Colouring Even Cycle Systems

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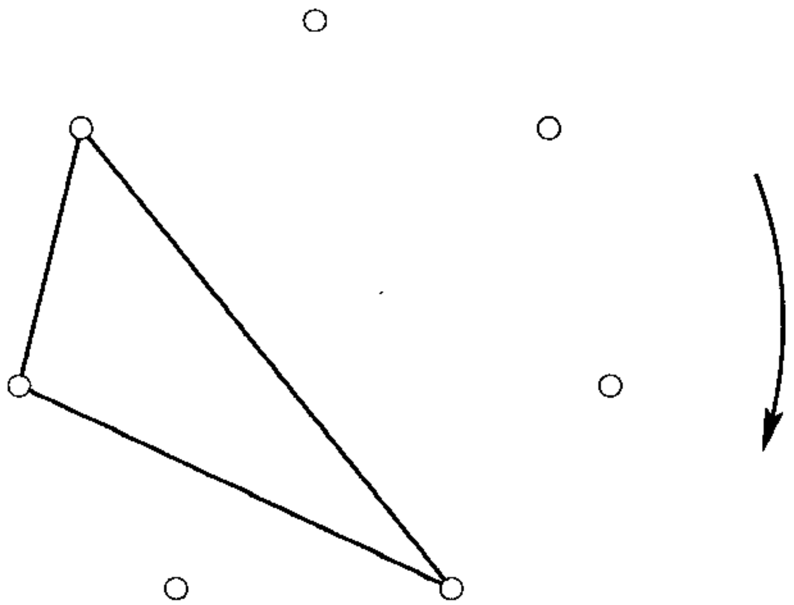
Definition:

An *m-cycle system of order n* is a partition of the edges of the complete graph K_n into cycles of length m.

Example: A cyclic 3-cycle decomposition of K_7:



Theorem (Alspach & Gavlas (2001), Šajna (2002)):

An m-cycle system of order n exists if and only if n is odd, m divides $\frac{n(n-1)}{2}$, and either $n = 1$ or $n \geq m$.

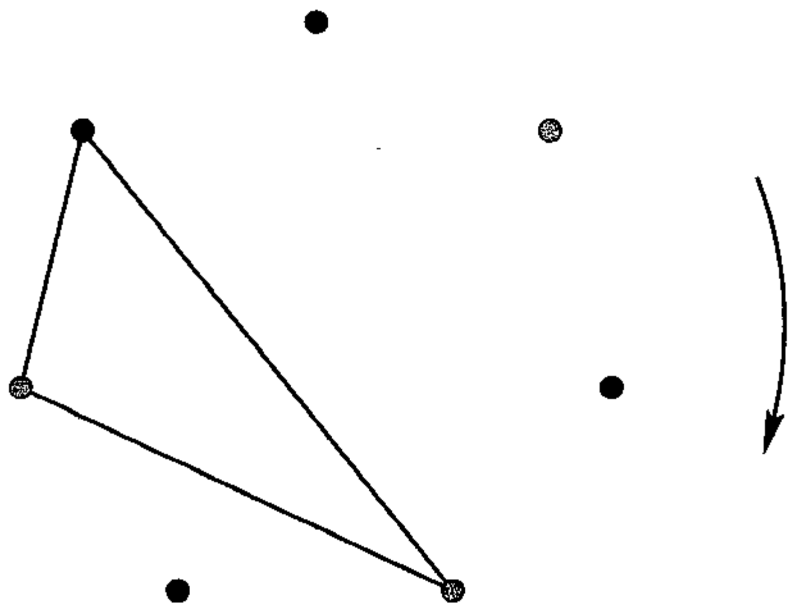
Cycle Length	Admissible Orders
$m = 3$	$n \equiv 1$ or $3 \pmod{6}$
$m = 4$	$n \equiv 1 \pmod{8}$
$m = 5$	$n \equiv 1$ or $5 \pmod{10}$

Definition:

A cycle system is *k-colourable* if its vertex set can be partitioned into k sets (i.e. colour classes) so that no cycle is monochromatic.

A cycle system is *k-chromatic* if it is k-colourable but is not $(k-1)$-colourable. The system's *chromatic number* is this k.

Example: A 3-colouring of a STS(7):



Lemma (Rosa (1970)):

Every 3-cycle system of order $n \geq 7$ has chromatic number at least 3.

Theorem (Rosa (1970)):

For every admissible order $n \geq 7$, there exists a 3-chromatic 3-cycle system of order n.

Theorem (Mathon, Phelps, & Rosa (1983)):

For every admissible order n, $7 \leq n \leq 15$, all 3-cycle systems of order n are 3-chromatic.

Observation (Rosa (1970)):

For any positive integer k, there is a 3-cycle system with chromatic number at least k.

Theorem (de Brandes, Phelps and Rödl (1982)):

For every integer $k \geq 3$, there exists an integer $n_3(k)$ such that for any admissible order $n \geq n_3(k)$, there is a k-chromatic 3-cycle system of order n.

Theorem (Milici and Tuza (1999)):

For each integer $m > 3$, every m-cycle system of order $2m + 1$ is 2-colourable. For each integer $m \geq 10$, every m-cycle system of order $4m + 1$ is 2-colourable.

Theorem (Milici and Tuza (1996)):

For every integer $m \geq 3$, there is an m-cycle system which is not 2-colourable.

Theorem: (Burgess and Pike):

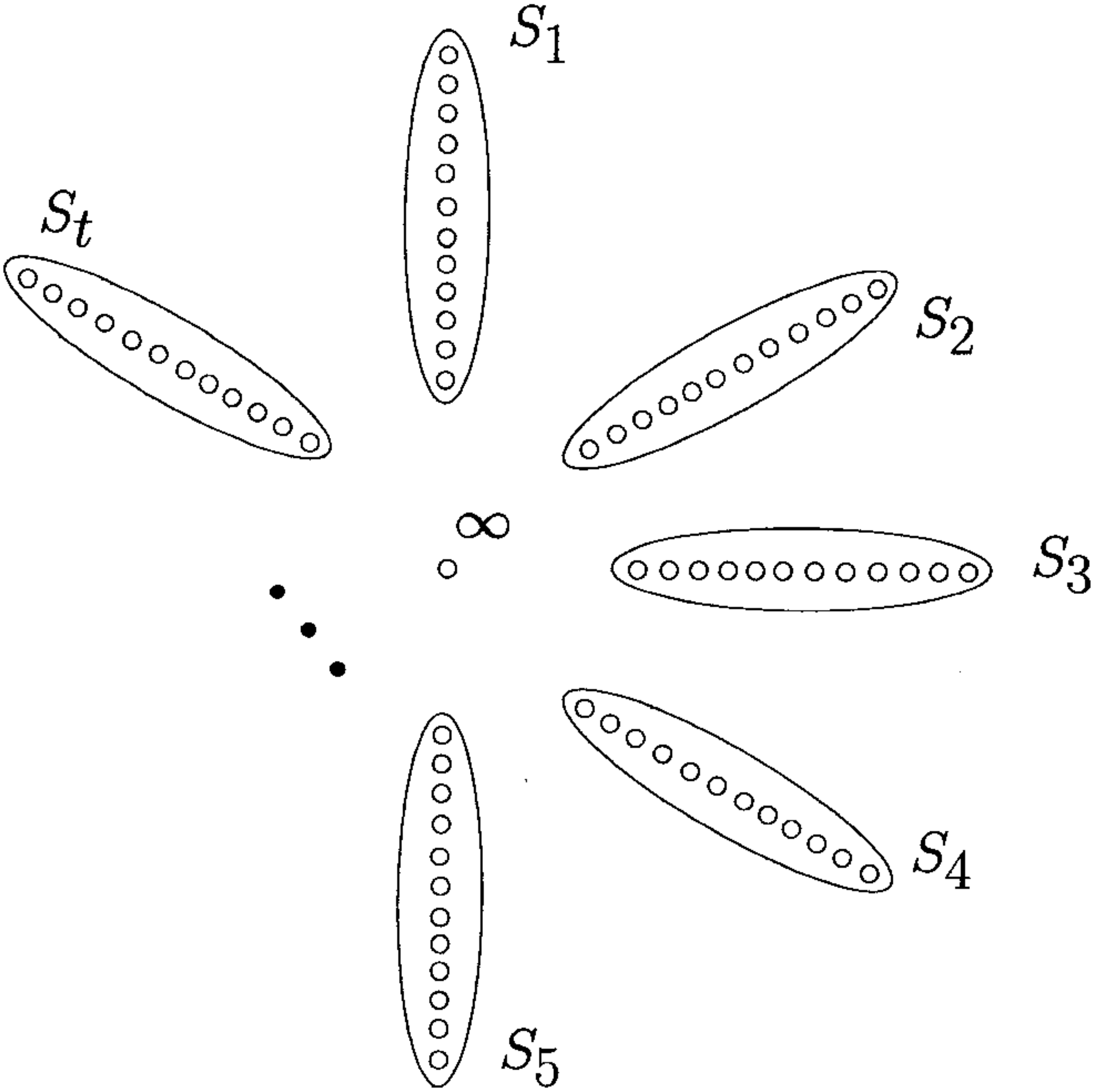
For any integer $k \geq 2$, there is an integer w_k such that for any admissible order $n \geq w_k$, there is a k-chromatic 4-cycle system of order n.

Moreover, letting $n_4(k)$ be the smallest such w_k which is an admissible order, $n_4(k)$ is the smallest order for which there exists a k-chromatic 4-cycle system.

Theorem (Burgess and Pike):

For any integers $k \geq 3$ and $r \geq 3$, there is a k-chromatic $(2r)$-cycle system.

A 3-chromatic 6-cycle system:

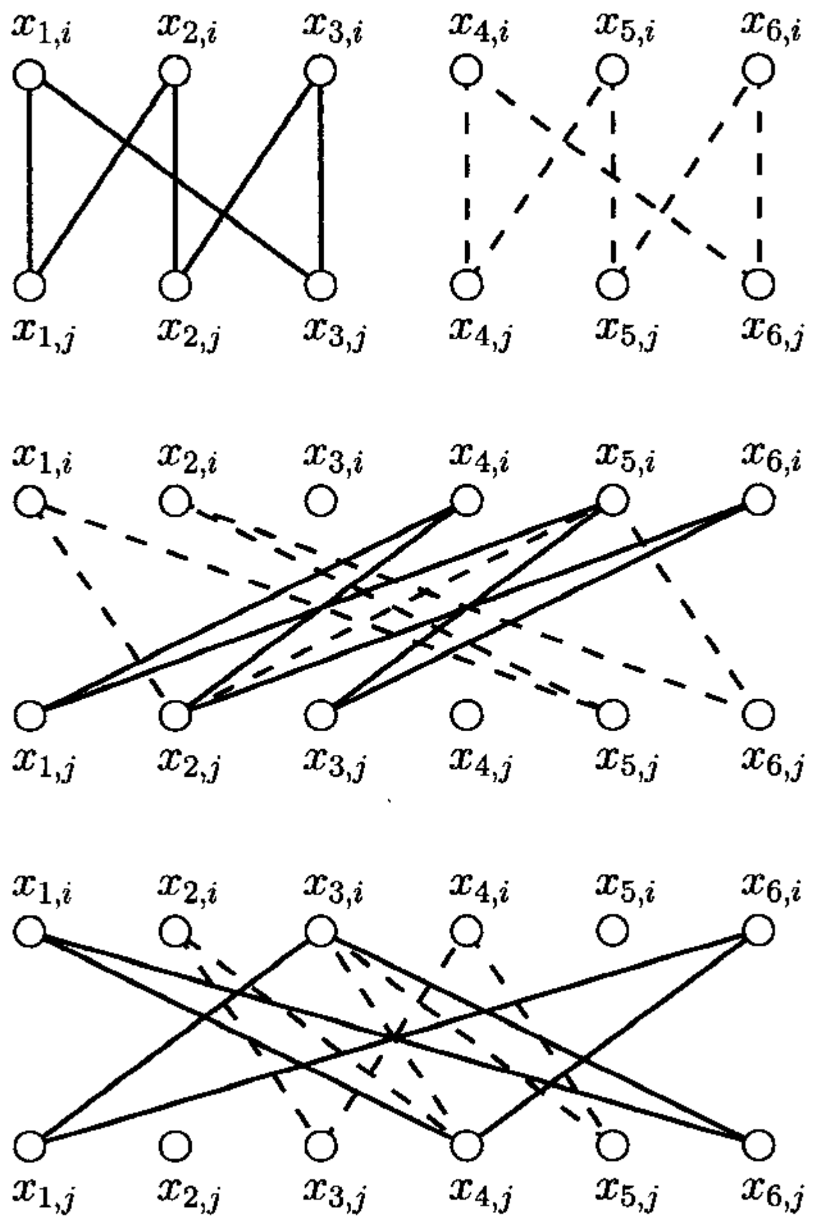


For each $i \in \{1, \ldots, t\}$, let

$$S_i = \{x_{1,i}, x_{2,i}, \ldots, x_{12,i}\}$$

and let $X_i = \{x_{1,i}, \ldots, x_{6,i}\}$.

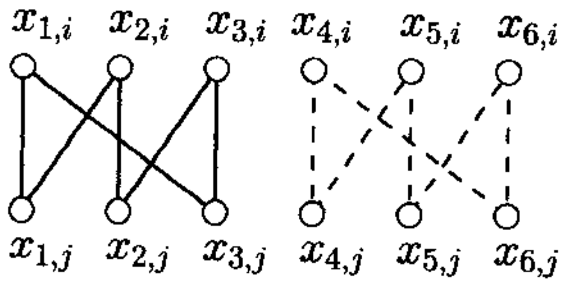
A 6-cycle decomposition, $\mathcal{D}_1$, of $K_{6,6}$ (between X_i and X_j):



If X_i and X_j have $\mathcal{D}_1$ between them, they cannot both be coloured $ccc{*}{*}{*}$, and they cannot both be coloured ${*}{*}{*}ccc$.

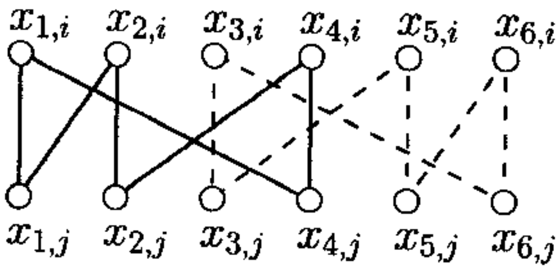
The ten 6-cycle decompositions of $K_{6,6}$ used:

$\mathcal{D}_1$:



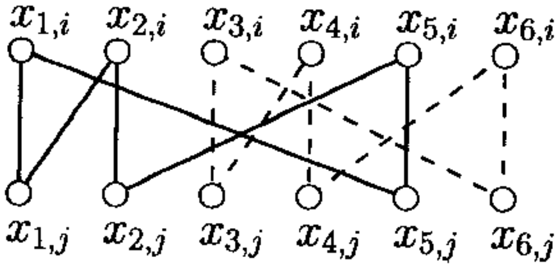
$\mathcal{P}_{c,1} = \{ccc{*}{*}{*}, {*}{*}{*}ccc\}$

$\mathcal{D}_2$:



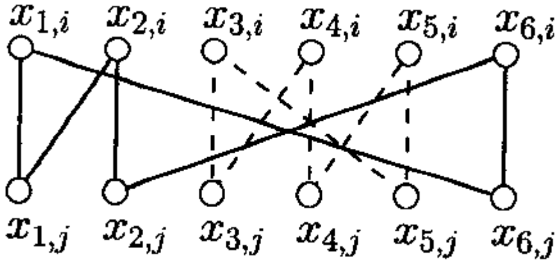
$\mathcal{P}_{c,2} = \{cc{*}c{*}{*}, {*}{*}c{*}cc\}$

$\mathcal{D}_3$:



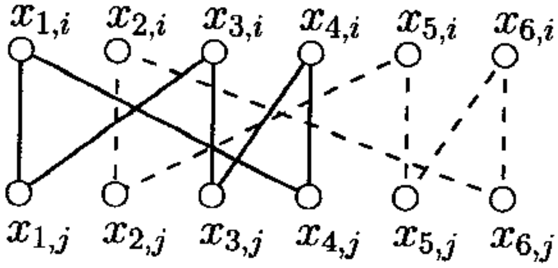
$\mathcal{P}_{c,3} = \{cc{*}{*}c{*}, {*}{*}cc{*}c\}$

$\mathcal{D}_4$:



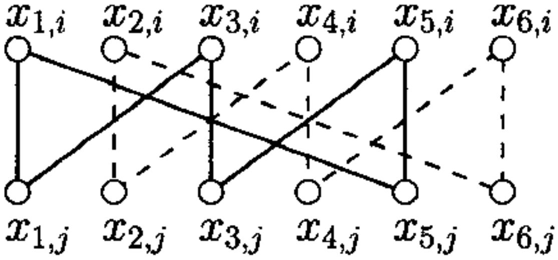
$\mathcal{P}_{c,4} = \{cc{*}{*}{*}c, {*}{*}ccc{*}\}$

$\mathcal{D}_5$:



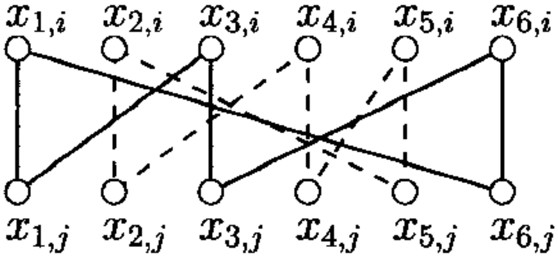
$\mathcal{P}_{c,5} = \{c{*}cc{*}{*}, {*}c{*}{*}cc\}$

$\mathcal{D}_6$:



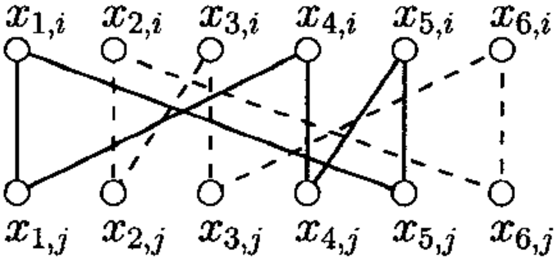
$$\mathcal{P}_{c,6} = \{c{*}c{*}c{*},\ {*}c{*}c{*}c\}$$

$\mathcal{D}_7$:



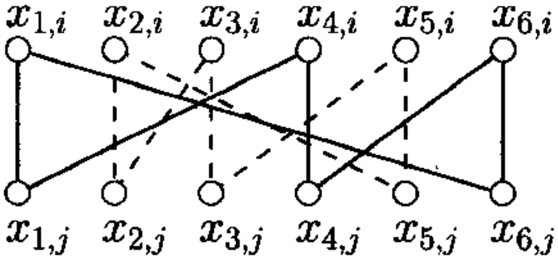
$$\mathcal{P}_{c,7} = \{c{*}c{*}{*}c,\ {*}c{*}cc{*}\}$$

$\mathcal{D}_8$:



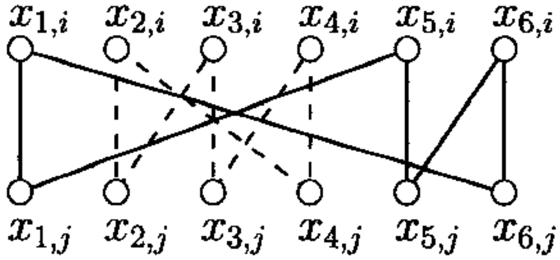
$$\mathcal{P}_{c,8} = \{c{*}{*}cc{*},\ {*}cc{*}{*}c\}$$

$\mathcal{D}_9$:



$$\mathcal{P}_{c,9} = \{c{*}{*}c{*}c,\ {*}cc{*}c{*}\}$$

$\mathcal{D}_{10}$:



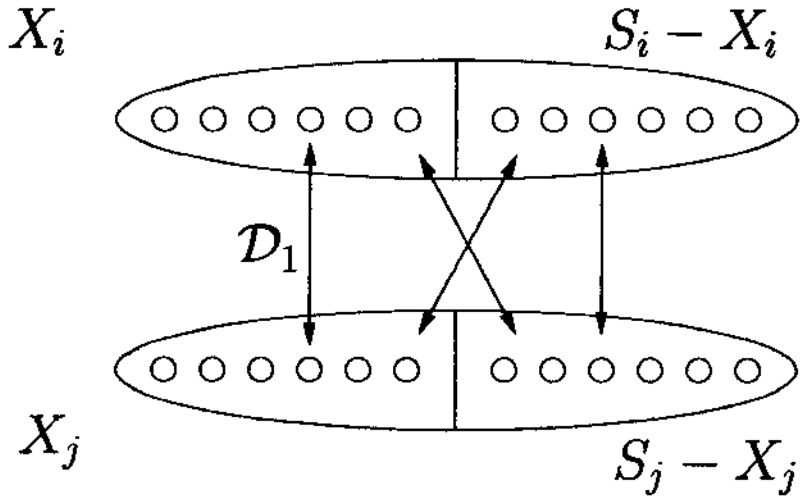
$$\mathcal{P}_{c,10} = \{c{*}{*}{*}cc,\ {*}ccc{*}{*}\}$$

For any pattern $P \in \mathcal{P}_{c,\alpha}$, no two sets X_i and X_j with $\mathcal{D}_\alpha$ between them can both have colouring matching P.

Any 2-colouring of X_i matches a pattern in one of the pattern sets $\mathcal{P}_{c,\alpha}$, $c \in \{1, 2\}$, $\alpha \in \{1, \dots, 10\}$.

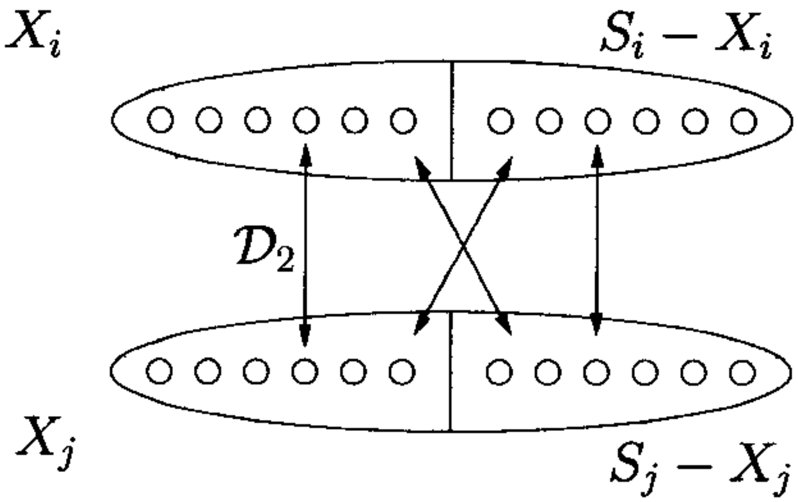
Constructing the system:

Take five sets S_1, S_2, S_3, S_4, S_5.



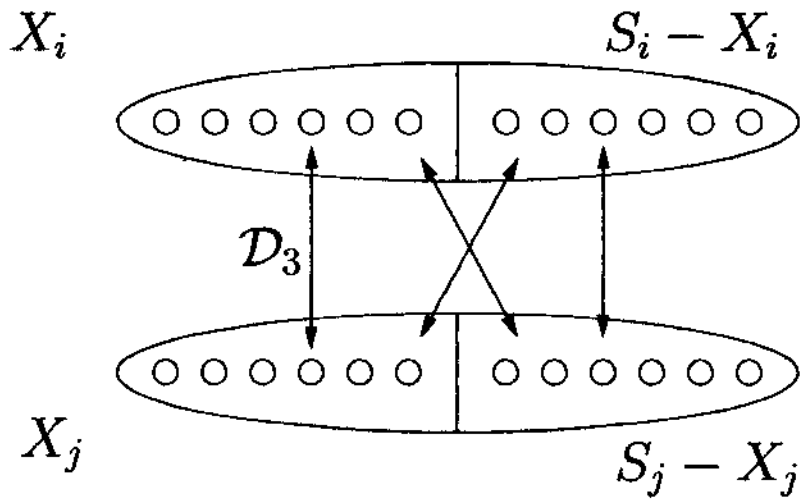
Resulting configuration: R_1.

Take five copies of R_1.



Resulting configuration: R_2

Take five copies of R_2.



Resulting configuration: R_3

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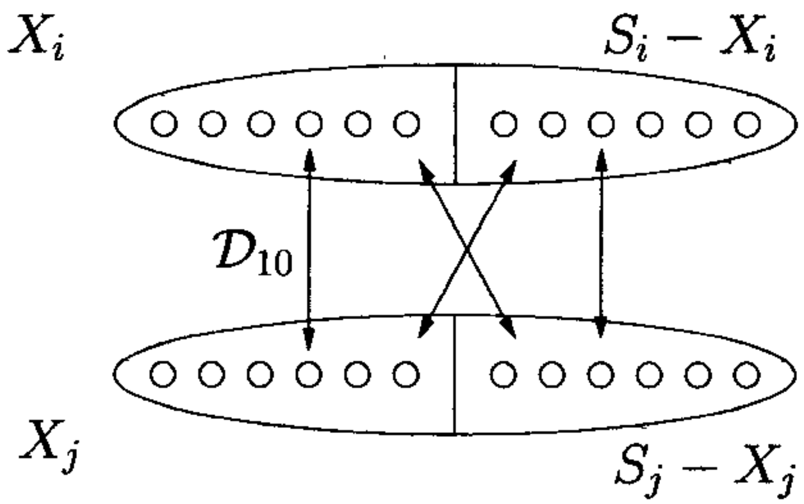
Take five copies of R_9.



Resulting configuration: R_{10}

Can we 2-colour the system?

R_{10} contains five copies of R_9; any X_i and X_j in different copies have $\mathcal{D}_{10}$ between them. For any $P \in \mathcal{P}_{1,10} \cup \mathcal{P}_{2,10}$, at most one copy of R_9 has a set X_i with colouring matching P.

So (at least) one copy of R_9 has no set X_i coloured to match any pattern in $\mathcal{P}_{c,10}$, $c \in \{1, 2\}$.

This copy of R_9 contains five copies of R_8; any X_i and X_j in different copies have $\mathcal{D}_9$ between them. For any $P \in \mathcal{P}_{1,9} \cup \mathcal{P}_{2,9}$, at most copy of R_8 has a set X_i with colouring matching P.

So (at least) one copy of R_8 has no set X_i coloured to match any pattern in $\mathcal{P}_{c,9}$ or $\mathcal{P}_{c,10}$, $c \in \{1, 2\}$.

Continuing similarly, we get that there must be a set X_i whose colouring matches no pattern in $\mathcal{P}_{c,\alpha}$ for any $c \in \{1, 2\}$, $\alpha \in \{1, \ldots, 10\}$.

A 3-colouring: