Constructing Optimal Solutions to the Minimum Cost 2-edge-connected Spanning Subgraph Problem

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Outline

1 Introduction

2 Ear Decompositions

3 Necklaces

4 Min-cut Cactii

5 Summary

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Preliminaries

- $K_n = (V, E)$ - the complete undirected graph on n vertices

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Definition

A multigraph G is *2-edge-connected* if, for every pair, $\{u, v\}$, of distinct vertices of G, there exist two edge-disjoint $\{u, v\}$-paths.

The Problem

Definition

The *Minimum Cost 2-edge-connected Spanning Subgraph Problem* with respect to c (henceforth called 2EC(c)) is the problem of finding a 2-edge-connected spanning multigraph of K_n which has minimum total edge-cost with respect to c.

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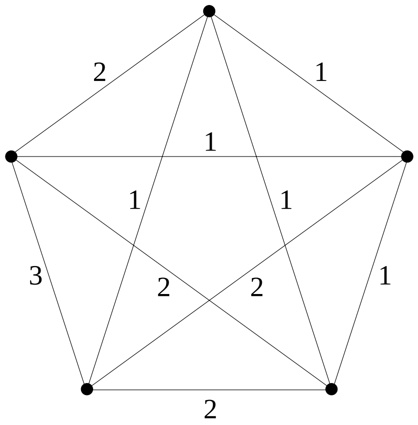
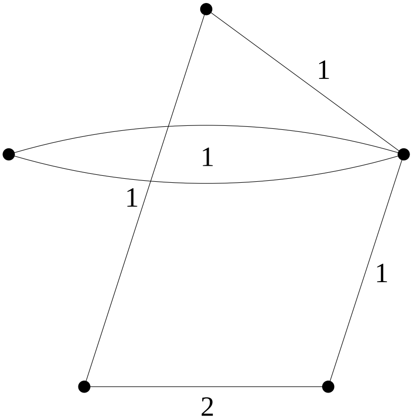
- Also called Network Survivability Problem
- 2EC(c) is NP-complete!

An Example of 2EC



An Example of 2EC



Metric Costs

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A cost function c is *metric* if for every distinct triple, $u, v, w \in V$, we have that

$$c_{uv} \leq c_{uw} + c_{wv}.$$

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The best approximation algorithm (Frederickson and Ja'Ja, 1982) for 2EC(c) when c is metric has a performance guarantee of $\frac{3}{2}$.

A Family of Optimal Solutions

Theorem (Monma, Munson, and Pulleyblank, 1990)

If c is a metric cost function then there is an optimal solution to $2EC(c)$ which is simple and

- *is edge-minimally 2-edge-connected,*
- *is 2-vertex-connected,*
- *every vertex has degree 2 or 3, and*
- *removing any pair of edges leaves a bridge in one of the resulting components.*

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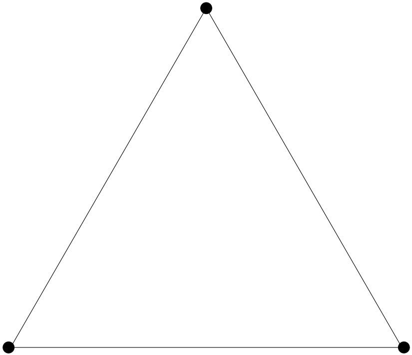
Let $\mathcal{M}$ denote the set of all graphs which have the above properties.

Some Graphs in $\mathcal{M}$



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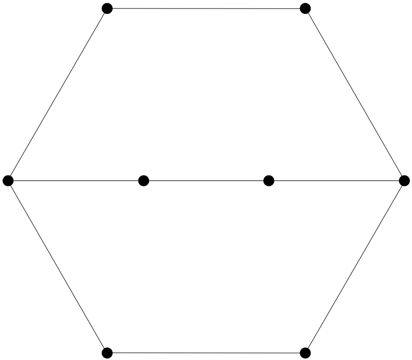

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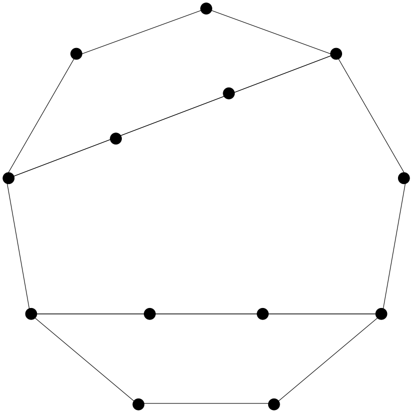

Purpose of our Research

We want to learn more about the graphs in $\mathcal{M}$ and their construction in the hopes that we can find a new approximation algorithm for 2EC(c) when c is metric.

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Ear Decompositions

Definition

A graph G has an *ear decomposition* if there is a sequence of subgraphs, $H_0, H_1, \ldots, H_k$, of G such that

- H_0 is a cycle,
- $H_k = G$, and
- for each $0 \leq i \leq k-1$, H_{i+1} is obtained by adding a path, P_i, of G to H_i where the endpoints of P_i are the only vertices of H_i in P_i.

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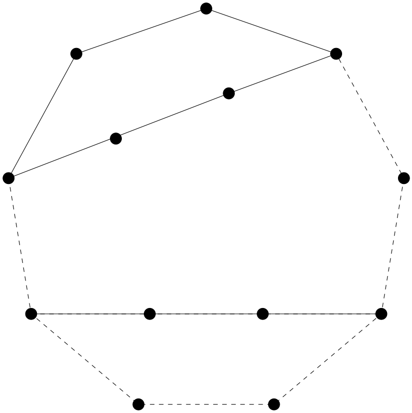
Theorem (Whitney, 1932)

A graph G is 2-edge-connected if and only if it has an ear decomposition.

An Example of an Ear Decomposition



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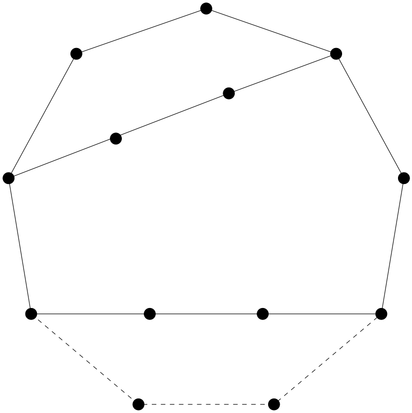

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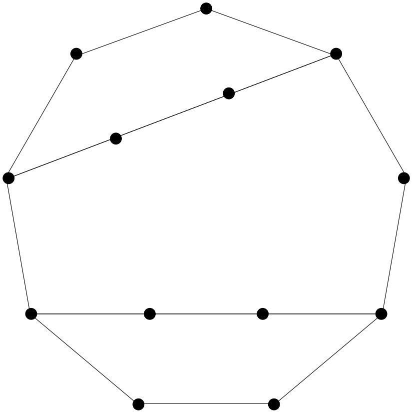

Ear Decompositions of Graphs in $\mathcal{M}$

Theorem

Let G be 2-edge-connected and let $H_0, H_1, \ldots, H_k$ be an ear decomposition of G. $G \in \mathcal{M}$ if and only if $H_i \in \mathcal{M}$ for every $0 \leq i \leq k$.

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Let $G \in \mathcal{M}$ and let $P_0, P_1, \ldots, P_{k-1}$ be the paths in an ear decompostion of G. For each $0 \leq i \leq k - 1$

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- the endpoints of P_i must be distinct in H_i,

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- the endpoints of P_i must have degree 2 in H_i, and

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- the endpoints of P_i must be distinct in H_i,
- the endpoints of P_i must have degree 2 in H_i, and
- $|P_i| \geq 3$.

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Min-cuts in a Graph

Definition

In a graph which is k-edge-connected but not $(k+1)$-edge-connected, a *min-cut* is any $\delta(S)$ containing exactly k edges.

Necklaces

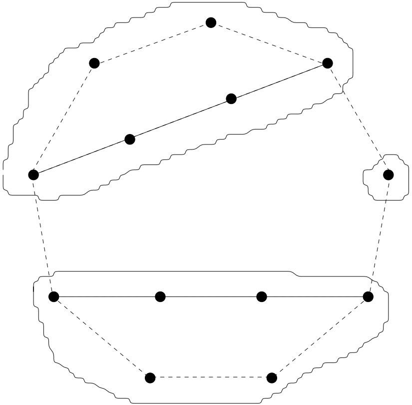
Definition

A *necklace* of a graph, G, which has a min-cut of size 2 is a partition $(V_0, V_1, \ldots, V_{k-1})$ of V such that for every $0 \leq i \leq k-1$,

- $\delta_G(V_i)$ is a min-cut,
- the edges of $\delta_G(V_i)$ have their other endpoints in V_{i-1} and V_{i+1} respectively, and
- there is no finer partition with these properties.

We call $V_0, \ldots, V_{k-1}$ the *beads* of the necklace.

An Example of a Necklace



Properties of Necklaces

Proposition

Let G be a graph with a min-cut $\delta_G(S)$ of size 2. Then S is the union of consecutive beads in some necklace of G.

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Proposition

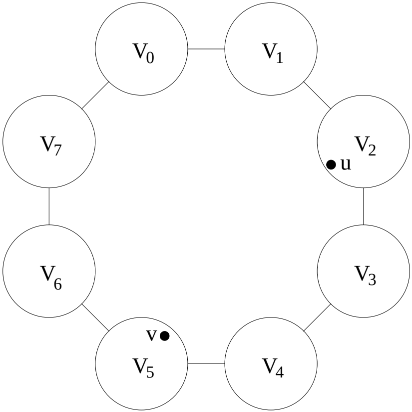
If $G \in \mathcal{M}$ then every edge of G is an inter-bead edge of a unique necklace and every necklace has at least three beads.

Ear Decompositions and Necklaces of Graphs in $\mathcal{M}$

Theorem

Let $H \in \mathcal{M}$ with distinct vertices, u and v, each of degree 2. Let P be a path of length at least 3 and let G be the graph obtained by identifying the endpoints of P with u and v. $G \in \mathcal{M}$ if and only if for any necklace $(V_0, V_1, \ldots, V_{k-1})$ of H where $u \in V_i$ and $v \in V_j$ we have that $i - j \neq \pm 1 \bmod k$ and $i - j \neq \pm 2 \bmod k$.

Adding an Ear to a Necklace



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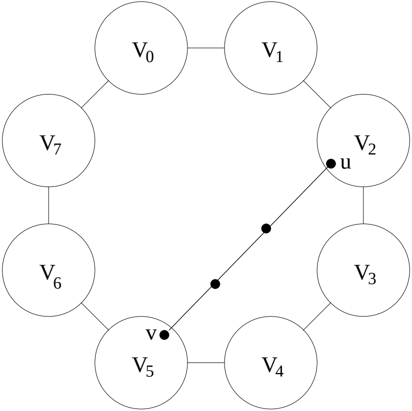

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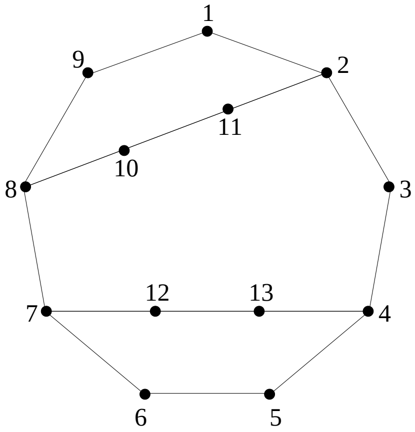
- the min-cuts in a cactus consist of pairs of edges in the same cycle
- the min-cuts in a graph $G \in \mathcal{M}$ consist of pairs of inter-bead edges in the same necklace
- Dinits, Karzanov, and Lomonosov (1976) used a cactus to describe min-cuts in a graph

A Graph and its Min-cut Cactus



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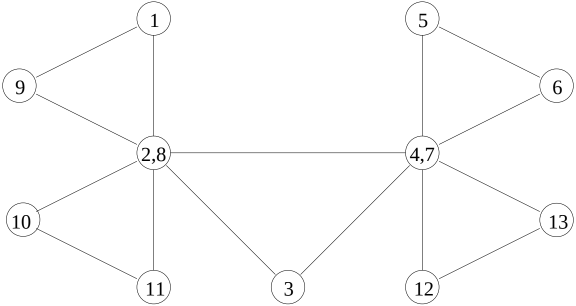
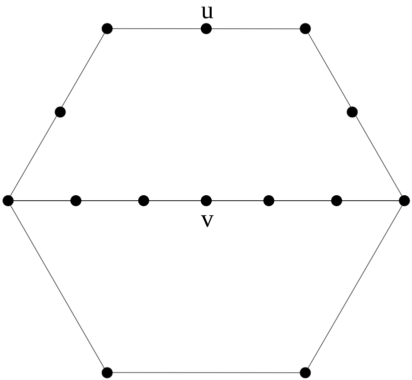


Constructing Graphs in $\mathcal{M}$

Proposition

Let $H \in \mathcal{M}$ with min-cut cactus $\mathcal{H}(H)$. Let u and v be vertices in distinct nodes of degree 2 of $\mathcal{H}(H)$ and let Q be a shortest path between them in $\mathcal{H}(H)$. Add a path of length at least 3 with endpoints u and v to obtain G. $G \in \mathcal{M}$ if and only if Q does not intersect any cycle of $\mathcal{H}(H)$ in exactly one or two edges.

Updating the Min-cut Cactus



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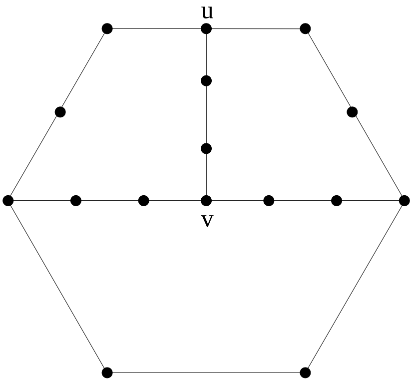

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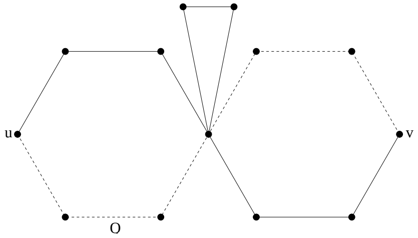

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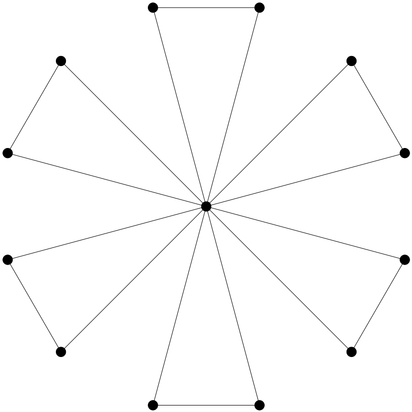

Bounding the Number of Edges

Proposition

If $G \in \mathcal{M}$ has n vertices and m edges then

$$m < \frac{6}{5}n.$$

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- The min-cut cactus can be easily updated
- Thank you for attending my talk!