{C}$ where $L\mathbb{C}^\times$ does act continuously with respect to the standard inner product.

Theorem A.1 *Let $L_c\mathbb{C}$ be a class of loops in $\mathbb{C}$ with the following properties:*

1. *there are continuous inclusions $\mathbb{C} \to L_c\mathbb{C} \to L^{1,\infty}\mathbb{C}$, where $\mathbb{C}$ corresponds to the constant loops and $L^{1,\infty}\mathbb{C}$ is the space of continuously differentiable loops;*

2. *the class of loops is preserved under products; thus $L_c\mathbb{C}^\times$ acts on $L_c\mathbb{C}$ and hence, via the adjoint map, on $L_c\mathbb{C}^*$;*

3. *$L_c\mathbb{C}$ is reflexive;*

4. *$L_c\mathbb{C}$ cannot be given the structure of a Hilbert space;*

then for any inner product on $L_c\mathbb{C}^*$ there is some $\alpha \in L_c\mathbb{C}^\times$ which acts unboundedly on $L_c\mathbb{C}^*$ with respect to the inner product topology.

Proof. Let $\langle \cdot, \cdot \rangle$ be a continuous inner product on $L_c\mathbb{C}^*$. Let H denote the Hilbert space completion of $L_c\mathbb{C}^*$ with respect to $\langle \cdot, \cdot \rangle$. The dual of the inclusion $L_c\mathbb{C}^*$ is a map $H^* \to L_c\mathbb{C}^* = L_c\mathbb{C}$.

Suppose that $L_c\mathbb{C}^\times$ acts continuously on $L_c\mathbb{C}^*$ with respect to the inner product topology. This implies that H^* is preserved in $L_c\mathbb{C}$ by $L_c\mathbb{C}^\times$. Suppose that $H^* \cap L_c\mathbb{C}^\times \neq \emptyset$. Because $L_c\mathbb{C}^\times$ is a group, this implies that $L_c\mathbb{C}^\times \subseteq H^*$. The linear span of $L_c\mathbb{C}^\times$ is $L_c\mathbb{C}$ so $H^* = L_c\mathbb{C}$. However, this implies that $H^* \to L_c\mathbb{C}$ is a continuous, linear bijection from a Hilbert space onto $L_c\mathbb{C}$ which contradicts the fourth assumption.

Thus we need to show that the other assumptions imply that $H^* \cap L_c\mathbb{C}^\times \neq \emptyset$. In other words, we need to show that there is an element in H^* which is never zero. To do this, we shall use the Banach-Steinhaus theorem as stated in [Sch71, III, §4.6]. As $L_c\mathbb{C}$ is reflexive, it is the dual of $L_c\mathbb{C}^*$. We shall write the evaluation of $\alpha \in L_c\mathbb{C}$ on $a \in L_c\mathbb{C}^*$ as $a(\alpha)$ rather than $\alpha(a)$ to avoid confusion with the notation $\alpha(\lambda)$ for the evaluation of α on $\lambda \in S^1$.

From the corollary to [Sch71, IV, §2.3], as the inclusion $L_c\mathbb{C}^* \to H$ is injective with weakly dense image, the map $H^* \to L_c\mathbb{C}^*$ is also injective with weakly dense image. Thus there is a sequence (α_n) in H^* which converges weakly to 1. That is, for all $a \in L_c\mathbb{C}^*$, $(a(\alpha_n))$ converges in $\mathbb{C}$ to $a(1)$. The space $L_c\mathbb{C}^*$ is reflexive, hence barrelled, and so the Banach-Steinhaus theorem applies. This states that (α_n) converges to 1 uniformly on each compact subset of $L_c\mathbb{C}^*$. We shall find a particularly convenient compact subset of $L_c\mathbb{C}^*$.

The norm on $L^{1,\infty}\mathbb{C}$ is $\|\gamma\|_{1,\infty} = \sup\{|\gamma(\lambda)|, |\gamma'(\lambda)|\}$. For $\lambda \in S^1$, there is an element e_λ of $L^{1,\infty}\mathbb{C}$ which evaluates a loop at time λ. If $\gamma \in L^{1,\infty}\mathbb{C}$ with $\|\gamma\|_{1,\infty} \le 1$ then γ is Lipschitz with constant $K \le 1$. Therefore $|e_\lambda(\gamma) - e_{\lambda'}(\gamma)|$ is less than or equal to the smaller angle between λ and λ'. Hence $\lambda \to e_\lambda$ is a continuous map from S^1 to $L^{1,\infty}\mathbb{C}^*$. Composing this with the dual of the map $L_c\mathbb{C} \to L^{1,\infty}\mathbb{C}$ defines a continuous map $S^1 \to L_c\mathbb{C}^*$. Its image is thus compact and therefore $(\alpha_n) \to 1$ uniformly on $\{e_\lambda : \lambda \in S^1\}$.

Hence there is some N such that for $n \ge N$, $|e_\lambda(\alpha_n) - e_\lambda(1)| < 1$ for all $\lambda \in S^1$. Thus $|\alpha_N(\lambda) - 1| < 1$ so $\alpha_N(\lambda) \ne 0$ for all $\lambda \in S^1$. Hence H^* contains an element which is never zero. □

A.1 Inner Products on Distribution Space

In this section we investigate those inner products on $\mathcal{S}^*$ which, under the isomorphism $\mathcal{S}^* \cong L\mathbb{C}^*$, are invariant under the circle action and the involution of reversing loops, and such that multiplication by z is continuous in the inner product topology. Our goal is to prove the following theorem:

Theorem A.2 *Let $\mathcal{C}_\theta$ be the set of inner products on $L\mathbb{R}^{2n^*}$ which are:*

1. *S^1-equivariant;*
2. *the involution which reverses loops is orthogonal,*
3. *multiplication by $\sin\theta$ and $\cos\theta$ are continuous.*

Then $\mathcal{C}_\theta$ is non-empty and convex.

To achieve this goal, we use the fact that as $\mathcal{S}^*$ is a sequence space we have a good presentation of elements of $\mathcal{S}^*$ and of operators acting on $\mathcal{S}^*$. We start by transferring the above mentioned operators from $L\mathbb{C}^*$ to $\mathcal{S}^*$.

Definition A.3 *Define the operators R_λ for $\lambda \in S^1$, ι, and z on $\mathcal{S}$ to be the operators corresponding under the Fourier isomorphism $\mathcal{S} \cong L\mathbb{C}$ to rotation by λ, reversal of the circle, and multiplication by z, respectively. We shall use the same notation for their adjoints which act on $\mathcal{S}^*$.*

The maps $\lambda \to R_\lambda \in \mathcal{L}(\mathcal{S})$ and $\lambda \to R_\lambda \in \mathcal{L}(\mathcal{S}^*)$ define an action of the circle on $\mathcal{S}$ and $\mathcal{S}^*$ respectively. We shall refer to ι as the *natural involution* on $\mathcal{S}$ and $\mathcal{S}^*$.

For $p \in \mathbb{Z}$, let $e^p \in \mathcal{S}$ and $e_p \in \mathcal{S}^*$ both denote the sequence with a 1 in the pth place and zero elsewhere. The sets $\{e^p\}$ and $\{e_p\}$ are topologically free bases for $\mathcal{S}$ and $\mathcal{S}^*$ respectively.

Lemma A.4 *In terms of the bases $\{e^p\}$ and $\{e_p\}$, the operators R_λ, ι, and z are given by the formulæ:*

$$R_\lambda e^p = \lambda^p e^p \qquad \iota e^p = e^{-p} \qquad z e^p = e^{p+1}$$
$$R_\lambda e_p = \lambda^{-p} e_p \qquad \iota e_p = e_{-p} \qquad z e_p = e_{p-1}$$

Definition A.5 *Let $\mathcal{C}$ denote the cone of positive semi-definite, sesquilinear forms on $\mathcal{S}^*$ which are invariant under the action of the circle and under the action of the natural involution. Let $\mathcal{C}^+ \subseteq \mathcal{C}$ denote the subcone consisting of positive definite forms.*

Let $\mathcal{T}$ denote the cone of positive, rapidly decreasing sequences (a_p) such that $a_p = a_{-p}$ for all $p \in \mathbb{Z}$. Let $\mathcal{T}^+$ denote the subcone of strictly positive sequences.

Theorem A.6 *The map $(\cdot, \cdot) \to ((e_p, e_p))$ defines a bijection of cones from $\mathcal{C}$ to $\mathcal{T}$ such that $\mathcal{C}^+$ is carried onto $\mathcal{T}^+$.*

Proof. As the set $\{e_p : p \in \mathbb{Z}\}$ is a basis for $\mathcal{S}^*$, any sesquilinear form, $(\cdot, \cdot)$, on $\mathcal{S}^*$ is completely determined by the $\mathbb{Z} \times \mathbb{Z}$-indexed set of numbers $\{(e_p, e_q)\}$. We shall refer to this as the double sequence associated to $(\cdot, \cdot)$.

Suppose that $(\cdot, \cdot)$ is a sesquilinear form on $\mathcal{S}^*$ invariant under the action of R_λ for some $\lambda \in S^1$ not of finite order. Then for all $p, q \in \mathbb{Z}$, $(R_\lambda e_p, R_\lambda e_q) = (e_p, e_q)$. Using the formula from lemma A.4, the left-hand side of this equation is $\lambda^{q-p}(e_p, e_q)$. As λ is not of finite order, if $p \neq q$ this implies that $(e_p, e_q) = 0$. Thus the double sequence associated to $(\cdot, \cdot)$ is zero off the main diagonal.

Conversely, suppose that $(\cdot, \cdot)$ is a sesquilinear form on $\mathcal{S}^*$ such that the associated double sequence is zero off the main diagonal. For $a = (a^p) \in \mathcal{S}^*$, the number (a, a) is given by the formula $\sum |a^p| (e_p, e_p)$. Thus as $R_\lambda a = (\lambda^{-p} a^p)$, $(R_\lambda a, R_\lambda a) = (a, a)$ for any $\lambda \in S^1$. Hence $(\cdot, \cdot)$ is invariant under the circle action.

If, in addition, the natural involution acts unitarily – that is, the sesquilinear form is invariant under the action of the natural involution – then lemma A.4 shows that $(e_p, e_p) = (e_{-p}, e_{-p})$. The converse is immediate.

Let $(\cdot, \cdot)$ be a sesquilinear form which is invariant under the circle action and under the natural involution. Let $a_p = (e_p, e_p)$ for $p \in \mathbb{Z}$. The form $(\cdot, \cdot)$ is continuous and therefore defines a conjugate linear map $\mathcal{S}^* \to \mathcal{S}^{**} = \mathcal{S}$. Under this map, an element $b \in \mathcal{S}^*$ is taken to the sequence $((e_p, b))$. Let $\mathbb{1} \in \mathcal{S}^*$ denote the sequence consisting completely of 1s. Under the map $\mathcal{S}^* \to \mathcal{S}$ defined by the form, this element is taken to $((e_p, \mathbb{1})) = (a_p)$. Hence the sequence (a_p) is rapidly decreasing and thus the map in the statement of the theorem is well-defined.

Thus a sesquilinear form which is invariant under the circle action and under the natural involution is completely determined by the sequence $((e_p, e_p))$. It is simple to see that the sequence is positive if and only if the sesquilinear form is positive semi-definite, and that the sequence is strictly positive if and only if the sesquilinear form is positive definite. Thus the sequence is an element of $\mathcal{T}$ and is in $\mathcal{T}^+$ if and only if the original sesquilinear form were positive definite. Whence the map $\mathcal{C} \to \mathcal{T}$ is well-defined and injective. A simple check shows that this is a map of cones.

To show that the map is surjective, and hence a bijection, let $(a_p) \in \mathcal{T}$. Let $b = (b^p)$ and $c = (c^p)$ be elements of $\mathcal{S}^*$. There exist integers $m, n > 0$ such that $(p^{-m} b^p)$ and $(p^{-n} c^p)$ are bounded. As (a_p) is rapidly decreasing, the sequence $(p^{n+m+2} a_p)$ is bounded and hence $(p^{n+m} a_p)$ is summable. Hence $(b^p \overline{c^p} a_p)$ is a summable sequence and thus the formula:

$$(b, c) \to \sum_{p \in \mathbb{Z}} b^p \overline{c^p} a_p$$

is well-defined as a sesquilinear map $\mathcal{S}^* \times \mathcal{S}^* \to \mathbb{C}$. It is evidently positive semi-definite. To show continuity, it is sufficient to show that it is continuous when restricted to each space $\{(x^p) : (p^{-n} x^p) \text{ is bounded}\}$. Continuity of this restriction follows from the estimate:

$$\left| \sum_{p \in \mathbb{Z}} b^p \overline{c^p} a_p \right| \leq \sup\{\left|p^{-n} b^p\right|\} \sup\{\left|p^{-n} c^p\right|\} \sum_{p \in \mathbb{Z}} \left|p^{2n} a_p\right| .$$

Thus the sequence (a_p) defines a sesquilinear form on $\mathcal{S}^*$. It is clear that the associated double sequence for this form is zero off the main diagonal and the main diagonal is (a_p). Thus it is invariant under the circle action and the natural involution and so is an element of $\mathcal{C}$. It is the preimage of (a_p) under the map $\mathcal{C} \to \mathcal{T}$ showing that that map is a bijection. □

Any continuous inner product on $\mathcal{S}^*$ defines a Hilbert space completion, but the map from inner products to Hilbert space completions is not injective. Two inner products define the same Hilbert space completion if and only if the identity map on $\mathcal{S}^*$ extends to an isomorphism between the completions. This condition can be stated elegantly in terms of the sequences in $\mathcal{T}^+$ associated to the given inner products:

Lemma A.7 *Let $(a_p), (b_p) \in \mathcal{T}^+$. The Hilbert space completions defined by the inner products associated to (a_p) and (b_p) are equivalent if and only if the sequences (a_p/b_p) and (b_p/a_p) are bounded.*

We now turn to the operator z and determine the answer to the following question: for which inner products on $\mathcal{S}^*$ is the operator z continuous with respect to the inner product topology?

Proposition A.8 *Let $(a_p) \in \mathcal{T}^+$. Let $(\cdot, \cdot)$ be the associated inner product on $\mathcal{S}^*$. The operator z is continuous with respect to the inner product topology if and only if the sequence of ratios (a_p/a_{p+1}) is bounded.*

In this case, $\|z\|^2 = \sup\{a_p/a_{p+1}\}$.

Notice that as $a_p = a_{-p}$, the sequence (a_p/a_{p-1}) is just (a_p/a_{p+1}) in reverse order.

Proof. Let $\|\cdot\|$ be the norm defined by the inner product. Suppose that z is continuous with respect to the inner product topology on $\mathcal{S}^*$. In particular, $\|z e_{p+1}\| \leq \|z\| \|e_{p+1}\|$ for all p. From lemma A.4, $z e_{p+1} = e_p$. Thus for $p \in \mathbb{Z}$, $\sqrt{a_p} \leq \|z\| \sqrt{a_{p+1}}$. Hence the sequence (a_p/a_{p+1}) is bounded above by $\|z\|^2$.

Conversely, suppose that (a_p/a_{p+1}) is bounded above by, say, M. Let $b = (b^p) \in \mathcal{S}^*$, then:

$$\|zb\|^2 = \sum \left|b^{p+1}\right| a_p \leq \sum \left|b^{p+1}\right| M a_{p+1} = M \|b\|^2 .$$

Thus z is continuous with respect to $\|\cdot\|$ and so extends to a continuous linear operator on H. Moreover, $\|z\|^2 \leq M$.

Combining the two relationships for $\|z\|$ shows that $\|z\|^2 = \sup\{a_p/a_{p+1} : p \in \mathbb{Z}\}$ when either side exists. □

Corollary A.9 *Let $(a_p) \in \mathcal{T}$ be such that (a_p/a_{p+1}) is bounded. For each $q \in \mathbb{Z}$, the operator z^q is continuous with respect to the inner product topology and $\|z^q\|^2 = \sup\{a_p/a_{p+q}\}$.*

Corollary A.10 *Let $\mathcal{C}_z$ be the subset of $\mathcal{C}$ consisting of those inner products for which the operation of multiplication by z is continuous, $\mathcal{T}_z$ the corresponding subcone of $\mathcal{T}$. Then $\mathcal{C}_z$ is a non-empty subcone of $\mathcal{C}^+$.*

Proof. The set $\mathcal{T}_z$ consists of those sequences $(a_p) \in \mathcal{T}^+$ for which (a_p/a_{p+1}) is bounded. This is non-empty as the sequence $(2^{-|p|})$ lies in $\mathcal{T}_z$.

Clearly, if $(a_p) \in \mathcal{T}_z$ then for any $t > 0$, $(ta_p) \in \mathcal{T}_z$. If $(a_p), (b_p) \in \mathcal{T}_z$ then there exist $M, N > 0$ such that $a_p/a_{p+1} \leq M$ and $b_p/b_{p+1} \leq N$ for all p. Equivalently, $a_p \leq M a_{p+1}$ and $b_p \leq N b_{p+1}$. Let $R = \max\{M, N\}$, then $a_p + b_p \leq R(a_{p+1} + b_{p+1})$ so $((a_p + b_p)/(a_{p+1} + b_{p+1}))$ is bounded, hence lies in $\mathcal{T}_z$.

Therefore $\mathcal{T}_z$ is a subcone of $\mathcal{T}^+$ and so $\mathcal{C}_z$ is a subcone of $\mathcal{C}^+$. □

These inner products transfer to $L\mathbb{C}^*$ via the isomorphism $L\mathbb{C}^* \to \mathcal{S}^*$ but we can find a formula which is more natural on $L\mathbb{C}$.

Proposition A.11 *Let $(\cdot,\cdot) \in \mathcal{C}$. Let $(a_p) \in \mathcal{T}$ be the associated sequence. Thinking of $\mathcal{T}$ as a subset of $\mathcal{S}$, let $\gamma_a \in L\mathbb{C}$ be the image of (a_p) under the isomorphism $\mathcal{S} \cong L\mathbb{C}$.*

Under the isomorphism $\mathcal{S}^* \cong L\mathbb{C}^*$, the form $(\cdot,\cdot)$ is given by the formula $(b, c) \to b(\bar{c} \diamond \gamma_a)$ where $c \diamond \gamma_a \in L\mathbb{C}$ is the map $\lambda \to c(R_{\lambda^{-1}}\gamma_a)$.

Proof. The map $S^1 \times S^1 \to \mathbb{C}$ defined by $(\lambda, \mu) \to \gamma_a(\lambda^{-1}\mu)$ is the composition of smooth maps hence is smooth. Therefore by the exponential law for smooth maps, [KM97, I.3], its adjoint, $\lambda \to R_{\lambda^{-1}}\gamma_a$, is a smooth map $S^1 \to L\mathbb{C}$. The element $c \in L\mathbb{C}^*$ is a continuous linear map $L\mathbb{C} \to \mathbb{C}$, hence is smooth, so the map $\lambda \to c(R_{\lambda^{-1}}\gamma_a)$ is a smooth map $S^1 \to \mathbb{C}$. Thus the formula $(b, c) \to b(\bar{c} \diamond \gamma_a)$ makes sense. It is also evident that the map $c \to c \diamond \gamma_a$ is continuous and so $(b, c) \to b(\bar{c} \diamond \gamma_a)$ is at least separately continuous and thus completely determined by its effect on a basis.

Using the isomorphisms $L\mathbb{C} \cong \mathcal{S}$ and $L\mathbb{C}^* \cong \mathcal{S}^*$, we transfer these operators to $\mathcal{S}$ and $\mathcal{S}^*$. Under these isomorphisms, $R_{\lambda^{-1}}\gamma_a$ becomes the sequence $(\lambda^{-p}a_p)$ and so $e_q(R_{\lambda^{-1}}\gamma_a) = \lambda^{-q}a_q$. Thus $e_q \diamond \gamma_a$ is the sequence corresponding to the function $\lambda \to \lambda^{-q}a_q$ which is $a_q e_{-q}$. Therefore, $\Sigma e_p(e_q \diamond \gamma_a) = e_{-p}(a_q e_{-q}) = a_q \delta_{pq}$.

Hence the sesquilinear form on $L\mathbb{C}^*$, $(b, c) \to b(\bar{c} \diamond \gamma_a)$, corresponds to the original sesquilinear form on $\mathcal{S}^*$, $((b^p), (c^p)) \to \sum b^p \overline{c^p} a_p$. □

The inner products we consider on $\mathcal{S}^*$ and $L\mathbb{C}^*$ arise as inner products on the underlying real spaces and therefore give a classification of inner products on $L\mathbb{R}^*$ which are invariant under the circle action and the natural involution, and also of those for which the operations of multiplication by $\cos\theta$ and by $\sin\theta$ are continuous.

Proposition A.12 *The sesquilinear forms on $\mathcal{S}^*$ and $L\mathbb{C}^*$ considered above are the complexifications of sesquilinear forms on the underlying real spaces of both $\mathcal{S}^*$ and $L\mathbb{C}^*$.*

Proof. For $\mathcal{S}^*$, this is evident from the formula. For $L\mathbb{C}^*$, it follows from the invariance under ι together with the fact that ι intertwines the complex conjugation operators arising from $\mathcal{S}^*$ and $L\mathbb{C}^*$ (note that the isomorphism $L\mathbb{C}^* \to \mathcal{S}^*$ does not induce an isomorphism of real structures and thus the complex conjugation operators differ). □

Together with corollary A.10, this proves theorem A.2.

A.2 Polarisations

In this section we examine how the theory of polarisations, and thus of unitary structures, interacts with these inner products on the space of distributions. We examine an inner product on $L\mathbb{C}^{n*}$ determined by a sequence in $\mathcal{T}_z$. To pass from an inner product on $L\mathbb{C}^*$ to one on $L\mathbb{C}^{n*}$, we use the isomorphism $L\mathbb{C}^{n*} \cong L\mathbb{C}^* \otimes \mathbb{C}^n$ together with the the standard inner product on $\mathbb{C}^n$.

Lemma A.13 *Let J be the operator on $\mathcal{S}^*$ defined by $Je_p = -(-1)^{sign(p)} ie_p$. Let $(\cdot,\cdot) \in \mathcal{C}^+$ be an inner product on $\mathcal{S}^*$ and let H be the corresponding Hilbert space completion. Then the operator J defines a polarisation of H.*

This extends in a natural way to a polarisation of the Hilbert completion of $L\mathbb{C}^{n*}$.

Proof. Let $(a_p) \in \mathcal{T}^+$ be the sequence corresponding to the inner product. For $b = (b^p) \in \mathcal{S}^*$, it follows straight from the formula for J that $(Jb, Jb) = (b, b)$ and therefore J extends to a unitary operator on H. It satisfies $J^2 = -1$ and $J \pm iI$ are not finite rank. Therefore, it defines a polarisation on H.

To extend this to the Hilbert completion of $L\mathbb{C}^{n*}$, we observe that this completion is naturally isomorphic to $H \otimes \mathbb{C}^n$. The polarising operator J on H defines one on $H \otimes \mathbb{C}^n$ by taking $J \otimes I_n$. □

Definition A.14 *The polarisation $\mathcal{J}$ so defined on the completion of $L\mathbb{C}^{n*}$ is called the* standard polarisation.

Proposition A.15 *Let $(a_p) \in \mathcal{T}_z$. Let H be the associated Hilbert space completion of $L\mathbb{C}^{n*}$. The polynomial loop group $L_{pol}\,\mathrm{Gl}_n(\mathbb{C})$ acts continuously on H and preserves the polarisation. Thus there is some r such that $L_r\,\mathrm{Gl}_n(\mathbb{C})$ acts continuously on H and preserves the polarisation.*

Proof. Let J be the polarising operator on H as defined in lemma A.13. Let $\mathcal{L}_{\mathcal{J}}(H)$ be the set of all bounded linear operators A on H such that $[A, J]$ is Hilbert-Schmidt. It is clear that $\mathrm{Gl}_{\mathcal{J}}(H) = \mathrm{Gl}(H) \cap \mathcal{L}_{\mathcal{J}}(H)$. The norm of an element $A \in \mathcal{L}_{\mathcal{J}}(H)$ is the sum of the operator norm of of A and the Hilbert-Schmidt norm of $[A, J]$.

There is an isometry $M_n(\mathbb{C}) \to \mathcal{L}(H)$ given by $A(a \otimes v) = a \otimes Av$, thinking of H as the completion of $L\mathbb{C}^* \otimes \mathbb{C}^n$. This maps continuously into $\mathcal{L}_{\mathcal{J}}(H)$ since $A \in M_n(\mathbb{C})$ commutes with J.

The operator z acts continuously on H and $[J, z]$ is finite rank. It therefore lies in $\mathcal{L}_{\mathcal{J}}(H)$. Thus the image of $L_{\mathrm{pol}} M_n(\mathbb{C})$ in $\mathcal{L}(H)$ lies in $\mathcal{L}_{\mathcal{J}}(H)$. Since this is a Banach space, if r is the norm of z in $\mathcal{L}_{\mathcal{J}}(H)$, the inclusion $L_r M_n(\mathbb{C}) \to \mathcal{L}_{\mathcal{J}}(H)$ is well-defined and continuous.

The result then follows from the fact that the image of $L_r \mathrm{Gl}_n(\mathbb{C})$ lies in the intersection of $\mathrm{Gl}(H)$ with $\mathcal{L}_{\mathcal{J}}(H)$. □

Proposition A.16 *The inclusion $L_r \mathrm{Gl}_n(\mathbb{C}) \to \mathrm{Gl}_{\mathcal{J}}(H)$ is homotopic to the standard inclusion which factors through $L\,\mathrm{Gl}_n(\mathbb{C})$.*

Proof. Let $T : H \to L^2(S^1, \mathbb{C}^n)^*$ be the isometry which takes e_p to $\sqrt{a_p} e_p$. This identifies $\mathrm{Gl}_{\mathcal{J}}(H)$ with $\mathrm{Gl}_{\mathcal{J}}(L^2(S^1, \mathbb{C}^n)^*)$ and so defines the map $L_r \mathrm{Gl}_n(\mathbb{C}) \to \mathrm{Gl}_{\mathcal{J}}(L^2(S^1, \mathbb{C}^n)^*)$.

Let $\zeta_t : L^2(S^1, \mathbb{C}^n)^* \to L^2(S^1, \mathbb{C}^n)^*$ be the map defined by $\zeta_t(e_p) = a_p^{t/2} e_{p-1}$. As a_p is positive for all p, this is well-defined. The map ζ_0 is the (adjoint of the) map z. The map ζ_1 is the map $T^{-1} z T$. Therefore the two inclusions of $L_r \mathrm{Gl}_n(\mathbb{C}^n)$ are $\sum z^q A_q \to \sum \zeta_0^q A_q$ and $\sum z^q A_q \to \sum \zeta_1^q A_q$. The required homotopy is $F(\sum z^q A_q, t) = \sum \zeta_t^q A_q$. □

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