Calabi-Yau Crystals from Chern-Simons Gauge Theory

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Plan of the talk

- Introduction : Open/closed duality and Calabi-Yau crystals for $\mathbf{C}^3$
- Crystal model for the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$
- Derivation of the crystal from a unitary matrix model for Chern-Simons theory on S^3
- D-branes and knot invariants in terms of the crystal

Introduction 1: Open/closed duality

Topological open string $\stackrel{\text{dual}}{\leftrightarrow}$ Topological closed string

Prototype example (Gopakumar-Vafa): A-model

Open string on deformed conifold T^*S^3 with N D-branes wrapping S^3
(= Chern-Simons theory on S^3 via Witten's work)

$\Updownarrow$ dual

Closed string on resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$ with Kähler modulus $t = g_s N$

- T^*S^3 has a natural symplectic structure in which $S^3 \subset T^*S^3$ is a Lagrangian submanifold $\Rightarrow$ D-branes
- Large N duality implies in particular

$$\begin{aligned} & Z_{\text{open},T^*S^3}(g_s, N) \\ = \quad & Z_{\text{closed},\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbf{P}^1}(g_s, t = g_s N) \end{aligned}$$

Introduction 2: Calabi-Yau crystals

As a Calabi-Yau, take $\mathbf{C}^3$.

A-model partition function

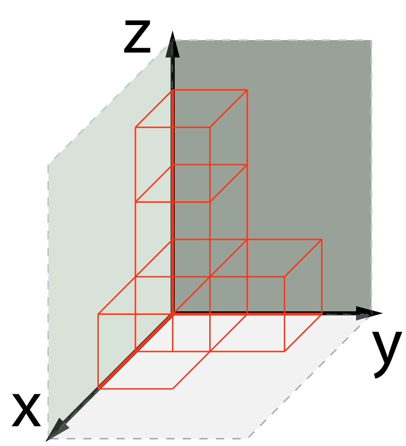
$$\begin{aligned} Z &= \prod_{j=1}^{\infty} \frac{1}{(1-q^j)^j}, \quad q := e^{-g_s} \\ &=: M(q) \end{aligned}$$

$M(q)$ is known as the **McMahon function**.

McMahon showed that $M(q)$ is the partition function of a classical statistical model:

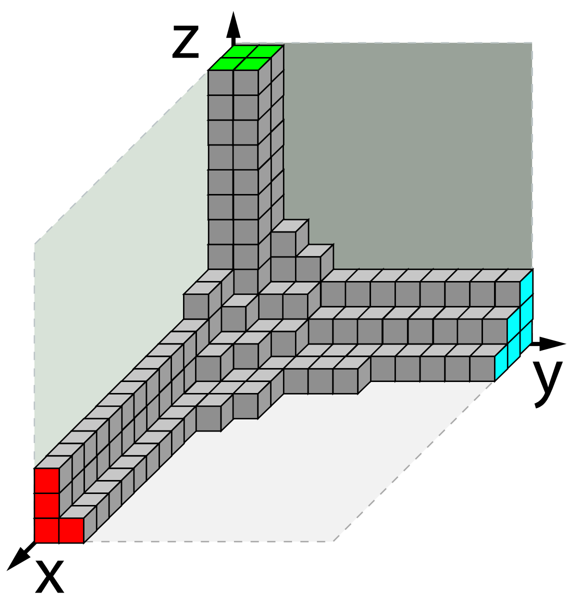
1. The positive octant of $\mathbf{R}^3$ and a cubic lattice.
   Cubic boxes = atoms.
   The positive octant = crystal.
2. Consider melted crystal configurations: Boxes are removed. Melting starts at the corner.
3. Melting rules can be formulated precisely.
4. A crystal has energy equal to the number of boxes removed.
5. Partition function

$$Z = \sum_{\text{confugurations}} q^{\text{number of boxes removed}}$$



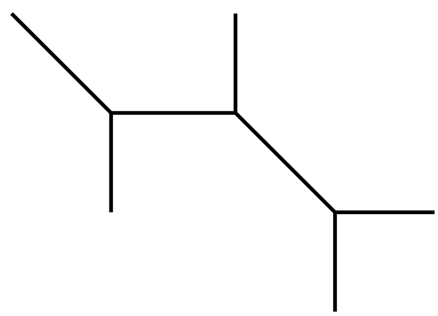
Crystal representations can be generalized to $\mathbf{C}^3$ with 3 non-compact D-branes (Okounkov-Reshetikhin-Vafa)

Asymptotic boundary conditions specified by Young diagrams μ, ν, and ρ. Need regularization.



This is precisely the toplogical vertex $C_{\mu\nu\rho}$.

Toplogical vertex and gluing rules
⇒ A-model amplitudes for any toric (non-compact) Calabi-Yau 3-fold



Iqbal, Nekrasov, Okounkov, and Vafa interpreted the topological vertex computation as the quantum foam formulation of the Kähler gravity (A-model topological closed string). Crystal atoms represent toric blow-ups and their generalizations.

CY crystals from CS theory

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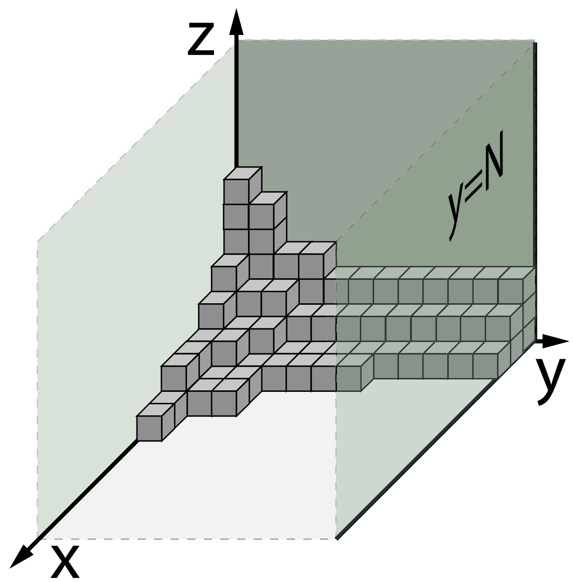
- Crystals represent violent fluctuations of topology and geometry.
- A geometric transition is a change of topology and geometry.

Question: Are they related?

-Indeed. It is possible to obtain new crystals from the open string side.

The model for $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1$: Crystal bounded by an extra wall:



Melting rules same as before, except atoms cannot be removed from $y > N$.

Will show:

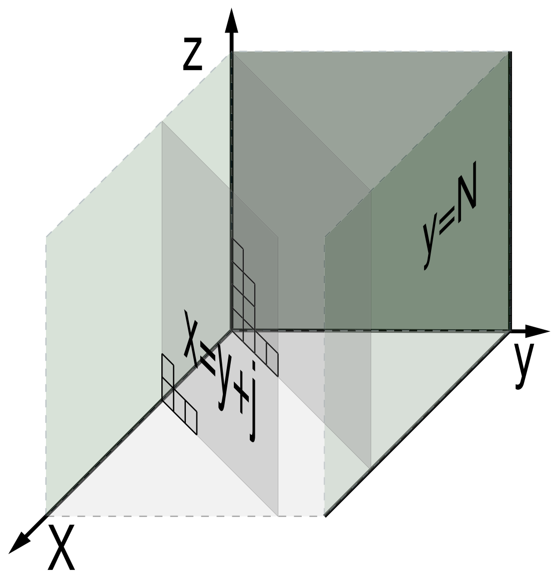
$$Z_{\text{crystal}} = M(q) e^{-\sum_{n>0} \frac{e^{-nt}}{n[n]^2}} (= Z_{\text{res.con.}}).$$

$([n] := q^{n/2} - q^{-n/2}, t := g_s N)$

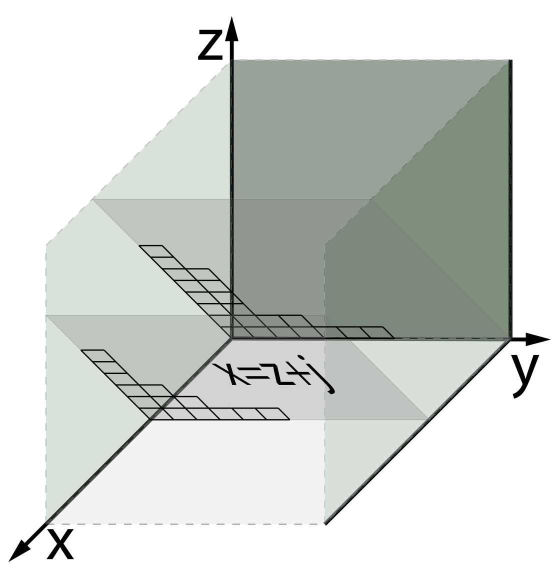
Different from Iqbal-Nekrasov-Okounkov-Vafa:

1. They got $Z = M(q)^{\mathbf{2}} e^{-\sum_{n>0} \frac{e^{-nt}}{n[n]^2}}$.
2. In their model, the distance between the pair of corners is not fixed. In ours, it is fixed.

Free fermions and two ways of slicing



Closed string slicing



Open string slicing

Computation via closed string slicing

One can represent the crystal in terms of a 2D free CFT.

$$\psi(z) = \sum_{r \in \mathbf{Z}+1/2} \frac{\psi_r}{z^{r+1/2}},\ \bar{\psi}(z) = \sum_r \frac{\bar{\psi}_r}{z^{r+1/2}},$$

$$\{\psi_r, \bar{\psi}_s\} = \delta_{r+s,0},$$

$$\phi(z) = x_0 - i\alpha_0 \log z + i \sum_{n \neq 0} \frac{\alpha_n}{n z^n},$$

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}.$$

- $\psi, \bar{\psi} \leftrightarrow \phi$ via bosonization: $i\partial\phi =: \psi\bar{\psi} :$
- Zero-charge fermionic Fock states are parameterized by Young diagrams $\mu = (\mu_1 \geq \mu_2 \geq ... \geq \mu_d > 0)$:

$$|\mu\rangle = \prod_{i=1}^{D} \bar{\psi}_{-(\mu_i - i + 1/2)} \psi_{-(\mu_i^t - i + 1/2)} |0\rangle$$

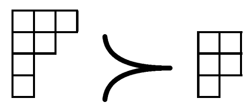
(D: the number of boxes in the diagonal)

- The Virasoro zero-mode counts the number of boxes: $L_0|\mu\rangle = |\mu||\mu\rangle$

The local melting rule can be formulated via the relation:

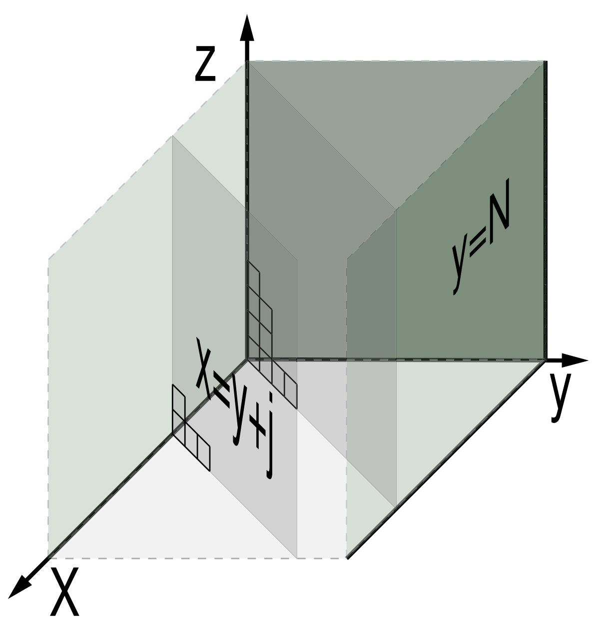
$$\lambda \succ \mu \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \lambda \supset \mu, (\lambda - \text{first row}) \subset \mu$$

(λ and μ are said to interlace)

For example, 

Then

$$\begin{aligned} & \{\text{Allowed configurations for } \mathbf{C}^3\} \\ = \quad & \{... \prec \mu^{(2)} \prec \mu^{(1)} \prec \mu^{(0)} \succ \mu^{(-1)} \succ \mu^{(-2)} \succ ...\}, \\ & \text{and} \\ & \{\text{Allowed configurations for } \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1\} \\ = \quad & \{... \prec \mu^{(1)} \prec \mu^{(0)} \succ \mu^{(-1)} \succ \mu^{(-2)} \succ ... \succ \mu^{(-N)} = \bullet\}, \end{aligned}$$



Closed string slicing

Let us define

$$\Gamma_\pm(z) = \exp \sum_{n>0} \frac{z^{\pm n}}{n} \alpha_{\pm n}.$$

These are useful because

$$\Gamma_+(1)|\lambda\rangle = \sum_{\lambda \succ \mu} |\mu\rangle, \quad \Gamma_-(1)|\lambda\rangle = \sum_{\mu \succ \lambda} |\mu\rangle.$$

Then

$$\begin{aligned} & Z(\text{crystal for } \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1) \\ = & \sum_{\text{allowed configurations}} q^{-\sum_{j \in \mathbf{Z}} |\mu^{(j)}|} \\ = & \langle 0| \left(\prod_{n=1}^{\infty} q^{L_0} \Gamma_+(1) \right) q^{L_0} \left(\prod_{m=1}^{N} \Gamma_-(1) q^{L_0} \right) |0\rangle \\ = & \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-1/2}) \prod_{m=1}^{N} \Gamma_-(q^{-(m-1/2)}) |0\rangle \end{aligned}$$

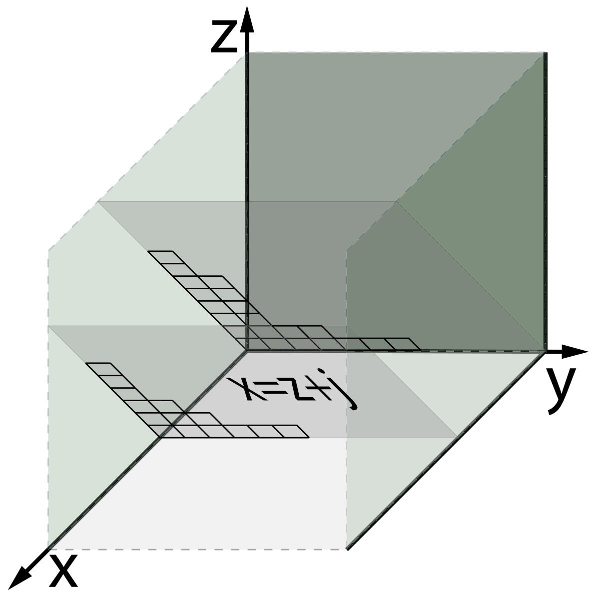
It is easy to evaluate this CFT correlator and find

$$Z = M(q) e^{-\sum_{n>0} \frac{e^{-nt}}{n[n]^2}}, \quad t = g_s N.$$

Closed string slicing computes closed string amplitudes.

Open string slicing

There is another way to implement the free field representation:



In this picture,

$$
\begin{aligned}
& \{\text{Allowed configurations for } \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1\} \\
= \quad & \{ \ldots \prec \mu^{(1)} \prec \mu^{(0)} \succ \mu^{(-1)} \succ \mu^{(-2)} \succ \ldots | (\mu^{(0)})_1 \leq N \}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& Z(\text{crystal for } \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{P}^1) \\
= \quad & \langle 0 | \prod_{n=1}^{\infty} \Gamma_+(q^{n-1/2}) \mathbf{1}_{\mu_1 \leq N} \prod_{m=1}^{\infty} \Gamma_-(q^{-(m-1/2)}) | 0 \rangle
\end{aligned}
$$

We will now show that open string slicing arises from Chern-Simons theory on S^3.

Unitary matrix model for CS theory on S^3

$$
\begin{aligned}
& Z(\text{Chern–Simons on } \mathrm{S}^3) \\
= \;& S_{00}(\text{Level } k \; U(N) \text{ current algebra}) \\
\simeq \;& \sum_{w \in W:\text{Weyl group}} \epsilon(w) q^{\frac{1}{2}(w(\rho)-\rho)^2}, \quad \rho : \text{Weyl vector}
\end{aligned}
$$

By massaging it, we get

$$
\begin{aligned}
Z(CS) &= \frac{1}{|W|} \int \left(\prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \vartheta_{00}(e^{i\theta_i}; q) \right) \left(\prod_{\alpha>0} 2 \sin \frac{\alpha \cdot \theta}{2} \right)^2 \\
&= \int_{U(N)} dU \det \vartheta_{00}(U; q), \quad dU : \text{Haar measure}
\end{aligned}
$$

Here we have introduced one of Jacobi's theta functions

$$
\vartheta_{00}(e^{i\theta}; q) := \sum_{m \in \mathbf{Z}} q^{\frac{m^2}{2}} e^{im\theta}
$$

Related to the matrix model of Marino via a modular transformation of ϑ_{00}.

Crystal from Chern-Simons theory

Via the product formula for the theta function, we get

$$Z(CS) = \left(\prod_{j=1}^{\infty}(1-q^j)\right)^N \int dU \exp\left[\sum_{n=1}^{\infty} \frac{\mathrm{Tr}U^n + \mathrm{Tr}U^{-n}}{n[n]}\right]$$

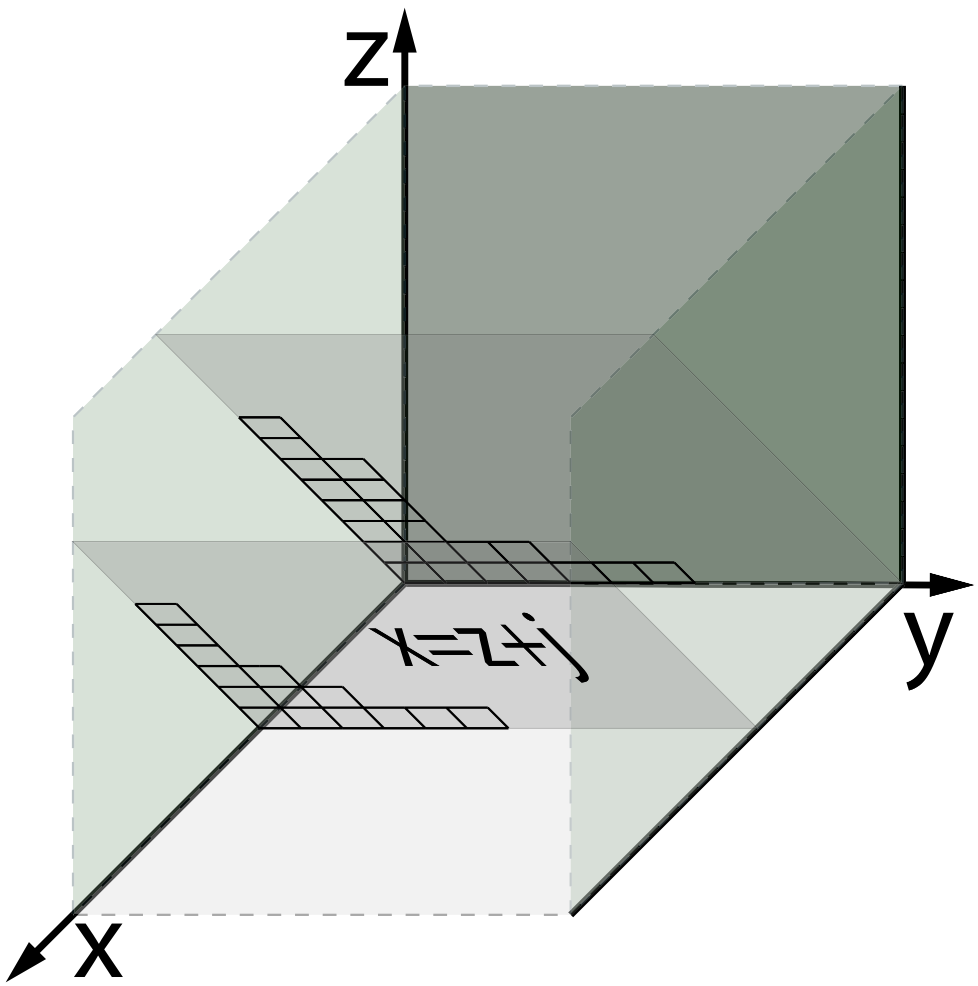
One can define states $|U\rangle$ such that

$$\int dU|U\rangle\langle U| = \mathbf{1}_{d\leq N} \; : \text{ Projects onto } \mu \text{ with at most } N \text{ rows}$$

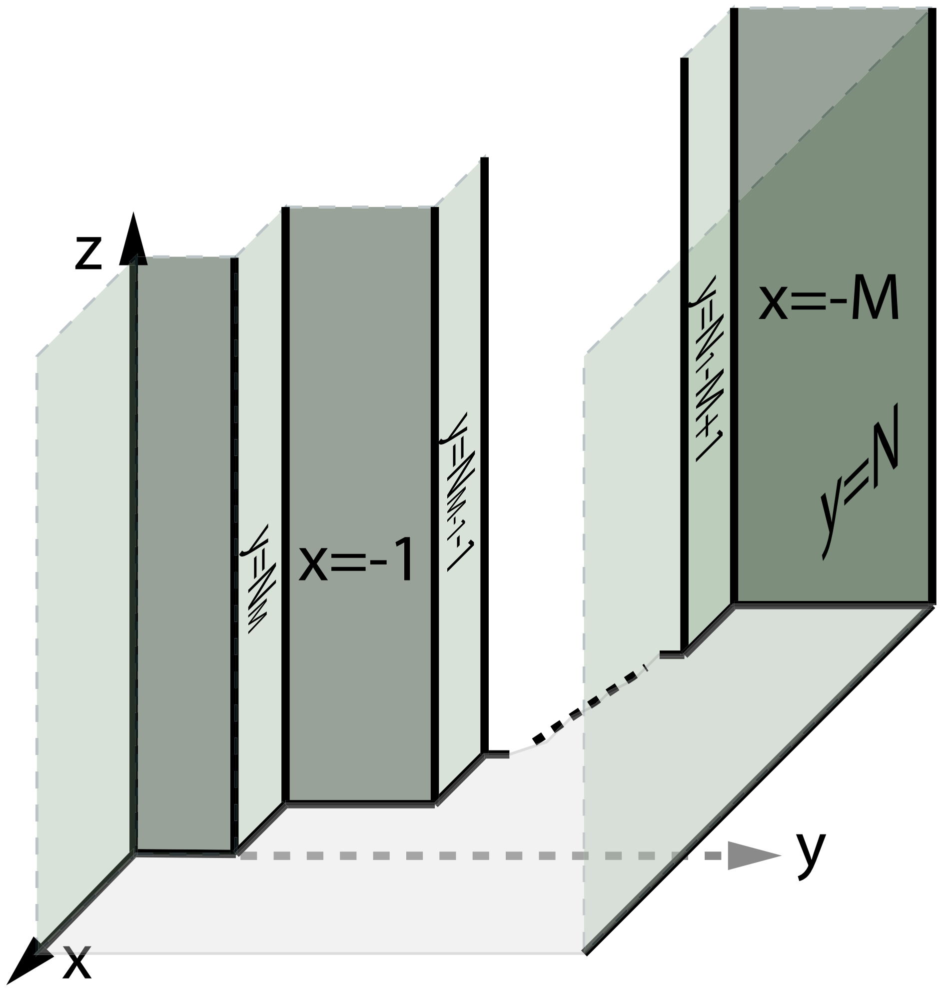
$$Z(CS) \simeq \langle 0|e^{\sum_{n=1}^{\infty} \frac{\alpha_n}{n[n]}}\mathbf{1}_{d\leq N}e^{\sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n[n]}}|0\rangle$$

$$Z(CS) \quad = \quad \langle 0| \prod_{n=1}^{\infty} \Gamma_{+}(q^{n-1/2}) \mathbf{1}_{\mu_1 \leq N} \prod_{m=1}^{\infty} \Gamma_{-}(q^{-(m-1/2)}) |0\rangle .$$

This free field correlator represents the open string slicing:



Adding D-branes



- Closed string slicing
  ⇒ Explicit computations of the D-brane amplitude.
- Open string slicing
  ⇒ Matrix model representation of (Un)knot invariants.

From closed string slicing:

$$Z_\mu \simeq M(q)\prod_{i=1}^{M}\exp\left[-\sum_{n=1}^{\infty}\frac{e^{-n\tilde{t}}}{n[n]^2}\right] \times\prod_{i=1}^{M}\exp\left[\sum_{n=1}^{\infty}\frac{e^{-na_i}+e^{-n(\tilde{t}-a_i)}}{n[n]}\right]$$

Here $a_i := g_s(N_i+1/2), \tilde{t} := g_s(N+M)$, and $\mu_i^t = N_i - M + i, i = 1, ..., M$.

From open string slicing,

$$Z_\mu \simeq \int dU \det \vartheta_{00}(U;q)\mathrm{Tr}_\mu U$$

Summary

- Proposed new (fixed) crystal model for the resolved conifold.
- Derived these crystals from the Chern-Simons theory on S^3.
- Found a unitary matrix model formulation of the Chern-Simons theory on S^3.
- Generalized the crystals to include non-compact D-branes.

Different from INOV.

Open question: Generalize beyond the resolved conifold?