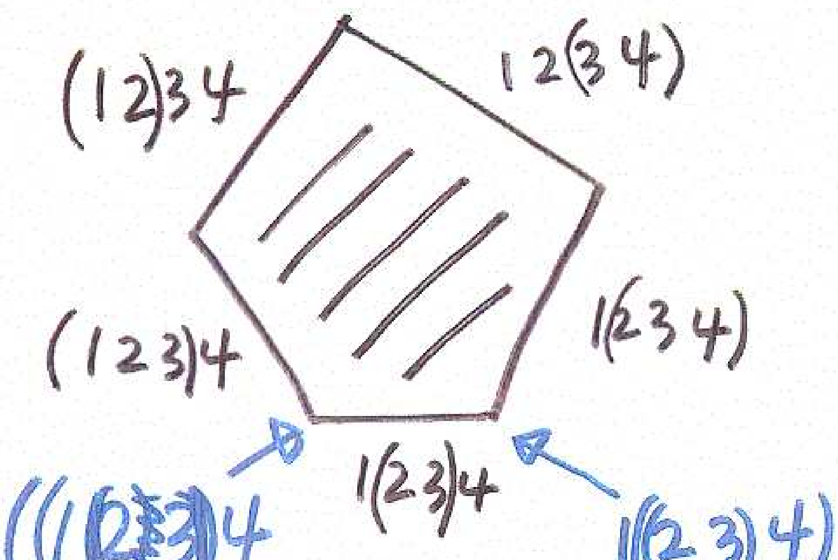
Stasheff Polytope K_n

: Possible ways of bracketting
n letters fit together

K_3

K_4



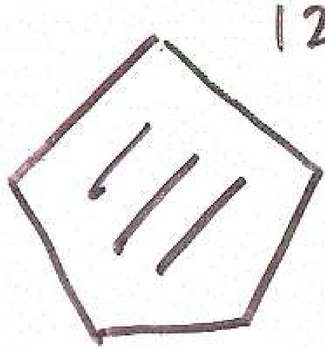
$$\mathcal{M}^{b,\,main}_{0,n+1,0} = \left\{ (D^2, \vec{z}) \;\middle|\; z_i \in \partial D^2,\ z_0, \cdots, z_n \text{ cyclically ordered} \right\} \Big/ \mathrm{Aut}(D^2)$$

Then

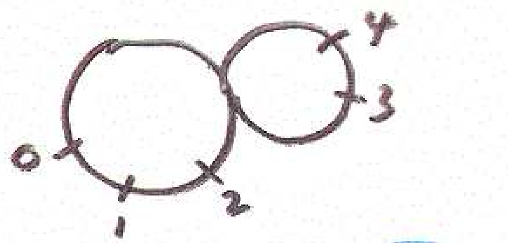
$$\overline{\mathcal{M}}^{b,\,main}_{0,n+1,0} \cong K_n$$

K_3 : (12)3 —— 1(23)

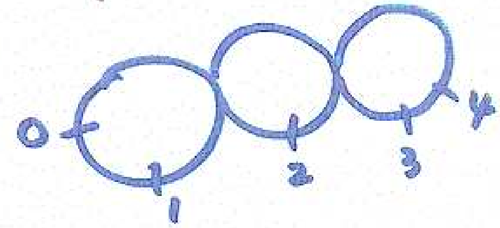
K_4 :



12(34)



1(2(34))



1(234)

A_∞-algebra of a Lagrangian Submanifold

(Fukaya-Oh-Ohta-Ono)

Using the moduli spaces $\overline{\mathcal{M}}^{b,\,main}_{0,n+1,0}(\beta)$.

P : symp. manifold

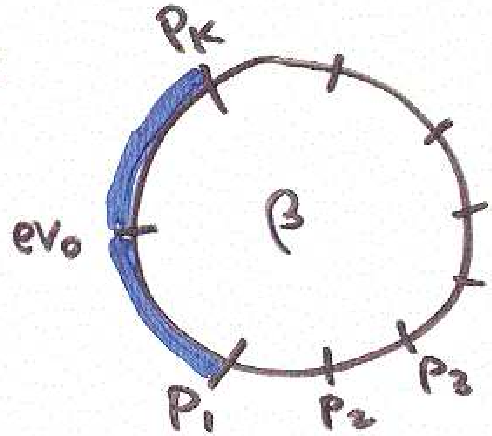
L : Lagrangian submanifold , $\beta \in \pi_2(P, L)$

$C^\circ(L : \Lambda_{nov})$: Currents (realized by geometric chains in L)

$P \xrightarrow{f} L$

A_∞-operation : $m_{1,0}(P) = \partial P$

$m_{k,\beta}(P_1, \cdots, P_k) =$



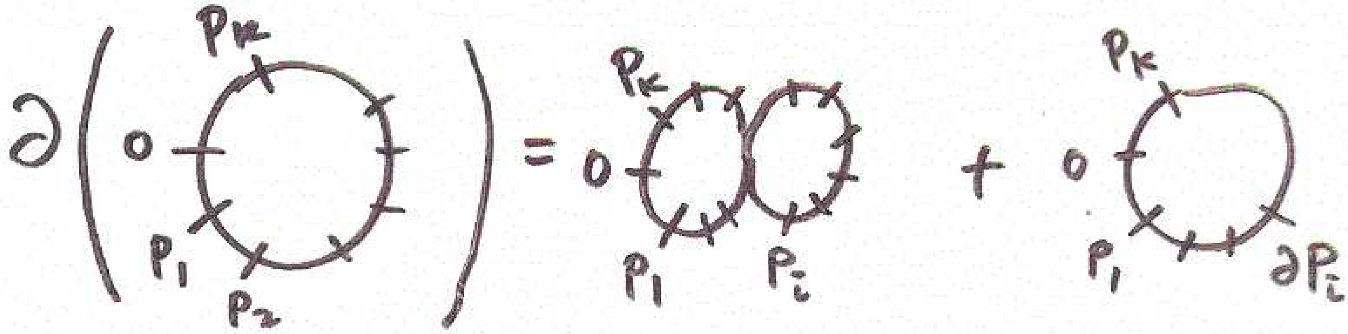
for all J-holo disc intersecting $P_1, \cdots, P_k$ at $z_1, \cdots, z_k$

Define $m_k = \sum_{\beta \in \pi_2(M,L)} m_{k,\beta} \otimes T^{\omega(\beta)} q^{\mu(\beta)}$.

They satisfy A_∞-formula

$$\sum_{\substack{k_1+k_2=n \\ k_i \geq 0}} (-1)^{\varepsilon_{i-1}} m_{k_1}(P_1, \cdots, P_{i-1}, m_{k_2}(P_i \cdots), \cdots P_k) = 0$$

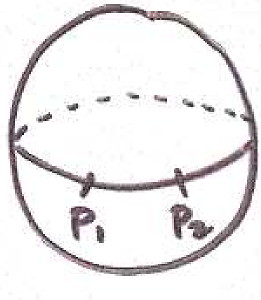
pf)



$\Updownarrow$

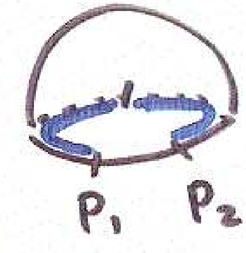
$$m_{1,0}(m_k(P_1, \dots, P_k)) = \sum m_{k_1}(P_1 \cdots P_{i-1}, m_{k_2}(P_i \cdots) \cdots P_k) + m_k(P_1, \cdots, m_{1,0} P_i, \cdots, P_k).$$

Example) Equator $S^1 \subset \mathbb{C}P^1$



S^1 $\quad m_2(P_1, P_2)$

upper

$= \Big($ 

$+$ 

$\Big) T^{\omega(D^2)} g$

lower

$= ($  $+$  $) T^{\omega(D^2)} g$

$= [$  $] T^{\omega(D^2)} g$

$= [S^1] T^{\omega(D^2)} g = 1 \cdot T^{\omega(D^2)} g$

Other homotopy classes in $\pi_2(\mathbb{C}P^1, S^1)$ do not contribute.

(Compare : $pt \cap pt = \phi$)

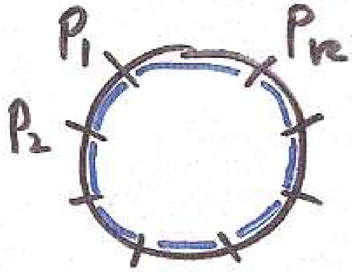


Image of whole boundary of J-hol. discs :

$$l_k(P_1, \ldots, P_k)$$
$$= \sum_{\sigma \in S_k} (-1)^{\epsilon(\sigma, P)} m_k(P_{\sigma(1)}, \ldots, P_{\sigma(k)})$$

$\Rightarrow$ l_∞ - algebra

$$0 = \sum_{\substack{k_1+k_2=n \\ \sigma \in (k_2, n-k_2) \\ \text{shuffle}}} \pm l_{k_1}(l_{k_2}(P_{\sigma(1)}, \ldots, P_{\sigma(k_2)}), \cdots P_{\sigma(k)})$$

THM (C) "Divisor equation"

$$l_{k,\beta}(P_1, \ldots, P_k) = (P_i \cdot \partial\beta)\, l_{k-1,\beta}(P_1, \ldots, \hat{P_i}, \ldots, P_k)$$

for $(n-1)$-dim cycle $P_i \subset L$

$(P_i \in H^1(L))$

Note : · Signed sum.

· not true for m_*

Thm (C-Oh) P : Fano toric manifold

L : Lagrangian torus fiber

N : # of Cod-1 faces of moment polytope of P

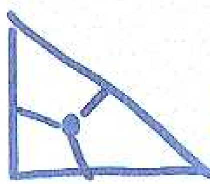
$\Rightarrow \exists$ N holo. discs of Maslov index 2

AND $$m_0 = \sum_{k=1}^{N} T^{\omega(\beta_k)} \underset{T^{2\pi}=e^{-1}}{``="} \sum_{k=1}^{N} e^{-y_k - \langle \theta, v_k \rangle}$$

$$\qquad\qquad\qquad\qquad \| $$

$$W(\theta)$$

LG superpotential



Thm (C) For some generators $c_1, \cdots, c_n \in H^1(L)$,

$$\ell_m(c_{i_1}, \cdots, c_{i_m}) = (-1)^{(n-1)m} \frac{\partial^m W(\theta)}{\partial \theta_{i_1} \cdots \partial \theta_{i_m}}$$

Thm (C) $HF^*(L, L)$ are Clifford algebras

$$m_2(c_i, c_j) + m_2(c_j, c_i) = \ell_2(c_i, c_j)$$

$$= \frac{\partial^2 W}{\partial \theta_i \partial \theta_j}$$

Outline

- A_∞-algebra for quantum cohomology ring

  : $(C^\bullet(P, \Lambda_{nov}), \{\mathfrak{q}_k\})$

  Homology of $\mathfrak{q}_1$: $H^*(P; \Lambda_{nov})$
  Singular cohomology

  $\mathfrak{q}_2 \to$ quantum Coh. ring st
  on $H^*(P; \Lambda_{nov})$

  $\mathfrak{q}_3 \to$ provides homotopy
  for associativity.

- A_∞-algebra of Hochschild cochains
  of A_∞-algebra of a Lagrangian Sub.
  (Getzler)
  (Fukaya Category w/ one object)

- A_∞-morphism between these A_∞-algebras.

A_∞-algebra for QC

Not entirely new : ① Fukaya's quantum Morse A_∞-Category

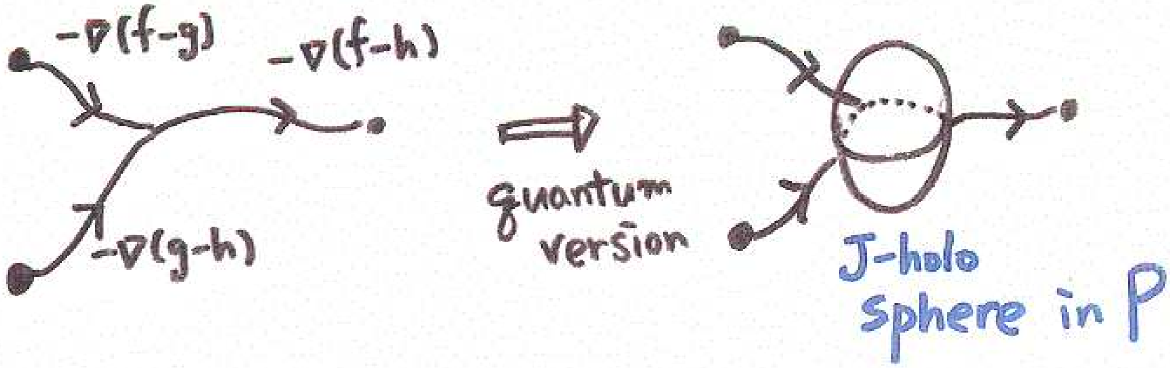
② (implicit in Kontsevich)

related to A_∞-structure of the diagonal Lag. sub $\Delta \subset P \times (-P)$

Not clear when we use "Kuranishi perturbation" to get good moduli spaces of J-holo discs.

(cf. FOOO in semi-positive case)

①



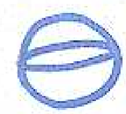
② $P \longleftrightarrow \Delta \subset P \times (-P)$

explicit if P is convex algebraic manifold.

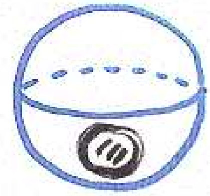
$\because$ J-holo disc : $(D^2, \partial D^2) \longrightarrow (\overset{J \times -J}{P \times (-P)}, \Delta)$

$\downarrow$ glue

J-hol. sphere in P :



J-hol. sphere :



any circle on S^2

$\downarrow$

J-holo disc

$\longrightarrow$

Fredholm regularity of J spheres $\Rightarrow$ Fred regul. of J-discs

THM) $HF^*(\Delta,\Delta) \cong QC$ of P. for Convex Proj. P.

product st.

$HF^*(\Delta,\Delta)$

$P \cong \Delta \subset M \times (-M)$

$p \to (p,p)$

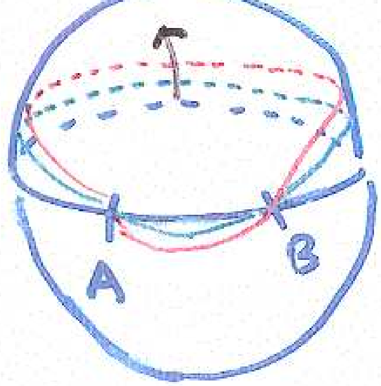
$A \to \tilde{A}$

$B \to \tilde{B}$

prod st. in QC

??

1-parameter family of discs through $\tilde{A}, \tilde{B}$ $\longleftarrow$ J-holo. sphere through A, B

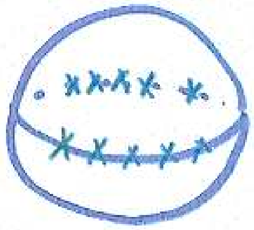


main component covers S^2

G

Sub moduli space $\mathcal{M}^c_{0,k} \subset \mathcal{M}_{0,k}$

$$\mathcal{M}^c_{0,k} := \{ (S^2, \vec{z}) \mid z_0, \cdots z_{k-1} \text{ lies on some circle in } S^2 \text{ cyclically counterclockwise} \} / \mathrm{Aut}(S^2)$$

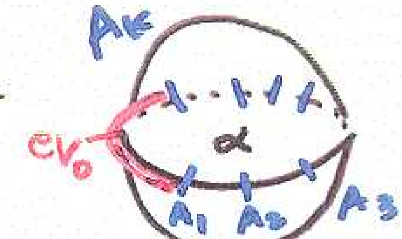


Actually $\overline{\mathcal{M}}^c_{0,k} \cong \overline{\mathcal{M}}^{b,\text{main}}_{0,k,0} \cong K_{k-1}$

$\Rightarrow A_\infty$-algebra

A_∞ operations on $C^*(P, \Lambda_{nov}) \ni A_i$

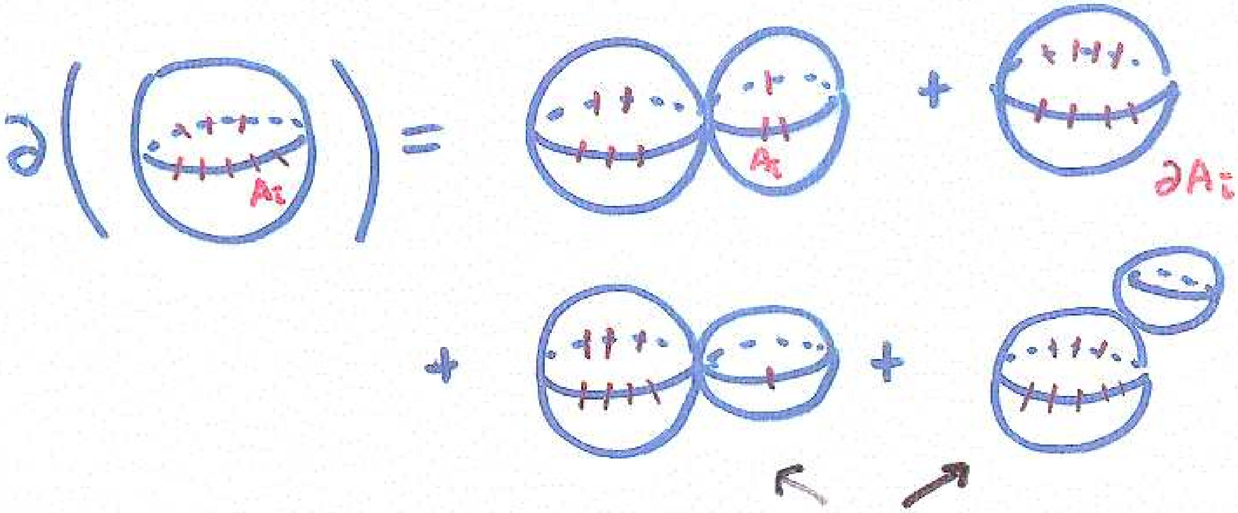
- $\mathfrak{q}_{1,0}(A) = \partial A$
- $\mathfrak{q}_{1,\beta}(A) = 0$
- $\mathfrak{q}_0 = 0$

$\mathfrak{q}_{k,\alpha}(A_1, A_2, \cdots A_k) =$ 

for all J-hol sphere in P intersecting $A_1, \cdots A_k$ at $z_1, \quad , z_k$

* No boundary condition

THM) $\displaystyle\sum_{\substack{k_1+k_2=k\\ k_i\geq 1}} \pm\, \mathfrak{q}_{k_1}(A_1, \cdots A_{i-1}, \mathfrak{q}_{k_2}(A_i \cdots A_{i+k_2-1}) \cdots A_k) = 0$ for all $k>0$, or $\mathfrak{q}\circ\mathfrak{q}=0$



Cod 2 : DO NOT APPEAR in A_∞ formula

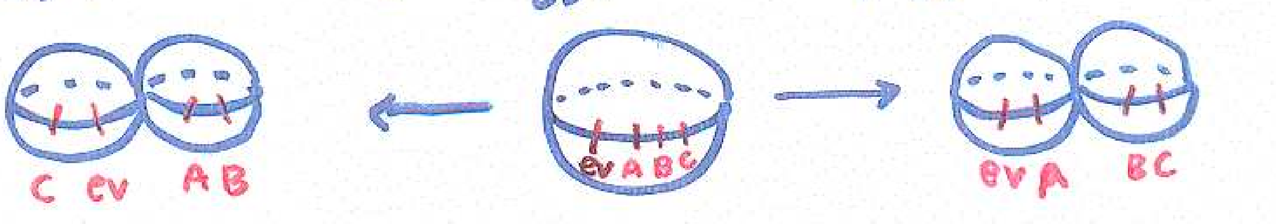
$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\mathfrak{q}_{1,\beta}=0$ $\qquad\qquad$ $\mathfrak{q}_0=0$

$\mathfrak{q}_1=\mathfrak{q}_{1,0}=\partial \Rightarrow$ Homology of $\mathfrak{q}_1 \cong$ singular coh. of P

$\mathfrak{q}_2 \Rightarrow$  quantum coh.
since $\overline{M}^{c}_{0,3}(\alpha)=\overline{M}_{0,3}(\alpha)$

$\mathfrak{q}_2(\mathfrak{q}_2(A,B),C) \longleftarrow \mathfrak{q}_3(ABC) \longrightarrow \mathfrak{q}_2(A,\mathfrak{q}_2(B,C))$



Hochschild Cohomology

- Deformation of an assoc. algebra A $\Rightarrow H^2(A,A)$
- (PS) Deformation of an A_∞ algebra A $\Rightarrow H^*(A,A)$

: write $\tilde{m}_k = m_k + t f_k$ / $t^2=0$, $ta = (-1)^{at} at$

If $\{\tilde{m}_k\}$ satisfies A_∞-equation

$$\tilde{m}_\bullet \circ \tilde{m}_\bullet = 0 \implies \underbrace{m \circ f - (-1)^f f \circ m}_{=[m,f]} = 0$$

$\Rightarrow f$ is Hoch. cocycle.

- A : graded vector space

The space of Hochschild cochains

$$C^\bullet(A,A) = \mathrm{Hom}(B(A), sA)$$

where $B(A) = \sum_{n=0}^{\infty} (sA)^{\otimes n}$

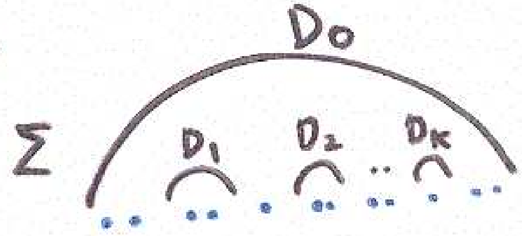
$$(sA)^i = A^{i+1}$$

Ex) $f \in C^\bullet(A,A) \Rightarrow f = \{f_k\}_{k\geq 0}$

$$f_k : A^{\otimes k} \to A$$

Brace operation on Hoch. cochains

$D_0\{D_1, \cdots, D_k\} \Rightarrow$



For A_∞ algebra (A, m),

(G) $C^\bullet(A,A)$ has an A_∞-structure

(Hoch. Co)

$$M[D_1, \dots, D_K] = \begin{cases} 0 & k=0 \\ m \circ D_1 - (-1)^{|D_1|} D_1 \circ m & k=1 \\ m\{D_1, \cdots D_k\} & k>1 \end{cases}$$

i.e. $\sum \pm M(D_1 \cdots M(D_i \cdots) \cdots) = 0$

Similar story for Fukaya category by Fukaya

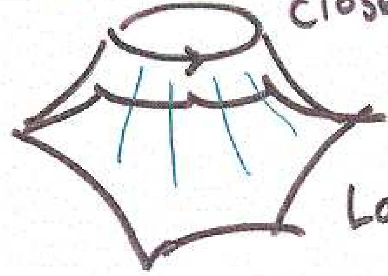
A_∞ functors, pre natural transformations, ..

$$M : \left(C^\bullet(A,A)\right)^{\otimes K} \to C^\bullet(A.A)$$

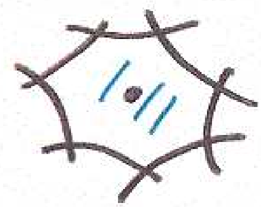
Open-closed map

Quantum Coh
$\downarrow$
Def. of Fukaya Cate.

Seidel
Hoffman-Ma
Kapustin-Li
others...

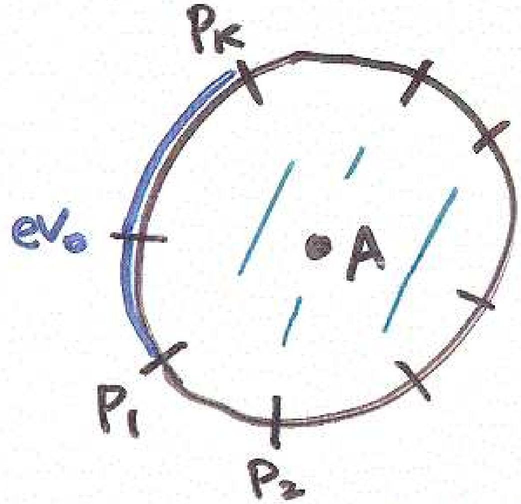


Here



Bott-Morse version

Further assume only 1 object in Fuk. category for simplicity



$A \in C^*(P;\mathbb{Q})$

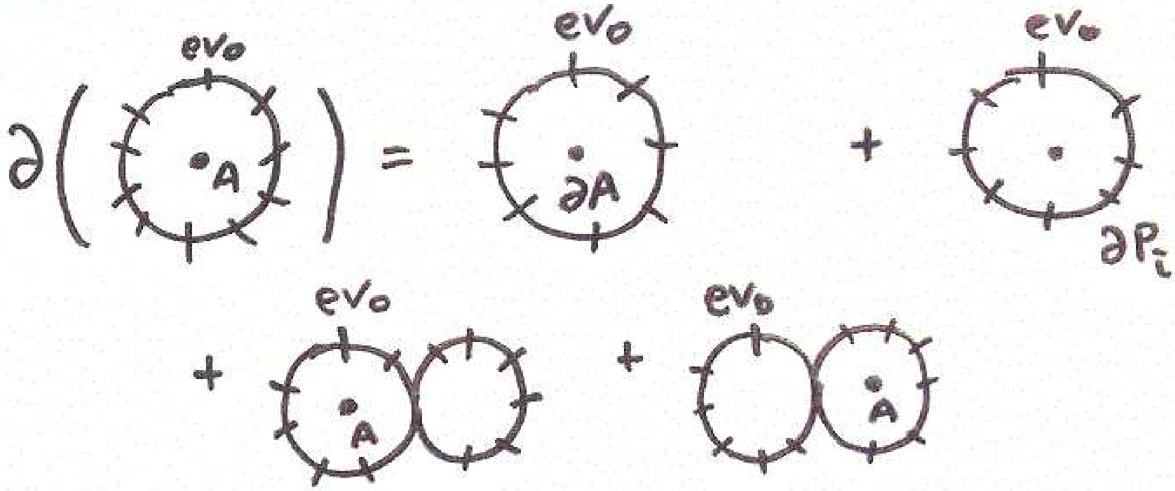
$\psi(A)(P_1, \dots, P_K) \to C^*(L, \Lambda_{nov})$

all J-holo discs intersect A at interior marked pt, $P_1 \cdots P_K$ at bdy marked pt.

$$\psi : C^{\bullet}(P, \Lambda_{nov}) \longrightarrow C^{\bullet}(C^{\bullet}(L, \Lambda_{nov}), C^{\bullet}(L, \Lambda_{nov}))$$

Chain in P $\longrightarrow$ Hoch cochain

$\boxed{1}$ ψ is a chain map



$$\Rightarrow \quad \psi(\partial A) = m \circ \psi(A) \pm \psi(A) \circ m$$

$$= [m, \psi(A)]$$

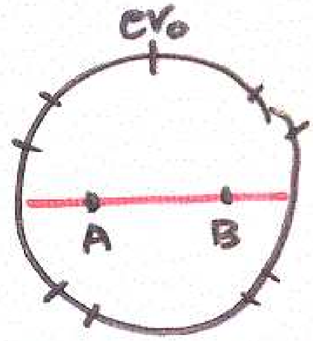
ψ induces a map between Homologies.

$$\mathrm{Sing}(M) \longrightarrow HH(\mathrm{Fuk})$$

$\boxed{2}$ ψ preserves ring structure.

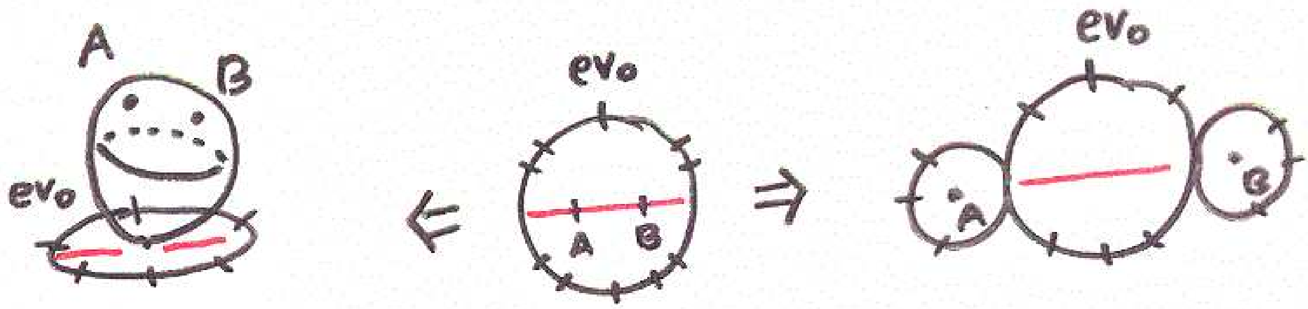
★ We need to use a submoduli space

Define $\psi(A, B) :=$



interior marked pts : symmetrically on real line.

provides the necessary homotopy



$\psi(A * B)$
↑
quantum coh. prod.
q_2

$m\{\psi(A), \psi(B)\}$
Hoch prod
M_2

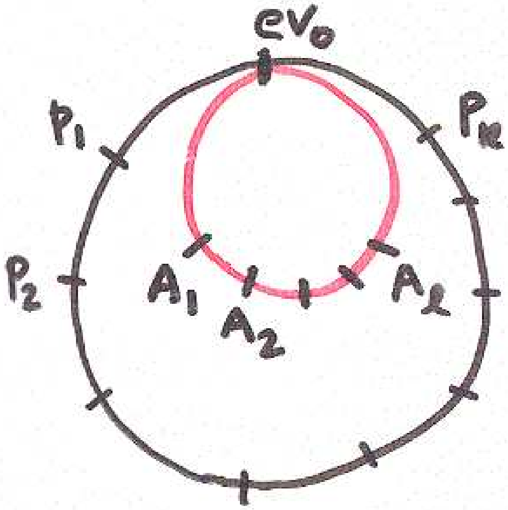
cod 1 limits

$+ [m, \psi(A,B)]$

$+ \psi(\partial A, B) + \psi(A, \partial B)$

$\boxed{3}$ ψ is an A_∞-homomorphism between two A_∞ algebras!

Define $\psi(A_1, A_2, \cdots, A_\ell)$ as Hoch. cochain



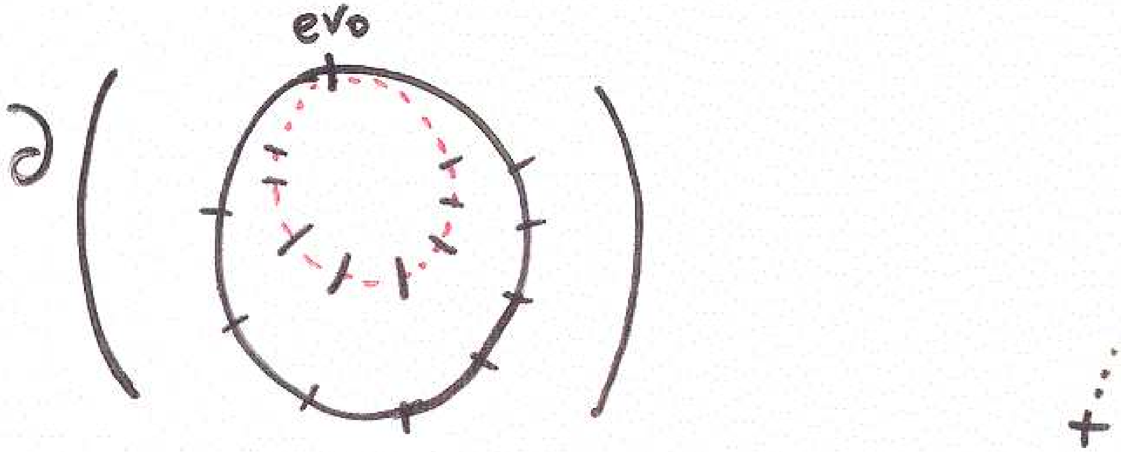
: any circle tangent to ∂D^2 at 0-th bdy marked pt.

← interior marked points lie on some circle cyclically counterclockwise

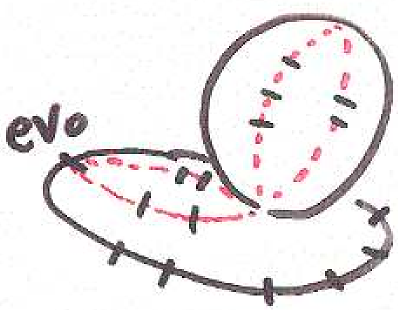
◉ For two A_∞-algebras $(C_1, \mathfrak{q})$, (C_2, M)

A_∞ homomorphism $f = (f_k) : (sC_1)^{\otimes k} \to sC_2$ deg 0

s.t



$=$



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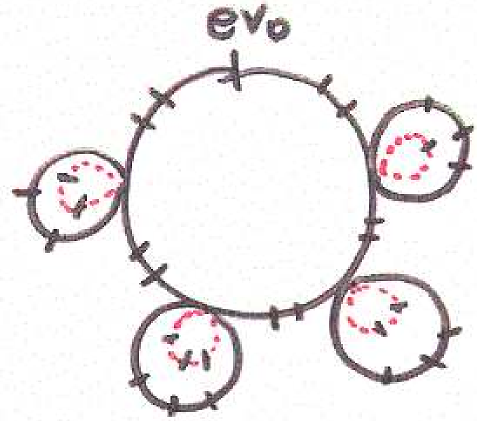
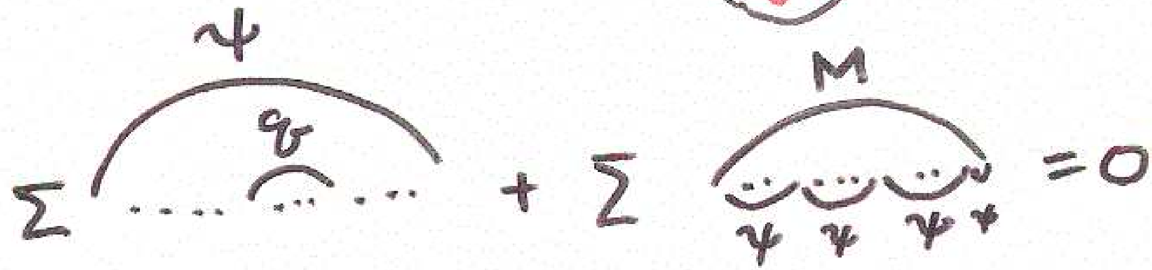


interior bubble

sphere

as 

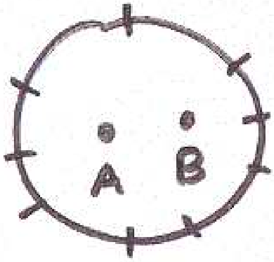


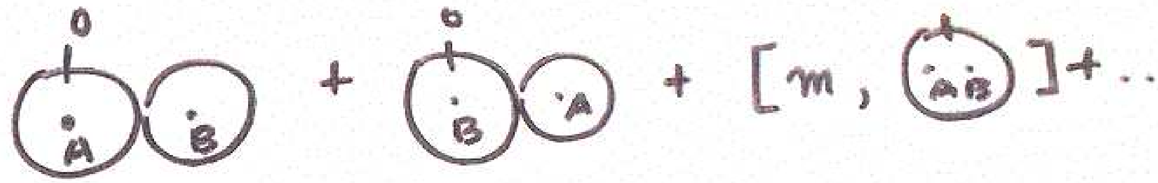
brace operation

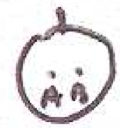
$m(\cdots \psi(\cdots)(\cdots) \cdots \psi(\cdots)(\cdots) \cdots)$

$\Rightarrow$ ψ is an A_∞-hom: $QC \to HH(Fuk)$.

Also,

$\partial($  $)$ without restriction shows that



$\Rightarrow \pm \{\psi(A), \psi(B)\} = [m,$  $]$ if $\partial A = 0$, $\partial B = 0$

$\Rightarrow$ Gerstenhaber bracket of the image vanishes in

$\psi : QC \to HH(Fuk)$

Well (?) Known Conjecture

ψ is an isomorphism.

Kapustin-Rozansky : Yes in B-model LG (Kontsevich's proposed category)

Kevin Costello $\to$ in 2 hours!

THANK YOU!