BIMODULES OVER SIMPLE FINITE-DIMENSIONAL

JORDAN SUPERALGEBRAS

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SUPERALGEBRA : $A = A_{\bar{0}} + A_{\bar{1}}$, $A_{\bar{i}} \cdot A_{\bar{j}} \subseteq A_{\bar{i+j}}$ a $Z/2Z$-graded algebra

EX. V vector space of countable dimension,

$G(V) = G(V)_{\bar{0}} + G(V)_{\bar{1}}$ Grassmann algebra over V,

$G(A) = A_{\bar{0}} \otimes G(V)_{\bar{0}} + A_{\bar{1}} \otimes G(V)_{\bar{1}} \leq A \otimes G(V)$ *Grassmann enveloping algebra of A*

$\mathcal{V}$ a variety of algebras (associative, Lie, Jordan,...)

DEF. $A = A_{\bar{0}} + A_{\bar{1}}$ is a $\mathcal{V}$-superalgebra if $G(A) \in \mathcal{V}$.

$J = J_{\bar{0}} + J_{\bar{1}}$ is a Jordan superalgebra if it satisfies

SJ1. *Supercommutativity* $a \cdot b = (-1)^{|a||b|} b \cdot a$,

SJ2. *Super Jordan identity*

$(a \cdot b) \cdot (c \cdot d) + (-1)^{|b||c|}(a \cdot c) \cdot (b \cdot d)+$

$(-1)^{|b||d|+|c||d|}(a \cdot d) \cdot (b \cdot c) =$

$((a \cdot b) \cdot c) \cdot d + (-1)^{|c||d|+|b||c}((a \cdot d) \cdot c) \cdot b+$

$(-1)^{|a||b|+|a||c|+|a||d|+|c||d|}((b \cdot d) \cdot c) \cdot a.$

JORDAN SUPERALGEBRAS

$A = A_{\bar{0}} + A_{\bar{1}}$ associative superalgebra

$A^{(+)} = (A, a \cdot b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ Jordan superalgebra

$J = J_{\bar{0}} + J_{\bar{1}} \leq A^{(+)}$ **special**. Otherwise **exceptional**

(A) $A^{(+)}$, $A = M_{m+n}(F)$ *full linear superalgebra*

(Q) $A^{(+)}$, $A = \{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \mid a, b \in M_n(F)\}$

If $\star : A \to A$ is an involution : $(a^\star)^\star = a$, $(ab)^\star = (-1)^{|a||b|}b^\star a^\star$.

$H(A, \star) = \{a \in A | a^\star = a\} \leq A^{(+)}$

(BC) $M_{m+2n}(F)$, $Q = \begin{pmatrix} I_m & 0 \\ 0 & S_{2n} \end{pmatrix}$,

$$S_{2n} = \begin{pmatrix} 0 & 1 & . & . & . \\ -1 & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & 0 & 1 \\ . & . & . & -1 & 0 \end{pmatrix}$$

$\star : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to Q^{-1} \begin{pmatrix} a^T & -c^T \\ b^T & d^T \end{pmatrix} Q$, $a \in M_m(F)$, $d \in M_{2n}(F)$,

$H(A, \star) = \mathbf{osp_{m,2n}(F)}$.

(P) $A = M_{n+n}(F)$, $\star : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d^T & -b^T \\ c^T & a^T \end{pmatrix}$,

$H(A, \star) = \{ \begin{pmatrix} a & b \\ c & a^T \end{pmatrix} \mid a, b, c \in M_n(F),\ b^T = -b,$

$c^T = c\}$.

(D) *A Superalgebra of a superform*

$V = V_{\bar{0}} + V_{\bar{1}}$, $<,>: V \times V \to F$ a supersymmetric bilinear form

$J = F1 + V = (F1 + V_{\bar{0}}) + V_{\bar{1}}$, $(\alpha 1 + v)(\beta 1 + w) = (\alpha\beta + < v, w >)1 + (\alpha w + \beta v)$.

($\mathbf{D_t}$) $J_t = (Fe_1 + Fe_2) + (Fx + Fy)$, $t \neq 0$

$e_i^2 = e_i, e_1 e_2 = 0, e_i x = \frac{1}{2} x, e_i y = \frac{1}{2} y, [x, y] = e_1 + t e_2.$

(J) All simple Jordan algebras

(F) The 10-dimensional exceptional Kac superalgebra

$$K_{10} = [(Fe_1 + \sum_{i=1}^{4} Fv_i) + Fe_2] + (\sum_{i=1}^{2} Fx_i + Fy_i)$$

$e_i^2 = e_i, \quad e_1e_2 = 0, \quad e_1v_i = v_i, e_2v_i = 0, v_1v_2 = 2e_1 = v_3v_4,$

$e_ix_j = \frac{1}{2}x_j, \quad e_iy_j = \frac{1}{2}y_j, \quad i,j = 1,2$

$y_1v_1 = x_2, \quad y_2v_1 = -x_1, \quad x_1v_2 = -y_2, \quad x_2v_2 = y_1,$

$x_2v_3 = x_1, \quad y_1v_3 = y_2, \quad x_1v_4 = x_2, \quad y_2v_4 = y_1,$

$[x_i, y_i] = e_1 - 3e_2, \quad [x_1, x_2] = v_1, \quad [y_1, y_2] = v_2,$

$[x_1, y_2] = v_3, \quad [x_2, y_1] = v_4.$

(K) The 3-dimensional Kaplansky superalgebra

$$K_3 = Fe + (Fx + Fy), \qquad e^2 = e, \quad ex = \frac{1}{2}x,$$

$ey = \frac{1}{2}y, \quad [x, y] = e.$

Theorem. *(Kac 77, Kantor 89) A simple finite dimensional Jordan superalgebra over an algebraically closed field of zero characteristic is isomorphic to one of the superalgebras A, BC, D, P, Q, D_t, F, K, J listed above or to a superalgebra obtained by the Kantor-double process*

Theorem. *(Racine, Zelmanov, J. of Algebra 270, 2003) Every simple Jordan superalgebra over an algebraically closed field F, ch$F = p > 2$, with its even part semisimple is isomorphic to one of the superalgebras mentioned above + Some additional examples in char 3*

Jordan Superalgebras defined by Brackets

$\Gamma = \Gamma_{\bar{0}} + \Gamma_{\bar{1}}$ an associative commutative superalgebra
$\{,\} : \Gamma \times \Gamma \to \Gamma$ a *Poisson bracket* if $\{\Gamma_{\bar{i}}, \Gamma_{\bar{j}}\} \subseteq \Gamma_{\overline{i+j}}$ and

(1) $(\Gamma, \{,\})$ is a Lie superalgebra,

(2) $\{ab, c\} = a\{b, c\} + (-1)^{|b||c|}\{a, c\}b$ (*Leibniz identity*)

Kantor Double Superalgebra

$J = \Gamma + \Gamma x, \quad a(bx) = (ab)x, \quad (bx)a = (-1)^{|a|}(ba)x,$
$(ax)(bx) = (-1)^{|b|}\{a, b\}, \quad J_{\bar{0}} = \Gamma_{\bar{0}} + \Gamma_{\bar{1}}x,\ J_{\bar{1}} = \Gamma_{\bar{1}} + \Gamma_{\bar{0}}x.$

Theorem. *(Kantor 1992) Let $\{,\}$ be a Poisson bracket* $\Longrightarrow \qquad J = \Gamma + \Gamma x$ *is a Jordan superalgebra.*

Kantor superalgebra

$\Gamma =$ Grassman algebra on $\xi_1, \ldots, \xi_n$
$\Gamma = \Gamma_{\bar{0}} + \Gamma_{\bar{1}}$, $\{f, g\} = \sum_{i=1}^{n} (-1)^{|f|} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}$

$$J = \Gamma + \Gamma x \begin{cases} n = 1 & J \simeq D(-1) \\ n \geq 2 & J_{\bar{0}} \quad \textit{is not semisimple} \end{cases}$$

CHENG-KAC JORDAN SUPERALGEBRAS

Z unital associative commutative algebra, $d : Z \to Z$ a derivation,

$CK(Z,d) = J_{\bar{0}} + J_{\bar{1}},\ J_{\bar{0}} = Z + \sum_{i=1}^{3} w_i Z,\ J_{\bar{1}} = xZ + \sum_{i=1}^{3} x_i Z$ free Z-modules of rank 4.

Even part $w_i w_j = 0, i \neq j, w_1^2 = w_2^2 = 1, w_3^2 = -1,$

Notation: $x_{i\times i} = 0, \quad x_{1\times 2} = -x_{2\times 1} = x_3 \qquad x_{1\times 3} = -x_{3\times 1} = x_2, \quad -x_{2\times 3} = x_{3\times 2} = x_1.$

Module action $f, g \in Z$

	g	$w_j g$
xf	$x(fg)$	$x_j(fg^d)$
$x_i f$	$x_i(fg)$	$x_{i\times j}(fg)$

Bracket on M

	xg	$x_j g$
xf	$f^d g - f g^d$	$-w_j(fg)$
$x_i f$	$w_i(fg)$	0

$CK(Z,d)$ is simple $\iff$ Z does not contain proper d-invariant ideals.

$$B(m) = F[a_1, \ldots, a_m \mid a_i^p = 0]$$

$$\mathbf{B(m,n) = B(m) \otimes G(n)} \qquad \mathbf{G(n) =< 1, \xi_1, \ldots, \xi_n >}$$

Theorem. *(M., Zelmanov, J. of Algebra 236, 2001)*

Let $J = J_{\bar{0}} + J_{\bar{1}}$ be a finite dimensional simple unital Jordan superalgebra over an algebraically closed field F, $\mathrm{ch}F = p > 2$, $J_{\bar{0}}$ not semisimple. Then

$\mathbf{J \simeq B(m,n) + B(m,n)x}$ a Kantor double or

$\mathbf{J \simeq CK(B(m), d)}$.

SPECIALITY

King, McCrimmon (J. Algebra 149, 1995)

- The Kantor Double of a bracket of vector field type ($\{a, b\} = a'b - ab'$ ′ a derivation) is special.

- The Kantor Double of $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$ on $F[x, y]$ is exceptional.

Shestakov (1993)

- A Kantor Double of Poisson bracket $<,>: \Gamma \times \Gamma \to \Gamma$ is special iff $<< \Gamma, \Gamma >, \Gamma >= (0)$.

- A Kantor Double of a Poisson bracket is i-special (homomorphic image of a special superalgebra)

Theorem. *(M., Shestakov, Zelmanov) A Kantor Double of a Jordan bracket is i-special.*

Assumption: $J = \Gamma + \Gamma x$ does not contain $\neq (0)$ nilpotent ideals

- If $\Gamma = \Gamma_{\bar{0}}$ then J is special iff $<,>$ is of vector field type.

- If $\Gamma_{\bar{1}}\Gamma_{\bar{1}} \neq (0)$ (at least 2 Grassmann variables) then J is exceptional.

- If $\Gamma = \Gamma_{\bar{0}} + \Gamma_{\bar{0}}\xi_1, < \Gamma_{\bar{0}}, \xi_1 >= (0), < \xi_1, \xi_1 >= -1$ then J is special iff $<,>$: $\Gamma_{\bar{0}} \times \Gamma_{\bar{0}} \to \Gamma_{\bar{0}}$ is of vector field type.

Theorem. *(M., Shestakov, Zelmanov) The Cheng-Kac superalgebra $CK(Z, d)$ is special*

The embedding extends McCrimmon embedding for vector field type brackets.

$W =< R(a), a \in Z, d >$ - differential operators on Z

$R = R_{\bar{0}} + R_{\bar{1}} = \mathcal{M}_{4\times 4}(W)$

Let J be a special Jordan superalgebra.

A specialization $u : J \longrightarrow U$ into an associative algebra U is said to be universal if $U =< u(J) >$ and for an arbitrary specialization $\varphi : J \to A$ there exists a homomorphism of associative algebras $\xi : U \to A$ such that $\varphi = \xi \cdot u$.

The algebra U is called the universal associative enveloping algebra of J.

An arbitrary special Jordan superalgebra contains a unique universal specialization $u : J \to U$.

U is equipped with a superinvolution * having all elements from $u(J)$ fixed, i.e., $u(J) \subseteq H(U, *)$.

We call a special Jordan superalgebra reflexive if $u(J) = H(U, *)$.

Theorem. $U(M_{m,n}^{(+)}(F)) \simeq M_{m,n}(F) \oplus M_{m,n}(F)$ *for* $(m, n) \neq (1, 1)$*;* $U(Q^{(+)}(n)) = Q(n) \oplus Q(n)$*,* $n \geq 2$*;* $U(osp(m, n)) \simeq M_{m,n}(F)$*,* $(m, n) \neq (1, 2)$*;* $U(P(n)) \simeq M_{n,n}(F)$*,* $n \geq 3$.

Theorem. *The embedding σ of the Cheng-Kac superalgebra is universal, that is, $U(CK(Z, D)) \cong M_{2,2}(W)$. The restriction of the embedding u (see above) to $P(2)$ is a universal specialization; $U(P(2)) \simeq M_{2,2}(F[t])$, where $F[t]$ is a polynomial algebra in one variable.*

The Jordan superalgebra of a superform

Let $V=V_{\bar{0}}+V_{\bar{1}}$ be a $Z/2Z$-graded vector space, dim $V_{\bar{0}}=m,\ dim V_{\bar{1}}=2m$; let $<,>: V\times V\to F$ be a supersymmetric bilinear form on V. The universal associative enveloping algebra of the Jordan algebra $F1+V_{\bar{0}}$ is the Clifford algebra $Cl(m)=<1,e_1,\dots,e_m|e_ie_j+e_je_i=0, i\neq j, e_i^2=1>$.

Consider the Weyl algebra $W_n=<1,x_i,y_i,1\leq i\leq n|[x_i,y_j]=\delta_{ij},[x_i,x_j]=[y_i,y_j]=0>$. Assuming $x_i,y_i,1\leq i\leq n$ to be odd, we make W_n a superalgebra. The universal associative enveloping algebra of $F1+V$ is isomorphic to the (super)tensor product $Cl(m)\otimes_F W_n$.

Specializations of $M_{1,1}(F)$

Theorem. $U(M_{1,1}(F))\simeq\begin{pmatrix} A & M_{12}\\ M_{21} & A\end{pmatrix}$. *The mapping*

$$u:\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}\to\begin{pmatrix}\alpha_{11} & \alpha_{12}+\alpha_{21}a^{-1}z_2\\ \alpha_{12}z_1+\alpha_{21}a & \alpha_{22}\end{pmatrix}$$

is a universal specialization.

Here a is root of the equation $a^2+a-z_1z_2=0$, $A=F[z_1,z_2]+F[z_1,z_2]a$ is a subring of K a quadratic extension of $F(z_1,z_2)$ generated by a and $M_{12}=F[z_1,z_2]+F[z_1,z_2]a^{-1}z_2$, $M_{21}=F[z_1,z_2]z_1+F[z_1,z_2]a$ are subspaces of K.

Let V be a Jordan bimodule over the (super)algebra J

V is a one-sided bimodule if $\{J, V, J\} = (0)$.

Then, the mapping $a \to 2R_V(a) \in End_F V$ is a specialization.

The category of one-sided bimodules over $J$is equivalent to the category of right (left) $U(J)$-modules.

Let e be the identity of J and let $V = \{e, V, e\} + \{1-e, V, e\} + \{1-e, V, 1-e\}$ be the Peirce decomposition. Then $\{e, V, e\}$ is a unital bimodule over J, that is, e is an identity of $\{e, V, e\} + J$. The component $\{1-e, V, e\}$ is a one-sided module, that is, $\{J, \{1-e, V, e\}, J\} = (0)$.

Finally, $\{1-e, V, 1-e\}$ is a bimodule with zero multiplication.

Remark One sided finite dimensional Jordan bimodules over $M_{1,1}(F)$ are not necessarily completely reducible.

Theorem. *(C.M. and I. Shestakov) If V is a unital bimodule over $M_{1,1}(F) \simeq D_{-1}$ and v is an element in $\{e_1, V, e_1\}$ (similarly in $\{e_1, V, e_1\}$) then the linear span of $v, w = (vx)y, vx, vy$ is a subbimodule of V.*

The multiplication is given by:

$e_1 v = v$, $e_2 v = 0$, vx, vy

$e_1 w = (\frac{1}{2} + \gamma)v$, $e_2 w = w + (\frac{-1}{2} + \gamma)v$, $wx = 2\gamma vx - \alpha vy$, $wy = \beta vx$.

$e_1 vx = \frac{1}{2}vx$, $e_2 vx = \frac{1}{2}vx$, $(vx)x = \alpha v$, $(vx)y = w$

$e_1 vy = \frac{1}{2}vy$, $e_2 v = \frac{1}{2}vy$, $(vy)x = 2\gamma v - w$, $(vy)y = \beta v$

with α, β and γ elements in F.

- If $\alpha\beta + \gamma^2 - \frac{1}{4} = 0$ the previous module is indecomposable, but not irreducible.

- A unital irreducible bimoduleover D_{-1} has either dimension 4 or dimension 2

Specializations of superalgebras $D(t)$

Clearly, $D(-1) \cong M_{1,1}(F)$, $D(0) \cong K_3 \oplus F1$, $D(1)$ is a Jordan superalgebra of a superform.

Let $osp(1,2)$ denote the Lie subsuperalgebra of $M_{1,2}(F)$ which consists of skewsymmetric elements with respect to the orthosympletic superinvolution. Let x, y be the standard basis of the odd part of $osp(1,2)$.

Theorem. *(I. Shestakov) The universal enveloping algebra of K_3 is isomorphic to $U(osp(1,2)/id([x,y]^2 - [x,y]))$, where $U(osp(1,2))$ is the universal associative enveloping algebra of $osp(1,2)$ and $id([x,y]^2 - [x,y]))$ is the ideal of $U(osp(1,2))$ generated by $[x,y]^2 - [x,y]$.*

Clearly, if $\mathrm{ch}F = 0$ then K_3 does not have nonzero specializations that are finite dimensional algebras. If $\mathrm{ch}F = p > 0$ then K_3 has such specializations. For example, $K_3 \subseteq CK(F[a|a^p = 0], d/da)$.

$t \neq -1, 0, 1$

Theorem. *(I. Shestakov) The universal enveloping algebra of $D(t)$ is isomorphic to $U(osp(1,2)/id([x,y]^2 - (1+t)[x,y] + t)$.*

Corollary. *If $chF = 0$ then all finite dimensional one-sided bimodules over $D(t)$ are completely reducible.*

Theorem. *(C.M and E. Zelmanov). Let $chF = 0$. Then:*

a) If $\frac{t=-m}{m+1}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $V_1(t)$ and $V_1(t)^{op}$.

b) If $\frac{t=-m+1}{m}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $V_2(t)$ and $V_2(t)^{op}$.

c) If t can not be represented as $\frac{-m}{m+1}$ or $\frac{-m+1}{m}$, where m is a positive integer, then $D(t)$ does not have nonzero finite dimensional specializations.

Let $V = V_{\bar{0}} + V_{\bar{1}}$ be a finite dimensional irreducible right module over the associative superalgebra $U(D(t))$. The elements $E = \frac{2}{1+t}x^2$, $F = -\frac{2}{1+t}y^2$ and $H = -\frac{2}{1+t}(xy + yx)$ span the Lie algebra sl_2.

Let's consider an infinite dimensional Verma type module $\tilde{V} = \tilde{v}U(D(t))$ defined by one generator $\tilde{v}$ and the relations: $\tilde{v}H = \lambda\tilde{v}$, $\tilde{v}e_1 = \tilde{v}$, $\tilde{v}y^2 = 0$.

Then the system of relators of $\tilde{V}$: $\tilde{v}e_1 - \tilde{v} = 0$, $\tilde{v}y^2 = 0$, $\tilde{v}yx - t\tilde{v} = 0$ + relators of $D(t)$: $e_1^2 - e_1 = 0$, $xe_1 + e_1x - x = 0$, $ye_1 + e_1y - y = 0$, $xy - yx - t - (1-t)e_1 = 0$. Hence the irreducible elements $\tilde{v}, \tilde{v}y, \tilde{v}x, i \geq 1$ form a basis of this module that we will denote as $\tilde{V}_1(t)$.

If $\tilde{v}y = 0$ then the irreducible elements $\tilde{v}, \tilde{v}x^i, i \geq 1$ form a basis of the module that will be denoted as $V_2(t)$.

Changing parity we get two new bimodules $\tilde{V}_1(t)^{op}$ and $\tilde{V}_2(t)^{op}$.

Unital bimodules over D_t char$F = 0$

Maria Trushina (also in case char$F = p$)

Definition. *For $\sigma \in \{\bar{0}, \bar{1}\}$, $i \in \{0, 1, \frac{1}{2}\}$, $\lambda \in F$, a Verma module $V(\sigma, i, \lambda)$ is a unital D_t-bimodule presented by one generator v of parity σ and the relations: $vR(e_1) = iv$, $vR(y) = 0$, $vH = \lambda v$.*

Notice that $V(\sigma, i, \lambda)^{op} = V(1 - \sigma, i, \lambda)$

1. For an arbitrary $\lambda \in F$, $V(\sigma, \frac{1}{2}, \lambda) \neq (0)$.

2. $V(\sigma, 1, \lambda) = (0)$ unless $\lambda = \frac{-2}{t+1}$.

3. $V(\sigma, 0, \lambda) = 0$ unless $\lambda = \frac{-2t}{t+1}$.

4. $V(\sigma, 1, -\frac{2}{t+1}) \neq (0)$.

5. $V(\sigma, 0, -\frac{2t}{1+t}) \neq (0)$.

6. Every nonzero Verma bimodule $V(\sigma, i, \lambda)$ contains a largest proper subbimodule $M(\sigma, i, \lambda)$. Hence there exists a unique irreducible D_t-bimodule

$$Irr(\sigma, i, \lambda) = V(\sigma, i, \lambda)/M(\sigma, i, \lambda)$$

generated by an element of the highest weight λ.

7. Every finite dimensional irreducible D_t-bimodule is isomorphic to $Irr(\sigma, i, \lambda)$ for some σ, i, λ.

Theorem. *If $t \neq -1$ is not of the type $-\frac{m}{m+2}$, $m \geq 0$; $-\frac{m+2}{m}$, $m \geq 1$; or 1, then the only unital finite dimensional irreducible D_t-bimodules are*

$$Irr(\sigma, \frac{1}{2}, m),\ m \geq 1 \qquad (*).$$

If $t = 1$ then add the one dimensional bimodules $Irr(\sigma, \frac{1}{2}, 0)$, $\sigma = \bar{0}, \bar{1}$ to the series $(*)$.

If $t = -\frac{m+2}{m}$, $m \geq 1$, then add the bimodules $V(\sigma, 1, m)$, $\sigma = \bar{0}, \bar{1}$ to $(*)$.

If $t = -\frac{m}{m+2}$, $m \geq 0$, then add the bimodules $V(\sigma, 0, m)$, $\sigma = \bar{0}, \bar{1}$ to $(*)$.

Corollary. *The only finite dimensional irreducible bimodules of the (nonunital) Kaplansky superalgebra* K_3 *are* $Irr(\sigma, \frac{1}{2}, m),\ m \geq 1$ *and* $Irr(\sigma, 0, 0)$. *We have:*

$$\dim Irr(\sigma, \tfrac{1}{2}, 1) = 3,\ \dim Irr(\sigma, 0, 0) = 1,$$

$$\dim Irr(\sigma, \tfrac{1}{2}, m) = 4m \text{ if } m \geq 2.$$

Let V' denote the sub-bimodule of $V(\sigma, i, m)$ generated by $vR(x)^{2m+1}$. The quotient module $W(\sigma, i, m) = V(\sigma, i, m)/V'$ is finite dimensional

Theorem. *Suppose that* t *is not of the type* $-\frac{m}{m+2}$, $-\frac{m+2}{m}$, 0, 1, -1, m *positive integer. Then every finite dimensional unital bimodule* V *over* D_t *is completely reducible.*

Theorem. *If* $t = -\frac{m+1}{m-1}$ *or* $t = -\frac{m-1}{m+1}$, $m \geq 2$, *then* $W(\sigma, \frac{1}{2}, m)$, $\sigma = \bar{0}$ *or* $\bar{1}$, *are the only finite dimensional indecomposible* D_t*-bimodules, which are not irreducible.*

Bimodules over P(n)

Let us notice that $P(n)$ consists of the symmetric elements of $Mn, n(F)$ with respect to the following superinvolution:

$$\star : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}$$

Examples of $JP(n)$ unital bimodules are:

1. The regular bimodule $R = JP(n)$,

2. The bimodule of skewsymmetric elements $S = Skew(M_{n,n}(F), \star) = \{\begin{pmatrix} a & h \\ k & -a^t \end{pmatrix} \mid a \in M_n(F),\ h^t = h,\ k^t = -k\}$

Theorem. *If $n \geq 3$, then every unital bimodule over $JP(n)$ is completely reducible. The only irreducible bimodules are $S = Skew(M_{n,n}(F), \star)$ and the regular one R (and the opposite ones).*

Remark If $n = 2$ not every module is completely reducible. Cheng- Kac is an indecomposable bimodule that is not irreducible.

Bimodules over $M_{n,m}(F)^{(+)}$

Definition. *Let $A = A_{\bar 0} + A_{\bar 1}$ an associative superalgebra. A graded mapping $\star : A \to A$ is a "pseudoinvolution" if $(a^\star)^\star = (-1)^{|a|}a$, $(ab)^\star = (-1)^{|a||b|}b^\star a^\star$ for homogeneous elements a, b.*

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\star = \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix}$ is a pseudoinvolution in $A(n,m)$

If $\star : A \to A$ is a pseudoinvolution then

$$\star : A^{(+)} \to M_2(A)^{(+)}$$

$$a \to \begin{pmatrix} a & 0 \\ 0 & -a^\star \end{pmatrix}$$

is an embedding of Jordan superalgebras

- If W is a subspace of A satisfying

$$aw + (-1)^{|a||w|} w a^\star \in W \ \forall w \in W, \ \forall a \in A$$

then $W^{up} = \begin{pmatrix} 0 & W \\ 0 & 0 \end{pmatrix}$ is a Jordan module over $A^{(+)} \simeq \{\begin{pmatrix} a & 0 \\ 0 & -a^\star \end{pmatrix}\}$.

- If If W is a subspace of A satisfying

$$a^\star w + (-1)^{|a||w|} wa \in W \ \forall w \in W, \ \forall a \in A$$

then $W^{down} = \begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}$ a Jordan module over $A^{(+)} \simeq \{\begin{pmatrix} a & 0 \\ 0 & -a^\star \end{pmatrix}\}$.

Examples of modules over $A(m,n)$

1. The regular bimodules R

2. $W_1^{down} = \{\begin{pmatrix} K_n & b \\ -b^t & H_m \end{pmatrix} \ |b \in M_{n\times m}(F)\}$

3. $W_2^{down} = \{\begin{pmatrix} H_n & b \\ -b^t & K_m \end{pmatrix} \ |b \in M_{n\times m}(F)\}$

4. $W_1^{up} = \{\begin{pmatrix} H_n & b \\ -b^t & K_m \end{pmatrix} \ |b \in M_{n\times m}(F)\}$

5. $W_2^{up} = \{\begin{pmatrix} K_n & b \\ -b^t & H_m \end{pmatrix} \ |b \in M_{n\times m}(F)\}$

Theorem. *Every* $A(n, m)$*- bimodule (* $n \geq 2$, $n \geq m$*) is completely reducible. There exist, up to opposite, five unital bimodules over* $A(n, m)$ *that are the given above*

Remark

Unital bimodules over Poisson brackets superalgebras have been studied by A. Stern.

- Every finitely generated bimodule is finite dimensional

- If J has $n > 4$ generators, then every irreducible J-bimodule is either isomorphic to the regular one or to its opposite.

- The same results for unital irreducible bimodules over K_{10}