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Basics of Equivariant Cohomology.

$X \rightsquigarrow H^*(X)$
smooth cx proj. variety

$G \circlearrowright X \rightsquigarrow H^*_G(X)$ moral: although this is bigger, maybe easier to compute.

First definition of $H^*_G(X)$

Take EG a contractible space with a free G-action
$$\begin{array}{c} EG \\ \downarrow \\ BG = EG/G \end{array}$$

Consider $G \circlearrowright EG \times X$ by $g\cdot(p,x) = (gp, g^{-1}x)$

Define $X_G := EG \times X / G = EG \times_G X$

Then $H^*_G(X) := H^*(X_G)$ [indep. of choice of EG]

$\boxed{Qs:}$ - What is EG?
- properties of EG?
~~how to compute?~~

EG :

FACT: For $n>0$, if the principle G-bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ is n-connected

$(S^m \to E$ contractible for $m \le n)$

Then $H^m_G(X) = H^m(E \times_G X)$ for $m \le n$, $\dim X \le n$.

Ex: $G = S^1$

G acts freely on $S^{2n-1} \subseteq \mathbb{C}^n$
with $S^{2n-1}/S^1 \cong \mathbb{C}P^{n-1}$

$\begin{array}{c} S^{2n-1} \\ \downarrow \\ \mathbb{C}P^{n-1} \end{array}$ is $(2n-2)$-connected

$$\left.\begin{array}{l} \text{Taking } ES^1 = \lim S^{2n-1} = S^\infty \\ \qquad\quad\; BS^1 = \lim \mathbb{C}P^{n-1} = \mathbb{C}P^\infty \end{array}\right\} \text{so } H^*_{S^1}(pt) = \mathbb{R}[t],\ t = \deg 2.$$

Similarly $BT = (\mathbb{C}P^\infty)^m$ for $T = (S^1)^m$

$$H^*_T(pt) = H^*(BT) = \mathbb{R}[t_1, \ldots, t_n] \quad \deg t_i = 2.$$

... can get a lot of mileage out of this!

__Proposition:__ G compact conn. Lie gp, $T \subset G$ max'l torus, $W = N(T)/T$, $G \circlearrowright X$.

Then $H^*_G(X) = H^*_T(X)^W$.

We have projections

$$\begin{tikzcd} EG \arrow[d] & EG \times X \arrow[l] \arrow[r] \arrow[d] & X \arrow[d] \\ BG & X_G \arrow[l, "\pi"'] \arrow[r, "\sigma"'] & X/G \end{tikzcd}$$

FACTS:

- we get a homomorphism $\pi^*: H^*_G(pt) \longrightarrow H^*_G(X)$, where $H^*_G(pt) = H^*(BG)$ and $H^*_G(X) = H^*(X_G)$,

  which gives $H^*_G(X)$ the structure of a $H^*_G(pt)$-module.
- For nice cases,

  $$H^*_T(X) \cong H^*(X) \otimes H^*_T(pt) \quad (\text{as v. sp.})$$

  and can recover $H^*(X)$ by setting $t_1 = \ldots = t_n = 0$.
- The fiber $\sigma^{-1}(Gx) \simeq BG_x$ where G_x is the stabilizer of x in G.

  In particular, if G acts freely on X, then the homomorphism

  $$\sigma^*: H^*(X/G) \longrightarrow H^*_G(X) \quad \text{is an iso}^{\underline{m}}.$$

  [in fact the whole point of the $EG \times X/G$ construction is to "free up" the action.]
- Also by inclusion of fibers $X \overset{i}{\hookrightarrow} X_G \to BG$, get $i^*: H^*_G(X) \longrightarrow H^*(X)$

... get a lot of properties on $H^*_T(-)$.

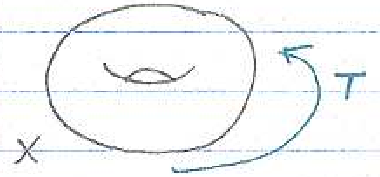
Second definition (via equivariant homology):

Def^n: An equivariant i-cycle is a free G-space E, $P = E/G$ of dim i pseudomfd and a diagram

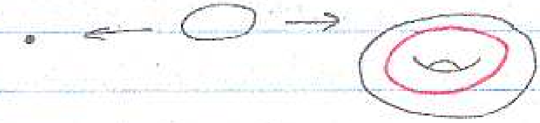
$$\begin{tikzcd} & E \arrow[dl] \arrow[dr] & \\ P & & X \end{tikzcd}$$

respecting G.

Ex: $X = S^1 \times S^1$
$T = S^1$.

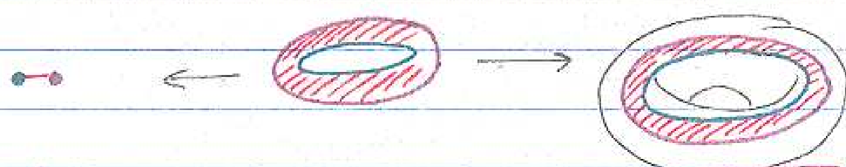


0-cycle



1-cycle

Cobordism between two 0-cycles:



Def^n: $\boxed{H_i^G(X) := \{\text{cobordism classes of eqvt } i\text{-cycles}\}}$

abelian gp: $-P$ has opposite orientation
$P_1 + P_2$ union.

so then $H_0^{S^1}(S^1 \times S^1) = \mathbb{R}$
$H_1^{S^1}(S^1 \times S^1) = \mathbb{R}$ } and that's all you have.

NOTE: This was a free action,

$$H_*^{S^1}(S^1 \times S^1) \cong H_*(S^1 \times S^1 / S^1 = S^1). \checkmark$$

(as expected)

Def^n: As a v.s. $\boxed{\begin{aligned} H_G^i(X) &= \text{v.s. dual } (H_i^G(X))^* \\ &= \mathrm{Hom}(H_i^G(X), \mathbb{R}) \end{aligned}}$.

with multiplication given by dualizing:

$$H_i^G(X) \otimes H_j^G(X) \xrightarrow[\cong]{} H_{i+j}^{G\times G}(X \times X) \xleftarrow[\Delta]{} H_{i+j}^G(X)$$

$$(g,g) \leftarrow\!\shortmid g$$

$$(x,x) \leftarrow\!\shortmid x$$

<u>Ex</u>: $X = pt$, $T = S^1$

2k-cycle $\quad \mathbb{C}P^k \leftarrow S^{2k-1} \longrightarrow pt$

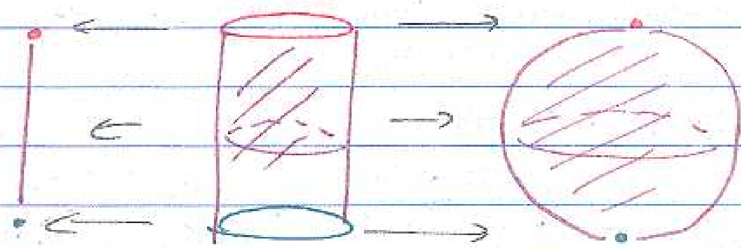
$$H_i^T(pt) = \begin{cases} \mathbb{R}[\mathbb{C}P^k] & \text{for } i = 2k \\ 0 & \text{otherwise} \end{cases}$$

and again $H_T^*(pt) = \mathbb{R}[t]$.

<u>Ex</u>: $X = S^2$, $T = S^1$

A relation in deg 0

cobordism



$$\rightsquigarrow [\mathbb{C}P_N^0] = [\mathbb{C}P_S^0]$$

In fact, $0 \to \mathbb{R} \longrightarrow H_*^T(N \cup S) \longrightarrow H_*^T(S^2) \to 0$

$$1 \longmapsto [\mathbb{C}P_N^0] - [\mathbb{C}P_S^0]$$

Dualizing,

$$0 \leftarrow \mathbb{R} \leftarrow H_T^*(N \cup S) \longleftarrow H_T^*(S^2) \leftarrow 0$$

$$H_T^*(N \cup S) = H_T^*(N) \oplus H_T^*(S) = \{(f,g) : f,g \in \mathbb{R}[t]\}$$

$$1 \leftarrow\!\shortmid 1_N - 1_S$$

<u>Conclusion</u>: $H_T^*(S^2) = \{(f,g) \mid f,g \in \mathbb{R}[t],\ f(0) = g(0)\}$