$\Gamma = SL(n, \mathbf{Z}), G = SL(n, \mathbf{R})$

$\Omega_n = G/\Gamma = \{$the space of unimodular $n \times n$ lattices

with discriminant one$\}$.

One can associate to an integrable function ψ on $\mathbf{R}^n$ a function $\tilde{\psi}$ on $\Omega_n = G/\Gamma$ by setting

$$\tilde{\psi}(g\Gamma) = \sum_{v \in g\mathbf{Z}^n, v \neq 0} \psi(v), \ g \in G.$$

According to a theorem of Siegel

$$\int_{\mathbf{R}^n} \psi dm^n = \int_{G/\Gamma} \tilde{\psi} d\mu$$

where m^n is the Lebesgue measure on $\mathbf{R}^n$ and μ is the G-invariant probability measure on G/Γ.

Δ a lattice in $\mathbf{R}^n$

$L \subset \mathbf{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L

$d_\Delta(L) = d(L) \overset{\text{def}}{=}$ the volume of $L/L \cap \Delta$

$d(L) = \|v_1 \wedge \ldots \wedge v_\ell\|$ where $(v_1, \ldots, v_\ell)$ is a basis of $L \cap \Delta$

$d(L) \overset{\text{def}}{=} 1$ if $L = \{0\}$

$d_\Delta(\mathbf{R}^n) = 1$ iff $\Delta \in \Omega_n = SL(n, \mathbf{R})/SL(n, \mathbf{Z})$

$\alpha_i(\Delta) = \sup\{\frac{1}{d(L)} \mid L$ is a $\Delta-$rational subspace of dimension i$\}$, $0 \leq i \leq n$

$$\alpha(\Delta) = \max_{0 \leq i \leq n} \alpha_i(\Delta)$$

$$\tilde{f}(\Delta) < c(f)\alpha(\Delta)$$

for any bounded function f on $\mathbf{R}^n$ with compact support

$p \geq q \geq 1, n = p + q, n \geq 3$

$$Q_0(\sum_{i=1}^{n} v_i e_i) = 2v_1 v_n + \sum_{i=2}^{p} v_i^2 - \sum_{i=p+1}^{n-1} v_i^2$$

$a_t e_1 = e^{-t} e_1, a_t e_n = e^t e_n$, and $a_t e_i = e_i, 2 \leq i \leq n-1$

$\{a_t\} \subset H \stackrel{\text{def}}{=} SO(Q_0) \subset G = SL(n, \mathbf{R})$

$\hat{K} = SO(n)$ and $K = H \cap \hat{K}$

K is a maximal compact subgroup of H, and consists of all $h \in H$ leaving invariant the subspace spanned by $\{e_1 + e_n, e_2, \ldots, e_p\}$.

σ the normalized Haar measure on K

$$\int_K \sum_{v \in g\mathbf{Z}^n} f(a_t k v)\theta(k) d\sigma(k) = \int_K \tilde{f}(a_t k g \Gamma)\theta(k) d\sigma(k)$$

where f is a continuous function on $\mathbf{R}^n \backslash \{0\}$ and θ is a bounded measurable function on K.

Theorem. *(Eskin, Margulis, Mozes 1998)* (a) *If* $p \geq 3$, $q \geq 1$ *and* $0 < s < 2$, *or if* $p = 2$, $q \geq 1$ *and* $0 < s < 1$, *then for any lattice* Δ *in* $\mathbf{R}^n$

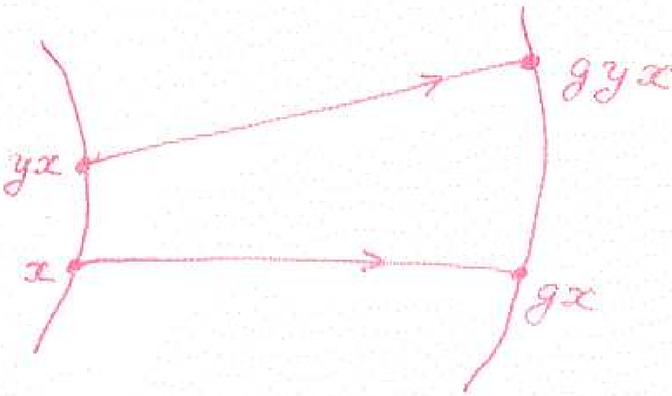
$$\sup_{t>0} \int_K \alpha(a_t k \Delta)^s d\sigma(k) < \infty.$$

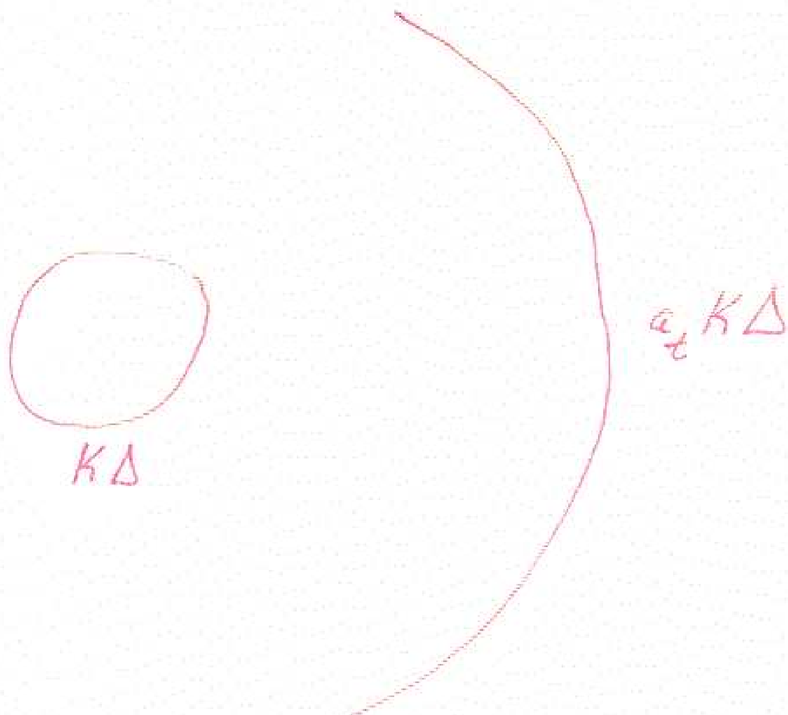
(ii) If $p = 2$ *and* $q = 2$, *or if* $p = 2$ *and* $q = 1$, *then for any lattice* Δ *in* $\mathbf{R}^n$

$$\sup_{t>1} \frac{1}{t} \int_K \alpha(a_t k \Delta) d\sigma(k) < \infty.$$

These bounds are uniform as Δ *varies over compact sets in the space of lattices.*

$$g y x = (g y g^{-1})(g x)$$





$a_t K\Delta$ locally "almost" coincides with an orbit of a unipotent subgroup

Theorem. *(Orbit closure theorem, Ratner 1991). Let G be a connected Lie group, Γ a lattice in G, and H a Lie subgroup of G that is generated by the Ad-unipotent subgroups contained in it. Then for any $x \in G/\Gamma$, there exists a closed connected subgroup $L = L(x)$ containing H such that $\overline{Hx} = Lx$ and there is an L-invariant probability measure supported on Lx.*

Special Cases (1) Horospherical subgroups

$$u_g = \{u \in G \mid g^i u g^{-i} \to 1 \text{ as } j \to +\infty\}.$$

Furstenberg 1972, Veech, Dani

(2) Solvable groups. Starkov (1989)

(3) Generic unipotent subgroups of $SL(3, \mathbf{R})$ (Dani, Margulis 1998). Suggest an approach for proving the Raghunathan conjecture in general.

Theorem. *(Measure classification theorem, Ratner 1991)* *Let G be a connected Lie group and Γ a discrete subgroup of H (not necessarily a lattice). Let H be a Lie subgroup of H that is generated by the Ad-unipotent subgroups contained in it. Then any finite H-ergodic H-invariant measure μ on G/Γ is homogeneous in the sense that there exists a closed subgroup F of G such that μ is F-invariant and* supp $\mu = Fx$.

Theorem. *(Uniform distribution theorem, Ratner 1991)* *If G is a connected Lie group, Γ is a lattice in G, $\{u(t)\}$ is a one-parameter Ad-unipotent subgroup of G and $x \in G/\Gamma$, then the orbit $\{u(t)x\}$ is uniformly distributed with respect to a homogeneous probability measure μ_x on G/Γ in the sense that for any bounded continuous function f on G/Γ*

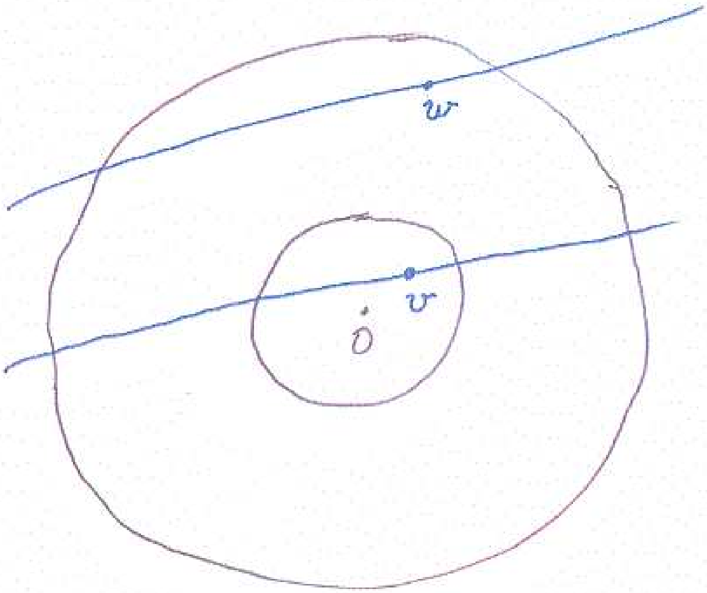
$$\frac{1}{T}\int_0^T f(u(t))dt \to \int_{G/\Gamma} f d\mu_x \text{ as } T \to \infty.$$

$$\beta(\Delta) = \inf\{\|v\| \mid v \in \Delta, v \neq 0\} = \alpha_1(\Delta)^{-1}$$

Theorem. (M. 1971) Let $\{u(t)\}$ be a one-parameter unipotent subgroup of $SL(n, \mathbb{R})$. For any lattice $\Delta \in \Omega_n$ there exists $\varepsilon(\Delta) > 0$ such that the set $\{t \geq 0 \mid \beta(u(t)\Delta) > \varepsilon(\Delta)\}$ is unbounded.

Theorem (Dani 1984-6). Let G be a connected Lie group, Γ a lattice in G, F a compact subset of G/Γ and $\varepsilon > 0$. Then there exists a compact subset K of G/Γ such that for any Ad-unipotent one-parameter subgroup $\{u(t)\}$ of G, any $x \in F$, and $T \geq 0$

$$\ell\{t \in [0, T] \mid u(t)x \in K\} > (1-\varepsilon)T.$$



To get from $B(0,\varepsilon)$ to the sphere of radius $1/2$ requires "comparable" amount of time as to get after that to the sphere of radius 1.

$$(A_t f)(x) = \int_K f(a_t kx) d\sigma(k), \; x \in X.$$

The main idea of the proof is to show that α_i^s *satisfy certain systems of integral inequalities which imply the desired bounds. If* $0 < s < 2, p \geq 3$ *and* $0 < i < n$ *or* $p = 2, q = 2$ *and* $i = 1$ *or* 3*, or if* $0 < s < 1$*, we prove that for any* $c > 0$ *there exist* $t = t(s, c) > 0$ *and* $w = w(s, c) > 1$ *such that*

$$A_t \alpha_i^s \leq \frac{c}{2} \alpha_i^s + \omega^2 \max_{0 < j \leq \min(n-i, i)} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}$$

$$A_t \alpha_i^* \leq \alpha_i^* + \omega^2 \sqrt{\alpha_{3-i}^*}, 1 \leq i \leq 2, \text{ for } (p, q) = (2, 1)$$

$$A_t \alpha_2^{\#} \leq \alpha_2^{\#} + \omega^2 \sqrt{\alpha_1 \alpha_3} \text{ for } (p, q) = (2, 2) \text{ and } i = 2$$

$$f_i(h) = \alpha_i(h\Delta), h \in H.$$

$$A_t f_i^s \leq \frac{c}{2} f_i^s + \omega^2 \max_{0<j\leq\min(n-i,i)} \sqrt{f_{i+j}^s f_{i-j}^s}.$$

$$q(i) \stackrel{\text{def}}{=} i(n-i); 2q(i) - q(i+j) - q(i-j) = -2j^2$$

$$A_t(\varepsilon^{q(i)} f_i^s) \leq \frac{c}{2}\varepsilon^{q(i)} f_i^s +$$

$$\omega^2 \max_{0<j\leq\min(n-i,i)} \varepsilon^{q(i)-\frac{q(i+j)+q(i-j)}{2}} \sqrt{\varepsilon^{q(i+j)} f_{i+j}^s \varepsilon^{q(i-j)} f_{i-j}^s}$$

$$\leq \frac{c}{2}\varepsilon^{q(i)} f_i^s + \varepsilon\omega^2 \max_{0<j\leq\min(n-i,i)} \sqrt{\varepsilon^{q(i+j)} f_{i+j}^s \varepsilon^{q(i-j)} f_{i-j}^s}$$

$$f_{\varepsilon,s}(h) = \sum_{0\leq i\leq n} \varepsilon^{q(i)} f_i^s(h) = \sum_{0\leq i\leq n} \varepsilon^{q(i)} \alpha_i(h\Delta)^s$$

$$A_t f_{\varepsilon,s} < 1 + d(\Delta)^{-s} + \frac{c}{2} f_{\varepsilon,s} + n\varepsilon\omega^2 f_{\varepsilon,s}$$

Taking $\varepsilon = \frac{c}{2n\omega^2}$ we see that

$$A_{t(s,c)} f_{\varepsilon,s} < c f_{\varepsilon,s} + b$$

where $b = 1 + d(\Delta)^{-s}$.

For every neighborhood V of e in H there exists a neighborhood U ov e in K such that

$$a_t U a_r \subset K V a_t a_r K$$

for any $t \geq 0$ and $r \geq 0$

$$\sup_{t>0} (A_t f_{\varepsilon,s})(e) < \infty$$

implies

$$\sup_{t>0} \int_K \alpha(a_t k \Delta)^s d\sigma(k) < \infty$$

Lemma 1. *Let*

$$F(i) = \{x_1 \wedge \ldots \wedge x_i \mid x_1, \ldots, x_i \in \mathbf{R}^{p+q}\} \subset \Lambda^i(\mathbf{R}^{p+q}).$$

If $0 < s < 2, p \geq 3$ *and* $0 < i < n$ *or* $p = 2$, $q = 2$ *and* $i = 1$ *or* 3, *or if* $0 < s < 1$, *then*

$$\lim_{t \to \infty} \sup_{v \in F(i), \|v\|=1} \int_K \frac{d\sigma(k)}{\|a_t k v\|^s} = 0.$$

Lemma 2. *For any two* Δ*-rational subspaces* L *and* M

$$d(L)d(M) \geq d(L \cap M)d(L + M)$$