Fields – Schubert Wkshp
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Mattieu Willems, Part II:

structure constants on $K^0(H, X)$, as defined previously.

Setup as before: $X = G/B$, etc.

$$G \longrightarrow \text{Cartan matrix}.$$

$$A = \{a_{ij}\}_{1 \le i,j \le \ell} \qquad a_{ii} = 2, \quad -a_{ij} \in \mathbb{N}$$

$$\text{eg } G = SL_n\mathbb{C} \quad A = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & & -1 \\ 0 & & & -1 & 2 \end{bmatrix}$$

- root system: $\mathfrak{h}$ = Lie algebra of H

simple roots $\{\alpha_i\}_{1 \le i \le \ell}$ a basis of $\mathfrak{h}^*$

simple coroots $\{\alpha_j^\vee\}_{1 \le j \le \ell}$ a basis of $\mathfrak{h}$

$$\alpha_i(\alpha_j^\vee) = a_{ji}$$

$$K(H,X) \longleftrightarrow A \qquad \{\alpha_i\}$$

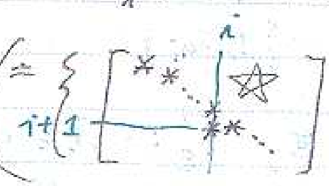
$H \subset B$ acts by $b(gB) = (bg)B$

$$X^H = W = N_G(H)/H.$$

$$\alpha_i \longleftrightarrow s_{\alpha_i} \in W, \quad s_{\alpha_i}^2 = 1.$$

Bott-Samelson varieties

$$\alpha_i \longleftrightarrow P_{\alpha_i} = Bs_{\alpha_i}B \cup B \subset G$$

$P_{\alpha_i} \supset B$ ($\simeq$ {  } $\subseteq SL_n\mathbb{C}$, for example)

$\Gamma(\alpha_i) := P_{\alpha_i}/B \cong \mathbb{CP}^1$. Now fix $N \ge 1$

Def^n: We choose N simple roots $\underbrace{(\mu_1, \dots, \mu_N)}_{\text{not necessarily distinct}}$ (Ex $\alpha_1, \dots, \alpha_n$)

$\Gamma(\mu_1, \dots, \mu_N)$ is the orbits of B^N acting on $P_{\mu_1} \times \dots \times P_{\mu_N}$ by

$(g_1, \dots, g_N) \cdot (b_1, \dots, b_N) = (g_1 b_1, b_1^{-1} g_2 b_2, \dots, b_{N-1}^{-1} g_N b_N)$

Example: $\mu_1, \mu_2 \quad \mu_1(\mu_2^\vee) = 0 \quad : \Gamma(\mu_1, \mu_2) = \mathbb{CP}^1 \times \mathbb{CP}^1$

Notation: $[g_1, \dots, g_N]$ class of $(g_1, \dots, g_N)$ in $\Gamma(\mu_1 \cdots \mu_N)$

Facts: 1) $\Gamma(\mu_1, \dots, \mu_N)$ is a smooth projective variety
2) $H \subset B \subset P_{\mu_i}$ acts on Γ by $g[g_1, \dots, g_N] = [gg_1, g_2, \dots, g_N]$

$$\begin{array}{ccc} \Gamma(\mu_1, \dots, \mu_N) & & [g_1, \dots, g_N] \\ \downarrow \pi_N = \text{fibration with fiber } \mathbb{CP}^1 & & \downarrow \\ \Gamma(\mu_1, \dots, \mu_{N-1}) & & [g_1, \dots, g_{N-1}] \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ \mathbb{CP}^1. & & \end{array}$$

(so can describe K-thy inductively)

Fixed points: $\Gamma = \Gamma(\mu_1, \dots, \mu_N)$

$$\mathcal{E} := \Gamma^H \simeq \prod_{1 \le i \le N} \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ \{1, s_{\mu_i}\}}}{P_{\mu_i}(H)}/H \simeq (\mathbb{Z}/2\mathbb{Z})^N$$

Cell decomposition: $\forall \varepsilon \in \mathcal{E}$, $\Gamma_\varepsilon := B \cdot \varepsilon \subset \Gamma$

$$\cong \mathbb{C}^{\ell(\varepsilon)} \qquad \ell(\varepsilon) = \#1\text{'s in } \varepsilon \in \mathcal{E}.$$

$$Y_\varepsilon := \overline{\Gamma_\varepsilon} \subset \Gamma$$

Proposition: Y_ε smooth, $Y_\varepsilon = \{[g_1, \dots, g_N],\ g_i = 1 \text{ if } \varepsilon_i = 0\}$

"Bott-Samelson variety"

1. $\Gamma = \bigsqcup_{\varepsilon \in \mathcal{E}} \Gamma_\varepsilon$

2. $Y_\varepsilon = \bigsqcup_{\varepsilon' \leq \varepsilon} \Gamma_{\varepsilon'}$. (take this as def^{n} of $\leq$?)

Connection with G/B : Define $g : \Gamma(\mu_1, \dots, \mu_N) \longrightarrow X$

$$[g_1, \dots, g_N] \longmapsto g_1 g_2 \cdots g_N B$$

Prop (Demazure): $w = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_N} \in W$ $N = \ell(w)$

Then $g : \Gamma(\mu_1 \cdots \mu_N) \longrightarrow X_w \ (\subset X)$ is an isomorphism between X_w° and $g^{-1}(X_w^\circ)$

THE POINT: $\Gamma(\mu_1 - \mu_N)$ is smooth. Note image of g is exactly X_w, which is not necessarily smooth.

Tasks:

1. $K(H, \Gamma)$
2. $g^* : K(H, X) \longrightarrow K(H, \Gamma)$

$K(H, \Gamma)$: if $N = 1$: $\Gamma(\mu_1) = \mathbb{C}P^1$

$K(H, \mathbb{C}P^1)$ is generated by $[1], [E_1]$ as before, with one relation $(E_1 - 1)(E_1 - e^{\mu_1}) = 0$.

$$E_1^2 = E_1 - e^{\mu_1}(1 - E_1) \quad ☆$$

Observe: $Z \subset Y$ algebraic $\mathbb{C}$ variety

closed subvariety (presume H-inv^t ...?)

$$\begin{array}{c} E \\ \pi \downarrow \\ Y \end{array}$$

$$\boxed{\chi(Z, E)(h) := \sum_{k \geq 0} (-1)^k \underbrace{\mathrm{tr}(h, H^k(E|_Z))}_{\text{trace of } h \text{ acting on } H^k(-)} \in R(H)}$$

finite sum, since $H^k(-) = 0$ for $k >> 1$

$\chi(Z, E) \in R[H]$. so $\chi(Z, -) : K^0(H, Y) \longrightarrow R(H)$

Now, back to example:
$$\left.\begin{aligned} \chi(\Gamma_0, E_1) &= 1 & \chi(\Gamma_0, 1-E_1) &= 0 \\ \chi(\Gamma_1, E_1) &= 0 & \chi(\Gamma_1, 1-E_1) &= 0 \end{aligned}\right\} \star\star$$

$\mathbb{C}P^1$: $\mu^0 = E_1$, $\mu^1 = 1 - E_1$ is a basis of $K(H, \Gamma)$ which satisfies $\star$, $\star\star$.

Now build by induction:

$$\begin{tikzcd} \Gamma(\mu_1, \dots, \mu_N) = \mathbb{P}(1 \oplus L_N) \arrow[d] \\ \Gamma(\mu_1, \dots, \mu_{N-1}) \arrow[u, dashed, bend left, "\pi_N"'] \end{tikzcd} \qquad \begin{tikzcd} L_N \text{ is a line bundle} \arrow[d] \\ \Gamma(\mu_1, \dots, \mu_{N-1}) \end{tikzcd}$$

Let E_N = tautological line bundle on Γ defined by π_N

<u>FACT</u> : $K(H, \Gamma)$ is a free module over $K(H, \Gamma(\mu_1 .. , \mu_{N-1}))$ (by π_N^*) with basis $\{1, E_N\}$, and one relation :
$$E_N^2 = E_N - L_N(1 - E_N).$$

$\forall \varepsilon \in \mathcal{E} = (\mathbb{Z}/2)^N$, $\boxed{\hat{\mu}^{\varepsilon} := \prod_{\varepsilon_i = 0} E_i \prod_{\varepsilon_j = 1} (1 - E_j)}$

and <u>these are a basis for $K(H, \Gamma)$ over $R(H)$</u>.

(Can find L_N explicitly in terms of $\hat{\mu}^{\varepsilon}$'s. ...)

use restrictions to fixed pts

$$\underline{\text{Th}^{m}}: \quad K(H,\Gamma) \cong R(H)[\overset{E_i \updownarrow}{x_1}, \ldots, x_N, x_1^{-1}, \ldots, x_N^{-1}] \Big/ I$$

$$I = \left\langle x_i^2 = x_i - e^{\mu_i} \prod_{j<i} x_j^{-\mu_i(\mu_j^\vee)} (1 - x_i) \right\rangle;$$

$$\left\{ \hat{\mu}^{\varepsilon} = \prod_{\varepsilon_i = 0} x_i \prod_{\varepsilon_j = 1} (1 - x_j) \right\}_{\varepsilon \in (\mathbb{Z}/2)^N} \quad \text{is a basis of } K(H,\Gamma)$$

Moreover $\chi(Y_{\varepsilon'}, \hat{\mu}_{\varepsilon}) = \delta_{\varepsilon\varepsilon'}$.

(can also compute str. constants for $\hat{\mu}_\varepsilon \hat{\mu}_{\varepsilon'}$'s ...)

<u>Prop</u>: (Kostant-Kumar)

$K(H,X)$ is a free module over $R(H)$ with a basis $\{\hat{\psi}^w\}_{w \in W}$ s.t.

$$\chi(X_v, \hat{\psi}^w) = \delta_{vw}$$

Then want str constants for $\hat{\psi}^v \hat{\psi}^w = \sum q^u_{vw} \hat{\psi}^u$.

Recall W generated by s_{α_i} with braid relations.

<u>Type A</u>: ① $s_i s_j = s_j s_i \qquad |i-j| > 1$

② $s_i^2 = 1$

③ $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

<u>Def^n</u>: 1. $\underline{W}$ new "product", monoid generated by $\{\underline{s_{\alpha_i}}\}_{1 \le i \le l}$ and with the relations ①, ③, ②': $\underline{s_{\alpha_i}}^2 = \underline{s_{\alpha_i}}$.

2. $W \longrightarrow \underline{W}$

$w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_k} \longmapsto \underline{s_{\alpha_1}} \cdots \underline{s_{\alpha_k}} = \underline{w}$ is a bijection.

$k = \ell(w)$

How to compute q_{uv}^{w} : $w = s_{\mu_1} \cdots s_{\mu_K}$

$$\begin{array}{ccc} \varepsilon & & w \\ \cap & & \cap \\ g : \Gamma = \Gamma(\mu_1, \ldots, \mu_K) & \longrightarrow & X \end{array}$$

$$g(\varepsilon) = \prod_{\varepsilon_i = 1} s_{\mu_i}$$

<u>Def</u>: $\underline{v}(\varepsilon) = \prod_{\varepsilon_i = 1} s_{\underline{\mu_i}} \in \underline{W}$

<u>MAGIC</u> : <u>Prop</u>: $g^* \hat{\psi}^{\mu} = \sum_{\underline{v}(\varepsilon) = \underline{w}} \hat{\mu}^{\varepsilon} \in K(H, \Gamma)$

so then just compute $g^* \hat{\varphi}^u \cdot g^* \hat{\varphi}^v$ in $K(H, \Gamma)$,
which we already know how to do.
and g^* is injective in an appropriate sense.

<u>Remarks</u>:
- Can get cancellations
- Haibao Duan : similar formulas, in cohomology.