MORE THAN EVERYTHING YOU WANT TO KNOW ABOUT CENTROIDS OF LIE ALGEBRAS

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I. Background and Motivation

$\mathcal{A}$: an algebra over $\mathbb{F}$

$$\mathrm{Cent}(\mathcal{A}) := \{\phi \in \mathrm{End}_{\mathbb{F}}(\mathcal{A}) \mid \phi(ab) = a\phi(b) = \phi(a)b \;\; \forall a, b \in \mathcal{A}\}$$

is the **centroid** of $\mathcal{A}$

The centroid is useful

- to construct derivations
- to study forms of algebras
  (e.g. for the Brauer group & division algebras for associative theory)

A Helpful Little Lemma

LEMMA If

- $\mathcal{A}$ is an algebra over $\mathbb{F}$
- $\mathcal{B}$ is a $\mathrm{Cent}(\mathcal{A})$-invariant ideal of $\mathcal{A}$ with $\mathrm{Cent}(\mathcal{B}) = \mathbb{F}\mathrm{id}$,

then

$$\mathrm{Cent}(\mathcal{A}) \quad = \quad \mathbb{F}\mathrm{id} \oplus \{\chi \in \mathrm{Cent}(\mathcal{A}) \mid \chi(\mathcal{B}) = 0\} \qquad \text{so}$$

$$\mathbf{Cent}(\boldsymbol{\mathcal{A}}) \quad \cong \quad \mathbb{F}\mathbf{id} \oplus \mathbf{Hom}_{\boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{B}}}\big(\mathcal{A}/\mathcal{B}, \mathbf{Ann}_{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{B}})\big)$$

($\mathrm{Ann}_{\mathcal{A}}(\mathcal{B})$ is an $\mathcal{A}/\mathcal{B}$-module under $(a + \mathcal{B})z = az$)

The Centroid of a Lie Algebra

$\mathcal{L}$ a Lie algebra over $\mathbb{F}$

$$\begin{aligned}\mathrm{Cent}(\mathcal{L}) &= \{\phi \in \mathrm{End}_{\mathbb{F}}(\mathcal{L}) \mid \phi([x,y]) = [x,\phi(y)] \ \ \forall x,y \in \mathcal{L}\} \\ &= \mathrm{End}_{\mathcal{L}}(\mathcal{L})\end{aligned}$$

Motivation for our investigations of the centroid:

$\mathcal{E}$: extended affine Lie algebra with core $\mathcal{K}$

(à la Allison, Azam, Berman, Gao, Pianzola)

$\mathcal{E}/\mathcal{K} \xleftrightarrow{\textbf{nondegenerately paired}} Z(\mathcal{E})$ (centre of $\mathcal{E}$)

$\mathcal{E}/\mathcal{K}$ built from $\mathrm{Cent}\big(\mathcal{K}/Z(\mathcal{K})\big)$

Our joint work (#0502561) with E. Neher

describes $\mathrm{Cent}(\mathcal{L})$ for $\mathcal{L}$ that are

(a) Lie algebras with a toral subalgebra (e.g. Kac-Moody algs.)

(b) Extended affine Lie algebras

(c) Root-graded Lie algebras

(d) Loop-like algebras

II. Lie Algebras $\mathcal{L}$ with Toral Subalgebras $\mathfrak{h}$

$\mathcal{L} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathcal{L}_\alpha$ where $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h, x] = \alpha(h)x \ \ \forall\, h \in \mathfrak{h}\}$

$\phi \in \mathrm{Cent}(\mathcal{L}) \Longrightarrow \phi(\mathcal{L}_\alpha) \subseteq \mathcal{L}_\alpha$ and $\phi(\mathfrak{h}) \subseteq Z(\mathcal{L}_0)$

THM. Let $\mathcal{B}$ be a $\mathrm{Cent}(\mathcal{L})$-invariant ideal such that

- $\dim(\mathcal{B} \cap \mathcal{L}_\alpha) = 1$ for some α and
- the ideal of $\mathcal{L}$ generated by $\mathcal{B} \cap \mathcal{L}_\alpha$ is $\mathcal{B}$.

Then (a) $\phi\,|_{\mathcal{B}} \in \mathbb{F}\mathrm{id}_{\mathcal{B}} \ \ \forall\, \phi \in \mathrm{Cent}(\mathcal{L})$

(b) $\mathbf{Cent}(\boldsymbol{\mathcal{L}}) = \mathbb{F}\mathbf{id} \oplus \mathbf{Hom}_{\boldsymbol{\mathcal{L}/\mathcal{B}}}\big(\boldsymbol{\mathcal{L}/\mathcal{B}}, \mathbf{C}_{\boldsymbol{\mathcal{L}}}(\boldsymbol{\mathcal{B}})\big)$

Application to Kac-Moody Lie Algebras

$\mathfrak{g}$: Kac-Moody Lie algebra with indecomposable Cartan matrix $\mathfrak{A}$

generators: e_i, f_i, h_i

relations: from $\mathfrak{A}$

$\mathcal{B} := \mathfrak{g}^{(1)} \Longrightarrow \dim\left(\mathcal{B} \cap \mathfrak{g}_{\alpha_i}\right) = 1$ for $\mathfrak{g}_{\alpha_i} = \mathbb{F}e_i$

THM. $\Longrightarrow$ $\mathbf{Cent(\mathfrak{g}) = \mathbb{F}id \oplus Hom_{\mathbb{F}}\left(\mathfrak{g}/\mathfrak{g}^{(1)}, C_{\mathfrak{g}}(\mathfrak{g}^{(1)})\right)}$

Special cases:

$\mathfrak{A}$ invertible $\Longrightarrow \mathfrak{g} = \mathfrak{g}^{(1)} \Longrightarrow \mathrm{Cent}(\mathfrak{g}) = \mathbb{F}\mathrm{id}$

$\mathfrak{A}$ affine $\Longrightarrow \mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathbb{F}d = \left(\mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]\right) \oplus \mathbb{F}c \oplus \mathbb{F}d$

$\Longrightarrow \mathrm{Cent}(\mathfrak{g}) = \mathbb{F}\mathrm{id} \oplus \mathrm{Hom}_{\mathbb{F}}\left(\mathbb{F}d, \mathbb{F}c\right)$

Group-Graded Lie Algebras and a Little More

- $\mathcal{L} = \bigoplus_{\lambda\in\Lambda} \mathcal{L}^\lambda$ and $\Lambda = \mathrm{span}_{\mathbb{Z}}\{\lambda \mid \mathcal{L}^\lambda \neq 0\}$
- $(\,|\,)$ nondegenerate symmetric invariant bilinear form on $\mathcal{L}$
- $\mathtt{SDer}(\mathcal{L}) = \{\partial \in \mathtt{Der}(\mathcal{L}) \mid (\partial x \mid y) = -(x \mid \partial y) \quad \forall\, x, y \in \mathcal{L}\}$
- $\mathtt{grSDer}(\mathcal{L}) = \bigoplus_{\lambda\in\Lambda} \Big(\mathtt{SDer}(\mathcal{L})\Big)^\lambda$
- $\mathcal{S} = \bigoplus_{\lambda\in\Lambda} \mathcal{S}^\lambda$ a graded subspace of $\mathtt{grSDer}(\mathcal{L})$
- $\mathcal{S}^{\mathtt{gr}*} = \bigoplus_{\lambda\in\Lambda} \left(\mathcal{S}^\lambda\right)^*$
- $\sigma_{\mathcal{S}} : \mathcal{L} \times \mathcal{L} \to \mathcal{S}^{\mathtt{gr}*}$

$$\boldsymbol{\sigma_{\mathcal{S}}(x, y)(d) = (dx \mid y) \qquad \forall d \in \mathcal{S},\ x, y \in \mathcal{L}}$$

- $\mathcal{K} = \mathcal{L} \oplus \mathcal{S}^{\mathtt{gr}*}$ with $[x \oplus c, y \oplus c']_{\mathcal{K}} = [x, y] + \sigma_{\mathcal{S}}(x, y)$

$$\forall x, y \in \mathcal{L}, \text{ and } c, c' \in \mathcal{S}^{\mathtt{gr}*}$$

II. Extended Affine Lie Algebras

THM. (a) If $\mathcal{K} = \mathcal{L} \oplus \mathcal{S}^{\texttt{gr}*}$ is perfect, then $\mathrm{Cent}(\mathcal{K})^{\lambda} = 0 \quad \forall \lambda \neq 0$.

(b) If $\mathcal{K}$ is a **Lie torus**, then $\mathrm{Cent}(\mathcal{K}) = \mathbb{F}\mathrm{id}$.

THM. Assume

- $\mathcal{E}$ is a *tame extended affine* Lie algebra
- $\mathcal{K}$ is the ideal generated by $\mathcal{E}_{\alpha}$ with $(\alpha|\alpha) \neq 0$ (the **core**)

Then (a) $\mathcal{K} \cong \mathcal{L} \oplus \mathcal{S}^{\texttt{gr}*}$ where $\mathcal{L} = \mathcal{K}/Z(\mathcal{K})$

(b) $\mathcal{K}$ is a Lie torus

(c) Hence $\mathbf{Cent}(\boldsymbol{\mathcal{E}}) = \mathbb{F}\,\mathbf{id} \oplus \mathbf{Hom}_{\boldsymbol{\mathcal{E}/\mathcal{K}}}\big(\boldsymbol{\mathcal{E}/\mathcal{K}}, \boldsymbol{Z(\mathcal{K})}\big)$

($\textit{tame}$ means $\mathrm{C}_{\mathcal{E}}(\mathcal{K}) = Z(\mathcal{K})$)

Affine Lie Algebras Revisited

- $\mathcal{E} = \mathbf{g} = \mathbf{g}^{(1)} \oplus \mathbb{F}d = \big(\mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]\big) \oplus \mathbb{F}c \oplus \mathbb{F}d$

- $\mathcal{K} = \mathbf{g}^{(1)} = \big(\mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]\big) \oplus \mathbb{F}c$

- $Z(\mathcal{K}) = \mathbb{F}c$

- $\mathcal{E}/\mathcal{K} = \mathbb{F}d$

$\mathrm{Cent}(\mathcal{E}) = \mathbb{F}\mathrm{id} \oplus \mathrm{Hom}_{\mathcal{E}/\mathcal{K}}\big(\mathcal{E}/\mathcal{K}, Z(\mathcal{K})\big)$ says

$\mathrm{Cent}(\mathbf{g}) = \mathbb{F}\mathrm{id} \oplus \mathrm{Hom}_{\mathbb{F}}\big(\mathbb{F}d, \mathbb{F}c\big)$

III. Root-graded Lie Algebras

Δ: a finite irreducible root system (possibly nonreduced)

DEFN. A Lie algebra $\mathcal{L}$ of char. 0 is **graded by $\boldsymbol{\Delta}$** if

(i) $\mathcal{L}$ contains a fin. dim'l split simple Lie algebra
$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{g}}} \mathfrak{g}_\alpha$ (*the grading subalgebra*);

(ii) $\mathfrak{h}$ is a toral subalg. of $\mathcal{L}$ and $\mathcal{L} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathcal{L}_\alpha$ relative to $\mathfrak{h}$;

(iii) $\mathcal{L}_0 = \sum_{\alpha \in \Delta} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$;

(iv) $\Delta_{\mathfrak{g}} = \Delta$ if $\Delta \neq \mathrm{BC}_r$ and $\Delta_{\mathfrak{g}} = \mathrm{B}_r, \mathrm{C}_r$, or D_r if $\Delta = \mathrm{BC}_r$

Defn. due to Berman-Moody ('92) for Δ reduced

and to Allison-B-Gao ('02) for $\Delta = \mathrm{BC}_r$.

Structure of Root-graded Lie Algebras

For $\Delta \neq \mathrm{BC}_r$:

- $$\boldsymbol{\mathcal{L} = (\mathfrak{g} \otimes A) \oplus (W \otimes B) \oplus D}$$

$W = 0$ for simply-laced & W is "little adjoint" $\mathfrak{g}$-module otherwise

- $\boldsymbol{\mathfrak{a} = A \oplus B}$ is the *c*oordinate algebra

$\mathfrak{a}$ is assoc., alternative, Jordan, or structurable depending on Δ

- $\boldsymbol{D = \langle \mathfrak{a}, \mathfrak{a} \rangle}$ where $\langle \alpha, \beta \rangle(\gamma) = D_{\alpha,\beta}(\gamma)$ (inner derivation)

Examples of Root-graded Lie Algebras

- $\mathcal{L} = \mathrm{g}^{(1)} = \big(\mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]\big) \oplus \mathbb{F}c$ (*affine*)
- $\mathcal{L} = \big(\mathfrak{g} \otimes \mathbb{F}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\big) \oplus C$ (*toroidal*)
- $\mathcal{K}$ *the core of an extended affine Lie algebra*

Centroid of a Root-graded Lie Algebra

THM. If $\Delta \neq \mathrm{BC}_r$, then

$$\mathbf{Cent}(\boldsymbol{\mathcal{L}}) \cong \boldsymbol{\mathcal{Z}}_{\mathfrak{a}} \qquad \text{where}$$

$\mathcal{Z}_{\mathfrak{a}}$ is the set of elements $\mathfrak{z} \in Z(\mathfrak{a}) \cap A$ so

- $\langle \mathfrak{z}\alpha, \beta \rangle = \langle \alpha, \mathfrak{z}\beta \rangle \;\; \forall\, \alpha, \beta \in \mathfrak{a}$
- $\sum_t \langle \alpha_t, \beta_t \rangle = 0 \Longrightarrow \sum_t \langle \mathfrak{z}\alpha_t, \beta_t \rangle = 0$

Map is $\mathfrak{z} \mapsto \Psi_{\mathfrak{z}}$ where

(i) $\Psi_{\mathfrak{z}}(x \otimes a) = x \otimes \mathfrak{z}a$

(ii) $\Psi_{\mathfrak{z}}(w \otimes b) = w \otimes \mathfrak{z}b$

(iii) $\Psi_{\mathfrak{z}}(\langle \alpha, \beta \rangle) = \langle \mathfrak{z}\alpha, \beta \rangle = \langle \alpha, \mathfrak{z}\beta \rangle$

Remark. $\langle \mathfrak{a}, \mathfrak{a} \rangle = D_{\mathfrak{a},\mathfrak{a}} \Longrightarrow \mathrm{Cent}(\mathcal{L}) = \mathcal{Z}_{\mathfrak{a}} = Z(\mathfrak{a}) \cap A.$
This is the case that $\mathcal{L}$ *is centreless.*

Affine Lie Algebras Re-Revisited

- $\mathcal{L} = \mathsf{g}^{(1)} = \big(\mathfrak{g} \otimes \mathbb{F}[t, t^{-1}]\big) \oplus \mathbb{F}c \quad (\textit{affine})$

- $\langle t^m, t^n \rangle = m\delta_{m,-n}c$

$\mathfrak{z} \in \mathcal{Z}_{\mathfrak{a}} = Z(\mathfrak{a}) \cap A = \mathbb{F}[t, t^{-1}]$ and $\langle \mathfrak{z}a, a' \rangle = \langle a, \mathfrak{z}a' \rangle$

From $\langle t^p t^m, t^n \rangle = \langle t^m, t^p t^n \rangle$ get

$(m + p)\delta_{m+p,-n} = m\delta_{m,-p-n}$, which implies $p = 0$

Thus, $\mathfrak{z} \in \mathbb{F}1$

$\mathrm{Cent}(\mathsf{g}^{(1)}) = \mathbb{F}\mathrm{id}$

Centroid of $\mathcal{A} \otimes \mathcal{B}$

$\mathrm{End}_{\mathbb{F}}(\mathcal{A}) \otimes \mathrm{End}_{\mathbb{F}}(\mathcal{B}) \to \mathrm{End}_{\mathbb{F}}(\mathcal{A} \otimes \mathcal{B})$,

where $f \otimes g \mapsto f \widetilde{\otimes} g$ and $(f \widetilde{\otimes} g)(a \otimes b) := f(a) \otimes g(b)$.

PROP.

(a) If $\mathcal{A}$ is perfect and $\mathcal{B}$ is unital, then

$$\mathrm{Cent}(\mathcal{A}) \otimes \mathrm{Cent}(\mathcal{B}) \cong \mathrm{Cent}(\mathcal{A}) \widetilde{\otimes} \mathrm{Cent}(\mathcal{B}) \subseteq \mathrm{Cent}(\mathcal{A} \otimes \mathcal{B})$$

(b) If also, either

(i) $\mathcal{A}$ is a finitely-generated $\mathrm{Mult}(\mathcal{A})$-module **OR**

(ii) $\mathrm{Cent}(\mathcal{A}) = \mathbb{F}\mathrm{id}$, then

$$\mathbf{Cent}(\mathcal{A} \otimes \mathcal{B}) \cong \mathbf{Cent}\,(\mathcal{A}) \otimes \mathbf{Cent}(\mathcal{B})$$

(Allison-Berman-Pianzola '04)

Prop. when $\mathcal{A}$ is fin.gen. Mult($\mathcal{A}$)-module and $\mathcal{B}$ is unital comm. assoc.

Centroids and Forms

COR. If

- $\mathcal{A}$ is perfect and $\operatorname{Cent}(\mathcal{A}) = \mathbb{F}\mathrm{id}$
- $\mathcal{B}$ is unital commutative associative
- $\mathcal{C}$ is unital subalgebra of $\mathcal{B}$ and

  $\mathcal{B}$ is a free $\mathcal{C}$-module with $\mathcal{C}$-basis containing 1
- $\mathcal{L} \otimes_{\mathcal{C}} \mathcal{B} = \mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$, *i.e.* $\mathcal{L}$ *is a* $\mathcal{C}$*-form of* $\mathcal{A} \otimes_{\mathbb{F}} \mathcal{B}$,

then $\boldsymbol{\operatorname{Cent}(\mathcal{L}) = \mathcal{C}\ \mathrm{id}}$

Centroid of an Twisted Loop Algebras

Example:

- $\mathcal{A}$: Lie algebra with automorphism σ of period m
- $\mathcal{A}_i := \{a \in \mathcal{A} \mid \sigma(a) = \zeta^i a\}$, ζ primitive mth root of 1
- $\mathcal{C} := \mathbb{F}[t^m, t^{-m}] \subset \mathcal{B} = \mathbb{F}[t, t^{-1}]$

(Allison-Berman-Pianzola '04)

$\mathcal{L}(\mathcal{A}, \sigma) := \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \otimes t^i$ is $\mathcal{C}$-form of $\mathcal{A} \otimes \mathcal{B}$.

So, $\mathrm{Cent}\big(\mathcal{L}(\mathcal{A}, \sigma)\big) = \mathcal{C}\,\mathrm{id}$