BLACK HOLES, QUANTUM YANG-MILLS THEORY AND NON-PERTURBATIVE TOPOLOGICAL STRINGS

WORK WITH

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RECENT CONJECTURE

BY OOGURI, STROMINGER AND VAFA

RELATES

- TOPOLOGICAL YANG-MILLS THEORY

  (B-TYPE D-BRANES ON X)

- GROMOV-WITTEN THEORY ON X

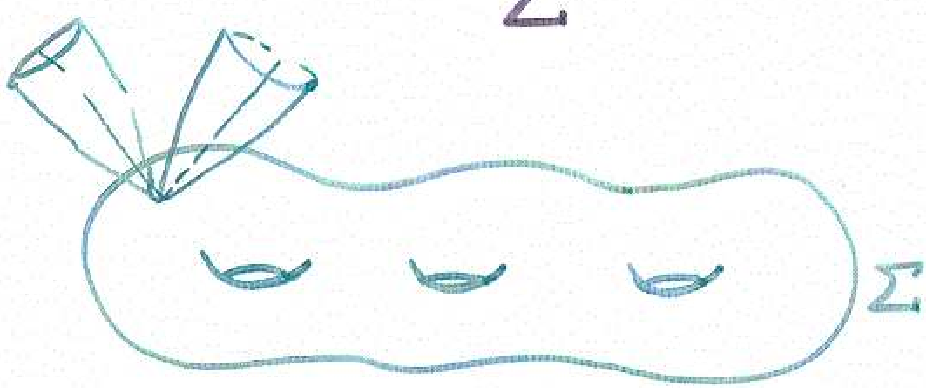
  (CLOSED TOPOLOGICAL A-MODEL)

"TOPOLOGICAL YM THEORY

$= \left| \text{GROMOV-WITTEN THEORY} \right|^2$"

I WILL DISCUSS THIS FOR A PARTICULAR CLASS OF CALABI-YAU MANIFOLDS

$$X = \begin{array}{c} L_1 \oplus L_2 \\ \downarrow \\ \Sigma \end{array}$$



X IS CALABI-YAU FOR

$$\deg(L_1) + \deg(L_2) = -\chi(\Sigma)$$

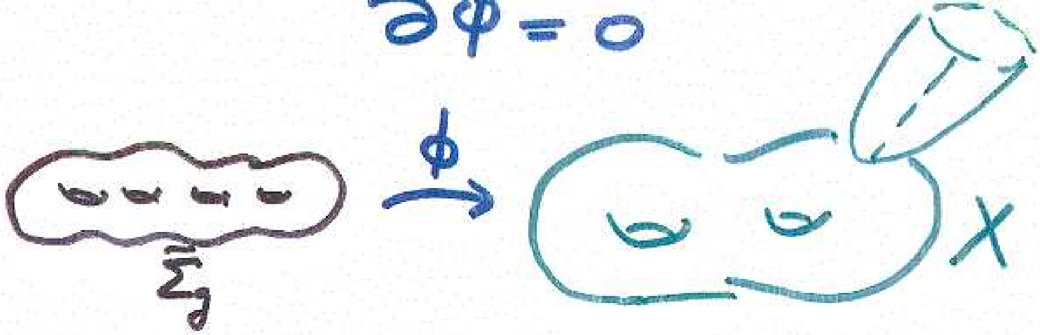
GROMOV - WITTEN THEORY

ON X IS COUNTING

HOLOMORPHIC MAPS

$$\phi : \Sigma_g \to X$$

$$\bar{\partial}\phi = 0$$



$\Rightarrow$ COMPUTES GW INVARIANTS

$$n_{d,g} \in \mathbb{Q}$$

FOR ϕ DEGREE d MAP.

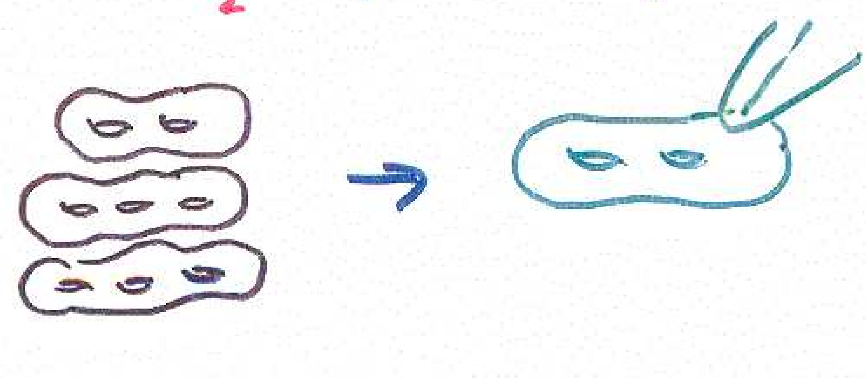
DEFINE GROMOV-WITTEN PARTITION FUNCTION Z^{GW}

$$Z^{GW}(t,\lambda) = \sum_{d,g} n_{d,g}\, \lambda^{2g-2} e^{-dt}$$

$\Rightarrow$ GENERATING FUNCTION OF DISCONNECTED GW INVARIANTS.

λ – GENUS COUNTING PARAMETER

$t = \int_{\Sigma} k$, Kahler form k.



TOPOLOGICAL YANG-MILLS THEORY

N D-BRANES WRAPPING A DIVISOR $\mathcal{D}$ IN X, WHERE $\mathcal{D}$ CORRESPONDS TO ZERO SECTION OF L_1:

$$\mathcal{D} = L_2 \to \Sigma$$

$\Rightarrow$ TOPOLOGICAL YM THEORY ON $\mathcal{D}$ WITH GAUGE GROUP $U(N)$:

$$Z^{YM} = \int \mathcal{D}A \; e^{-S_{YM}(A,\theta,\lambda)+\dots}$$

$$S_{YM} = \int_{\mathcal{D}} \frac{1}{\lambda} \operatorname{Tr} F \wedge F + \frac{\theta}{\lambda} \operatorname{Tr} F \wedge k_{\Sigma}$$

k_{Σ} - GENERATES $H^2(X, \mathbb{Z})$

THE PARTITION FUNCTION OF

THE YM THEORY IS COMPUTING

$$Z^{YM} = \sum_{m,n} \chi(\mathcal{M}_{m,n,N})\, e^{-\frac{m}{\lambda} - \frac{\theta n}{\lambda}}$$

WHERE

$\mathcal{M}_{m,n,N} =$ MODULI SPACE OF $U(N)$ INSTANTONS ON D WITH

$$m = \int_D \mathrm{Tr}\, F \wedge F\,, \quad n = \int_D \mathrm{Tr}\, F \wedge k_\Sigma$$

$\chi(\mathcal{M}_{m,n,N}) =$ # OF BOUND STATES OF N D4 BRANES ON D WITH n D2 BRANES ON Σ AND m D0 BRANES

WHY IS THIS EXCITING?

GW THEORY IS DEFINED

ONLY AS AN ASYMPTOTIC SERIES

IN λ

$$Z^{GW}(\lambda) = \sum_g Z_g^{GW} \lambda^{2g-2}$$

MAPS  $\rightarrow X$

GENUS g

TOPOLOGICAL YM THEORY

PROVIDES A NON-PERTURBATIVE

COMPLETION FOR GROMOV-WITTEN

THEORY.

LET ME GIVE A MORE
PRECISE STATEMENT
OF THE CONJECTURE
IN THIS CONTEXT...

THE CONJECTURE

STATES THAT

$$Z^{YM}(N,\theta,\lambda) = \left| Z^{GW}(\lambda,t) \right|^2$$

WHERE

$$t = \frac{\#}{2} N\lambda + i\theta$$

$$\# = \#(\Sigma \cap D)$$

THIS WOULD BE A PRECISE STATEMENT IF X WERE COMPACT. IN THE NON-COMPACT CASE

$$Z^{YM}(N,\theta,\lambda) = \sum_{\alpha} \left| Z_{\alpha}^{GW}(\lambda,t) \right|^2$$

α - "BOUNDARY CONDITIONS AT ∞ OF X"

I'LL NOW EXPLAIN

HOW GW THEORY ON X

AND THE CORRESPONDING

YM THEORY CAN BE SOLVED

EXPLICITLY, PROVIDING A

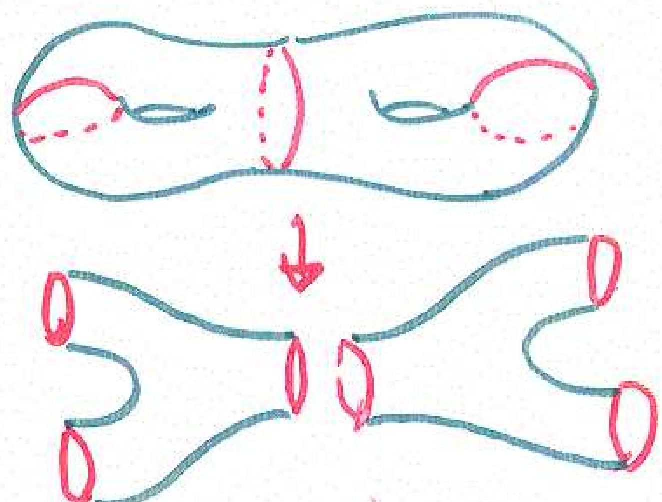
<u>TEST</u> OF THE CONJECTURE...

GW THEORY OF $X = \begin{matrix} L_1 \oplus L_2 \\ \downarrow \\ \Sigma \end{matrix}$

BRYAN AND PANDHARIPANDE :

CHOP X INTO PIECES

Σ :



ON WHICH ONE CAN SOLVE

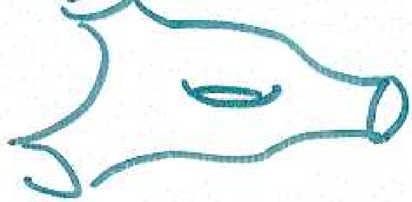
THE THEORY , AND

GET THE REST BY GLUING.

CONSIDER

$$X = \begin{array}{c} L_1 \oplus L_2 \\ \downarrow \\ \Sigma_{g,h} \end{array}$$

$\Sigma_{g,h}$ = RIEMANN SURFACE WITH h PUNCTURES (BOUNDARIES)



$\Sigma_{g=1,\,h=3}$

AND COUNT HOLOMORPHIC MAPS OF DEGREE d, WITH PRESCRIBED RAMIFICATIONS $\vec{k}_i$, $i=1,\dots,h$ (RELATIVE GW THEORY)

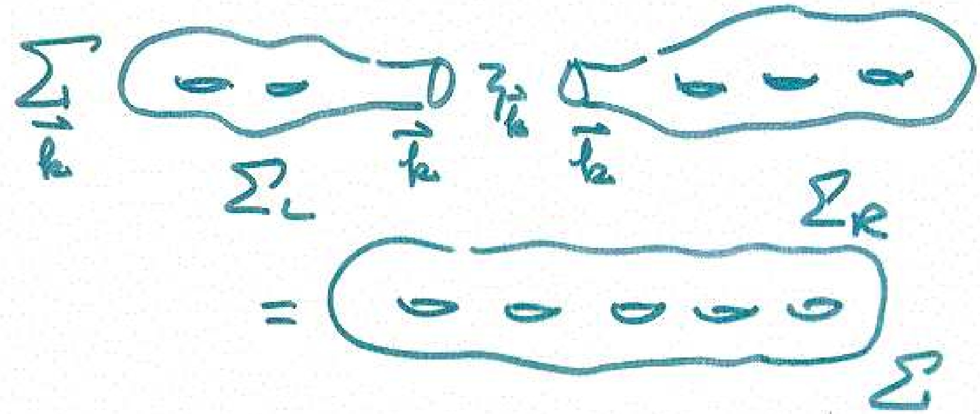
$$\Longrightarrow \quad Z^{GW,\,d}_{\vec{k}^{(1)},\dots,\vec{k}^{(h)}}\left(\Sigma_{g,h}\right)$$

$$\vec{k} = (1^{k_1}, 2^{k_2}, \dots) \qquad 1^d$$

$$d = \sum_j (k_j)\, j$$

THESE SATISFY GLUING RELATIONS

$$Z^d(\Sigma_L \cup \Sigma_R) = \sum_{\vec{k}} Z^d_{\vec{k}}(\Sigma_L)\,\eta_{\vec{k}}\, Z^d_{\vec{k}}(\Sigma_R)$$



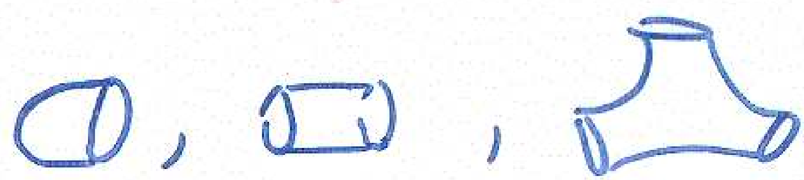
REALLY, ONE IS GLUING NOT ONLY $\Sigma_{L,R}$ BUT ALSO THE BUNDLE OVER IT

$$X_L \cup X_R = X$$

$$\deg L_i(X_L) + \deg L_i(X_R) = \deg L_i(X)$$

$$i = 1, 2$$

IN FACT, BP SHOW THAT
GROMOV-WITTEN THEORY
ON $\begin{array}{c} X \\ \downarrow \\ \Sigma \end{array}$ IS A TOPOLOGICAL
QUANTUM FIELD THEORY ON
Σ, SO ALL THAT ONE
NEEDS TO COMPUTE ARE
GW PARTITION FUNCTIONS OF

$\Sigma =$ 

IN WRITING $Z_{\vec{k}}(\Sigma)$

$\rightarrow$ PARTICULAR CHOICE OF BASIS $\{\vec{k}\}$ OF HILBERT-SPACE OF TQFT.

THERE IS A MORE "NATURAL" BASIS W/ STATES LABELED BY

$R \leftrightarrow$  YOUNG TABLEAUX

$$Z^{d}_{R} = \sum_{\vec{k}} \chi_R(\vec{k})\, Z^{d}_{\vec{k}}$$

CHARACTERS OF S_d SYMMETRIC GROUP

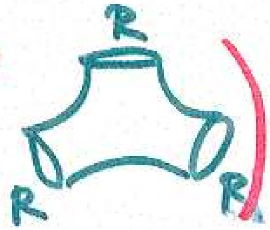
GLUING RULE IS

$$Z(\Sigma_L \cup \Sigma_R) = \sum_R Z_R(\Sigma_L)\, Z_R(\Sigma_R)\, e^{-\frac{t|R|}{d}}$$

IN THIS BASIS:

$Z\Big($ 

$\Big) = d_q(R)\, q^{\# C(R)/2}$

$Z\Big($ 

$\Big) = \dfrac{1}{d_q(R)}\, q^{\# C(R)/2}$

$Z\Big($ 

$\Big) = q^{\# C(R)/2}$

WHERE $\# = \deg(L_1) - \deg(L_2)$

$$q = e^{-\lambda}$$

$$d_q(R) = \prod_{\square \in R} \frac{1}{[h(\square)]_q}$$

$[n]_q \equiv q^{n/2} - q^{-n/2}$

"quantum dimension of symmetric group representation"

FROM THIS, BY GLUING

$(2g-2)$  $= (g-1)$ 

GET

$Z($  $) = \sum_R d_q(R)^{2-2g}\, q^{C(R)/2} \times e^{-|R|t}$

g handles

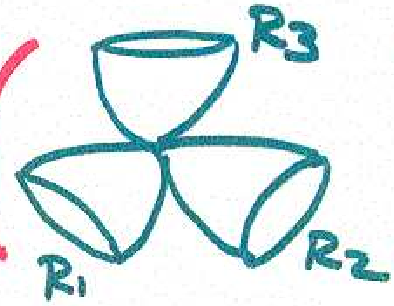
WHERE

$|R| = \#$ OF BOXES IN R

$=$ DEGREE OF A MAP

$$t = \int_\Sigma k$$

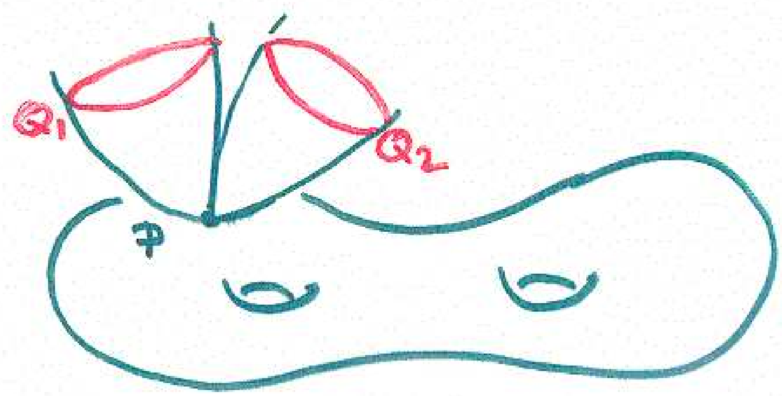
THIS IS VERY ANALOGOUS TO TOPOLOGICAL VERTEX WHERE HOLOMORPHIC MAPS TO $\mathbb{C}^3$ LOCALIZE TO VERTEX OF THE FORM

Z () $= C_{R_1 R_2 R_3}$

AND BY GLUING COMPUTES GW INVARIANTS OF TORIC CY.

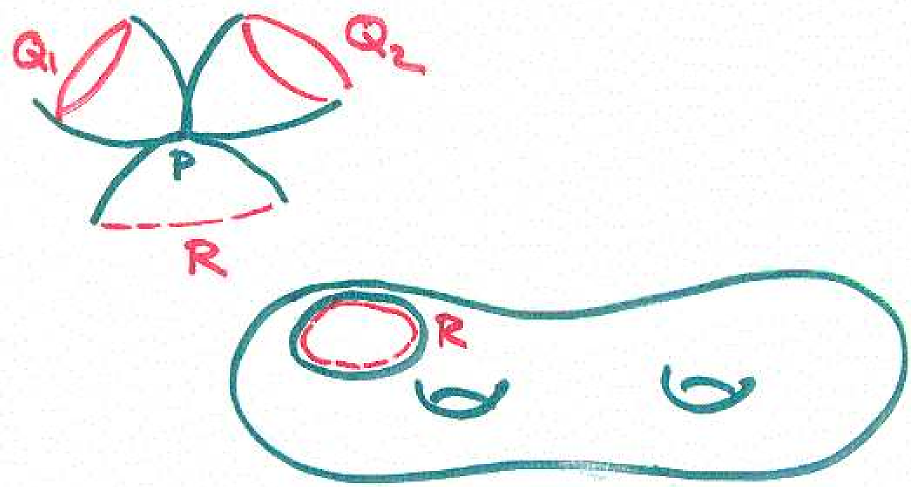
ACTUALLY, WE CAN COMBINE THE TWO THEORIES ...

CORRESPONDING TO HAVING D-BRANES OVER A POINT P ON Σ



$\Rightarrow$ BY GLUING

Σ_R



TOPOLOGICAL YM THEORY

RECALL: THEORY OF N D-BRANES WRAPPING A DIVISOR D

$$Z_{YM}(N,\lambda,\theta) = \int \mathcal{D}A \; e^{-S_{YM}(A,\lambda,\theta)}$$

$$S_{YM} = \frac{1}{\lambda}\int_D \mathrm{Tr}\, F\wedge F + \frac{\theta}{\lambda}\int_D \mathrm{Tr}\, F\wedge k_\Sigma$$

$$D = \begin{array}{c} L_2 \\ \downarrow \\ \Sigma \end{array}$$

THE TASK IS TO COMPUTE Z_{YM} ...

$\Rightarrow$ ON A GENERAL 4-MANIFOLD D

THIS IS DIFFICULT.

HOWEVER, $D = \begin{matrix} L^2 \\ \downarrow \\ \Sigma \end{matrix}$ HAS A

$\mathbb{C}^*$ SYMMETRY:

$\Rightarrow$ LOCALIZES TO INVARIANT MODES

<u>TYM THEORY ON D IS</u>

<u>EFFECTIVELY A 2-DIMENSIONAL</u>

<u>THEORY ON Σ!</u>

ROUGHLY, BY LOCALIZATION WE CAN INTEGRATE OVER THE FIBERS

$$\int_{\substack{\text{FIBER} \\ \text{AT } P}} F = \oint_{S^1_P} A = \Phi(P)$$

"QUANTUM" YM THEORY

$$Z^{YM} = \int \mathcal{D}A \, \mathcal{D}_H \phi \; e^{-S(\Phi, A, \theta, \lambda)}$$

$$S = \frac{1}{\lambda} \int_\Sigma \mathrm{Tr}\, \Phi F + \theta \,\mathrm{Tr}\, \Phi + \deg(L_2)\, \mathrm{Tr} \frac{\phi^2}{2}$$

THIS WOULD BE AN "ORDINARY" YM THEORY ON Σ, EXCEPT THAT Φ IS A HOLONOMY FIELD, SO MEASURE IS "EXOTIC". $e^{i\Phi}$

→ THE THEORY IS THE SAME AS CHERN-SIMONS ON S^1 BUNDLE OF DEGREE $\deg(L_2)$ OVER Σ.

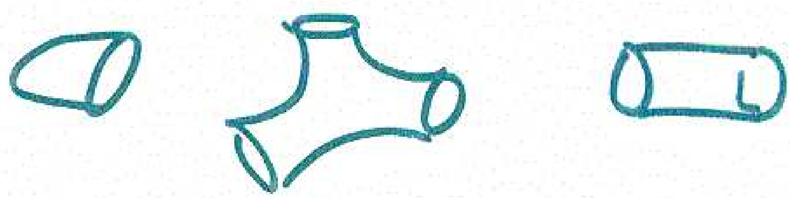
THE QUANTUM YM THEORY

IS SOLVABLE

E.G. BY CUTTING

AND GLUING FROM

BASIC PIECES



qYM PARTITION FN. $Z(\Sigma, U)$

ON A MANIFOLD WITH BOUNDARY

Σ

DEPENDS ON HOLONOMY

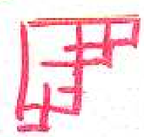
$$U = P e^{i\oint A}$$

ON THE BOUNDARY.

$\Rightarrow$ HILBERT SPACE $\mathcal{H}_{S^1}$ IS

SPANNED BY CHARACTERS

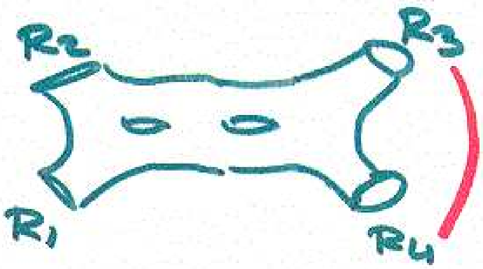
$\mathrm{Tr}_R\, U$ $\quad$ R - IRRED. REP OF $U(N)$



$$\underline{Z(\Sigma, U) = \sum_R Z(\Sigma)_R \,\mathrm{Tr}_R U}$$

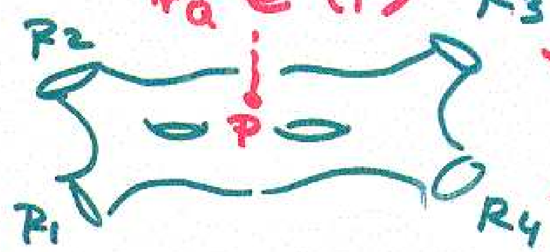
$$\int dU \; Z(\Sigma_L, U)\, Z_R(\Sigma_R, U^{-1}) = \sum_R Z(\Sigma_L)_R Z(\Sigma_R)_R$$

A CONSEQUENCE OF TOPOLOGICAL INVARIANCE IS THAT

$Z\Big($  $\Big) = 0$, UNLESS $R_1 = R_2 = R_3 = R_4$

E.G INSERTING

$\mathrm{Tr}_a\, e^{\phi(p)}$

$Z\Big($  $\Big)$

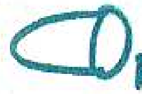
$\Rightarrow$ INDEPENDENT OF P,

AT i-th PUNCTURE

COMPUTES CASIMIRS OF

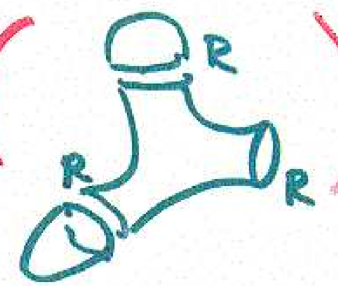
CORRESPONDING REPRESENTATION R_i.

THIS IMPLIES THAT THE THEORY IS COMPLETELY FIXED JUST BY KNOWING THE CAP AMPLITUDE.

IF $Z\Big($  $\Big) = C_R$

$Z\Big($  $\Big) = P_R$

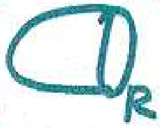
THAN

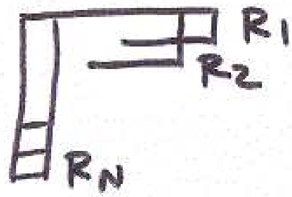
$C_R = Z\Big($  $\Big)$

$$= C_R^2 P_R$$

$$\rightarrow P_R \propto 1/C_R$$

THE THEORY ON A DISK IS EASY TO SOLVE BY EXPLICIT EVALUATION OF THE PATH INTEGRAL

$Z\Big($  $_R\Big) = \sum_{R \in \widehat{U(N)}} \dim_q(R)\, q^{\frac{\# C_2(R)}{2}}\, e^{i\theta |R|}$

$R =$ 

$$q = e^{-\lambda}$$

$\dim_q(R) =$ "quantum dimension of $U(N)$ representation"

$$= \prod_{1 \le i < j \le N} \frac{[R_i - R_j + j - i]_q}{[j-i]_q}$$

THIS ALLOWS US TO COMPUTE E.G

$$Z\Big($$ 

$$\Big) = \frac{1}{\dim_q(R)}\, q^{\#\frac{C_2(R)}{2}} e^{i\theta|R|}$$

$$Z\Big($$ 

$$\Big) = q^{\#\frac{C_2(R)}{2}} e^{i\theta|R|}$$

$$Z\Big($$ 
$$\Big) = \sum_R (\dim_q(R))^{2-2g}\, q^{\#\frac{C_2(R)}{2}} e^{i\theta|R|}$$

g-handles

$\#$ = Chern Class of S^1 BUNDLE OVER Σ $= \deg(L_2)$

$q = e^{-\lambda}$

NOTE HERE

$$Z^{YM} = Z^{YM}(q)$$

$$q = e^{-\lambda}$$

RELATED TO $N=4$ YM S-DUALITY

$\rightarrow Z^{YM}$ IS A MODULAR FORM:

ALSO HAS EXPANSION

$$Z^{YM} = Z^{YM}(\tilde{q}) \qquad \tilde{q} = e^{-1/\lambda}$$

$$= \sum_{m,n} X_{m,n,N} \, e^{-\frac{m}{\lambda} - \frac{n\theta}{\lambda}}$$

RELATION OF GW/qYM THEORIES

RECALL THAT THE CONJECTURE PREDICTS A RELATION BETWEEN THE 2 THEORIES

$$Z^{YM}(N,\theta,\lambda) = \sum_{\alpha} \left| Z_{\alpha}^{GW}(\lambda,t) \right|^2$$

FOR

$$t = \frac{\#}{2} N\lambda + i\theta$$

$$\begin{aligned} \# &= \#(\Sigma \cap D) \\ &= \deg(L_1) \end{aligned}$$

WHERE $D = \begin{array}{c} L_2 \\ \downarrow \\ \Sigma \end{array}$ IN X

IT IS APPARENT THAT Z^{YM} AND Z^{GW} ARE RELATED AT LARGE N

$$Z^{YM}\Big($$$$, \lambda, N, \theta \Big) =$$

$$= \sum_{R \in U(N)} \left(\dim_q(R) \right)^{2-2g} q^{\# \frac{C_2(R)}{2}} e^{-\theta |R|}$$

AND

$$Z^{GW}\Big($$$$, \lambda, t \Big) =$$

$$= \sum_{R \in U(\infty)} \left(d_q(R) \right)^{2-2g} q^{\# \frac{C(R)}{2}} e^{-t|R|}$$

$$\dim_q(R) \underset{N \to \infty}{\sim} d_q(R)\, q^{-\frac{C(R)}{2}} e^{N\lambda |R|}$$

SINCE GW AMPLITUDES

ARE EXPANSIONS IN e^{-t}, λ

WE SHOULD CONSIDER

Z^{qYM}

AT LARGE $N \propto \mathrm{Re}\, t/\lambda$.

IN PARTICULAR,

$$Z^{qYM}(N,\theta,\lambda) = \sum_{\alpha} \left| Z^{GW}(t,\lambda) \right|^2$$

UP TO TERMS

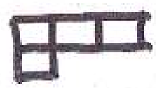
$$\mathcal{O}(e^{-N}) = \mathcal{O}(e^{-t/\lambda}, e^{-\bar{t}/\lambda})$$

THIS WOULD SUGGEST

$$\text{“} Z^{YM}(\theta, N, \Lambda) = Z^{GW}(t, \lambda) \text{”}$$

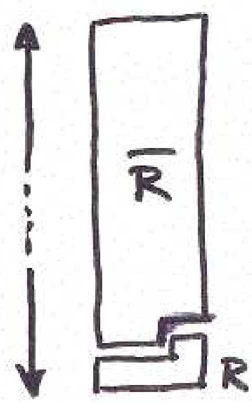
... HOWEVER, THIS IS <u>NOT RIGHT</u>.

FOR EVERY REPRESENTATION

R 

WITH SMALL # OF BOXES,

THERE IS A CONJUGATE REPRESENTATION

N ROWS 

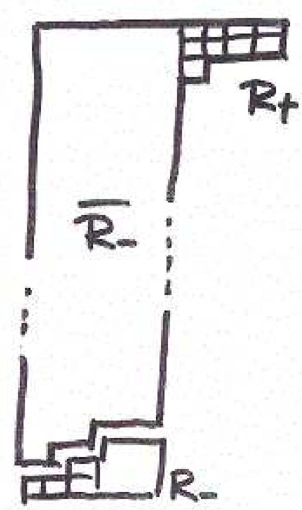
WITH ORDER N BOXES

$\Rightarrow$ WE <u>THREW AWAY</u> "HALF"
OF <u>THE THEORY</u>!

$\Rightarrow$ THE HILBERT SPACE OF THE THEORY SPLITS AT LARGE N

$$\mathcal{H}_{U(N)} \to \mathcal{H}^{+} \otimes \mathcal{H}^{-}$$

A GENERIC STATE IN $\mathcal{H}^{+} \otimes \mathcal{H}^{-}$ CORRESPONDING TO

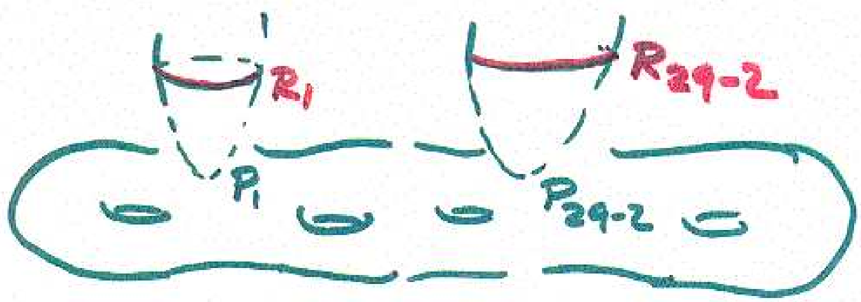


$$R \to R_+ \overline{R_-}$$

IT IS STRAIGHT-FORWARD TO SHOW

$$Z^{YM}(\Sigma_g ; N, \theta, \lambda) = \sum_{R_1, \dots, R_{2g-2}} \left| Z^{GW}_{R_1, \dots, R_{2g-2}}(t, \lambda) \right|^2$$

WHERE $Z^{GW}_{R_1, \dots, R_{2g-2}}$ COMPUTES



ON $\quad X = \begin{array}{c} L_1 \oplus L_2 \\ \downarrow \\ \Sigma \end{array}$

WITH $\quad t = \frac{\deg(L_1)}{2} N\lambda + i\theta$

WHAT ABOUT BLACK HOLES?

$D4 + D2 + D0$

BOUND STATES WHICH YM THEORY IS COUNTING ARE BLACK HOLES.

WHY THIS HAS
ANYTHING TO DO WITH
TOPOLOGICAL STRINGS
– THE INTUITION
BEHIND THE CONJECTURE –
WILL BE EXPLAINED
TOMORROW.

$N\,D4 + n\,D2 + m\,D0$

A BLACK HOLE IS SPECIFIED BY CHARGES

(N, n, m)

OF BOUND STATES WITH THESE CHARGES IS THE # OF WAYS OF "MAKING" THE BLACK HOLE

$\Longleftrightarrow$ # OF BLACK HOLE "MICROSTATES"