# Dynamic Networks and Evolutionary Variational Inequalities

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# *General Formulation of the set $\mathbf{K}$ for traffic network problems, spatial price equilibrium problems, financial equilibrium problems*

$$\mathbf{K} = \Big\{ u(t) \in L^2\left([0,T], \mathbf{R}^q\right) : \lambda(t) \le u(t) \le \mu(t) \text{ a.e.in } [0,T];$$

$$\left.\sum_{i=1}^{q} \xi_{ji}\, u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \xi_{ji} \in \{0,1\},\ i = 1,\dots,q,\ j = 1,\dots,l \right\}$$

$$\lambda, \mu, \rho \in L^p\left([0,T], \mathbf{R}^q\right): \text{given functions}$$

# Discretization Procedure

Partitions of $[0,T]$:

$$\pi_n = \left(t_n^0, t_n^1, \ldots, t_n^{N_n}\right), 0 = t_n^0 < t_n^1 < \ldots < t_n^{N_n} = T$$

$$\delta_n = \max\left\{t_n^j - t_n^{j-1} : j = 1, \ldots, N_n\right\}, \lim_n \delta_n = 0.$$

$$P_n\left([0,T], \mathbf{R}^m\right) = \left\{v \in L^\infty\left([0,T], \mathbf{R}^m\right) : v_{\left(t_n^{j-1}, t_n^j\right]} = v_j \in \mathbf{R}^m,\right.$$

$$\left. j = 1, \ldots, N_n\right\}$$

$$\mu_n v_{\left(t_n^{j-1}, t_n^j\right]} := \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} v(s)\, ds$$

$$\mathbf{K} = \Big\{ F(t) \in L^2\left([0,T], \mathbf{R}^m\right) : \lambda \le F(t) \le \nu \text{, a.e. in } [0,T],$$

$$\Phi F(t) = \rho,\ \ \lambda, \nu \ge 0 \Big\}$$

$$C\left[t, F(t)\right] = A(t) F(t) + B(t),$$

$$A(t) \in L^\infty\left([0,T], \mathbf{R}^{m^2}\right),\ B(t) \in L^2\left([0,T], \mathbf{R}^m\right)$$

$$\int_0^T \left\langle A(t)H(t)+B(t),\, F(t)-H(t)\right\rangle dt$$

$$=\sum_{j=1}^{N_n}\int_{t_n^{j-1}}^{t_n^j} \left\langle A(t)H(t)+B(t),\, F(t)-H(t)\right\rangle dt$$

$$H_j^n(t)\in \mathbf{K}:$$

$$\int_{t_n^{j-1}}^{t_n^j} \left\langle A(t)H_j^n(t)+B(t),\, F_j^n(t)-H_j^n(t)\right\rangle dt \geq 0,$$

$$\forall F_j^n(t)\in \mathbf{K}$$

Finite-dimensional Variational Inequality:

$$H_j^n \in \mathbf{K}_m : \left\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \right\rangle \geq 0, \ \forall F_j^n \in \mathbf{K}_m$$

where

$$A_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} A(t)\, dt; \quad B_j^n = \frac{1}{t_n^j - t_n^{j-1}} \int_{t_n^{j-1}}^{t_n^j} B(t)\, dt.$$

$$H_n(t) = \sum_{j=1}^{N_n} \chi\left(t_n^{j-1}, t_n^j\right) H_j^n$$ piecewise constant

approximation to the solution of EVI

# Theorem

$A(t)$ positive definite a.e. in $[0,T] \Rightarrow$ the set $U = \{H_n\}_{n\in\mathbb{N}}$ (weakly) compact, with feasible cluster points solutions to the EVI.

$$\mathbf{K} = \Big\{ F(t) \in L^2\left([0,T],\mathbf{R}^m\right) : \lambda(t) \le F(t) \le \nu(t) \text{ a.e.in } [0,T];$$
$$\lambda(t), \nu(t) \ge 0,\ \Phi F(t) = \rho(t) \text{ a.e. in } [0,T]\Big\}$$

$$\pi_n : \mathbf{K}_j^n = \Big\{ F_j \in \mathbf{R}^m :\ \overline{\lambda}_{j,n} \le F_j \le \overline{\nu}_{j,n} \text{ a.e.in } \left(t_{j-1},t_j\right);$$
$$\Phi F_j = \overline{\rho}_{j,n} \text{ a.e. in } \left(t_{j-1},t_j\right)\Big\}$$

**Lemma** $\mathbf{K}_n = \bigcap_j \mathbf{K}_j^n \to \mathbf{K}$

Initial problem:

$$H(t) \in \mathbf{K}(t)$$

$$\int_0^T \left\langle C\left[t, H(t)\right], F(t) - H(t) \right\rangle dt \geq 0 \quad \forall F(t) \in \mathbf{K}(t)$$

**Theorem**

$A(t)$ positive definite a.e. in $[0,T] \Rightarrow H_n(t) = \sum_{j=1}^{N_n} \chi\left(t_n^{j-1}, t_n^j\right) H_j^n$

has (weakly) cluster points solutions to the initial problem.

# Continuity assumption

$$F \text{ strictly monotone} \Rightarrow$$

$$u(t) \in C^0\left([0,\mathrm{T}], \mathbf{R}^q\right) \text{ unique solution to the EVI:}$$

$$\left\langle F\left(u(t)\right), v(t) - u(t)\right\rangle \geq 0, \ \forall t \in [0,T]$$

- Discretization the time interval [0,$T$];
- Sequence of PDS;
- Calculation of the critical points of the PDS;
- Interpolation of the sequence of critical points;
- Approximation of the equilibrium curve.

# Traffic Network


$$A(t)=\begin{bmatrix} 2+t & 0 \\ 0 & 1 \end{bmatrix},\ F(t)=\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix},\ B(t)=\begin{bmatrix} -\frac{3}{2} \\ -1 \end{bmatrix},$$

$$\mathbf{K}=\left\{F\in L^2\left([0,2],\mathbf{R}^2\right):0\le F_1(t)\le t;\ 0\le F_2(t)\le 3;\right.$$

$$\left.F_1(t)+F_2(t)=t \text{ a.e.in } [0,2]\right\}$$

$$0<\frac{2}{n}<\frac{4}{n}<\ldots<(j-1)\frac{2}{n}<j\frac{2}{n}<\ldots<(n-1)\frac{2}{n}<2$$

$$\left\langle A_j^n H_j^n + B_j^n, F_j^n - H_j^n \right\rangle \geq 0 \ \ \forall F_j^n \in \mathbf{K}_j^n$$

$$\mathbf{K}_j^n = \left\{ F_j^n \in \mathbf{R}^2 : 0 \leq F_{j1}^n \leq \frac{2j-1}{n},\ 0 \leq F_{j2}^n \leq 3,\ F_{j1}^n + F_{j2}^n = \frac{2j-1}{n} \right\}$$

$$\left[ \left( 2 + \frac{2j-1}{n} \right) H_{j1}^n - \frac{3}{2} \right] \left( F_{j1}^n - H_{j1}^n \right) + \left( H_{j2}^n - 1 \right) \left( F_{j2}^n - H_{j2}^n \right) \geq 0 \ \ \forall F_j^n \in \mathbf{K}_j^n$$

$$F_{j2}^n = \frac{2j-1}{n} - F_{j1}^n$$

$$\left[ \left( 2 + \frac{2j-1}{n} \right) H_{j1}^n - \frac{3}{2} - \frac{2j-1}{n} + H_{j1}^n + 1 \right] \left( F_{j1}^n - H_{j1}^n \right) \geq 0$$

$$H_{j1}^{n}=\begin{cases}\dfrac{2j-1}{n} & \text{if } \ 1\le j<\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\\[2ex] \dfrac{4j-2+n}{2(3n+2j-1)} & \text{if } \ j>\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\end{cases}$$

$$H_{j2}^{n}=\begin{cases}0 & \text{if } \ 1\le j<\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\\[2ex] \dfrac{-n^2+4(2j-1)n+2(2j-1)^2}{2n(3n+2j-1)} & \text{if } \ j>\dfrac{1}{2}\left(1+\dfrac{n}{2+\sqrt{6}}\right)\end{cases}$$

$$H_n(t)=\sum_{j=1}^{n}\chi\left[(j-1)\frac{2}{n},\, j\frac{2}{n}\right]H_j^n$$

# Direct Method

$$\left[(2+t)H_1(t)-\frac{3}{2}\right]\left[F_1(t)-H_1(t)\right]+\left[H_2(t)-1\right]\left[F_2(t)-H_2(t)\right]\geq 0$$

$$\forall F(t)\in \mathbf{K}=\left\{F(t)\in L^2\left([0,2],\mathbf{R}^2\right):\, 0\leq F_1(t)\leq t;\right.$$

$$\left. 0\leq F_2(t)\leq 3;\ F_1+F_2(t)=t, \text{ a.e. in } [0,2]\right\}$$

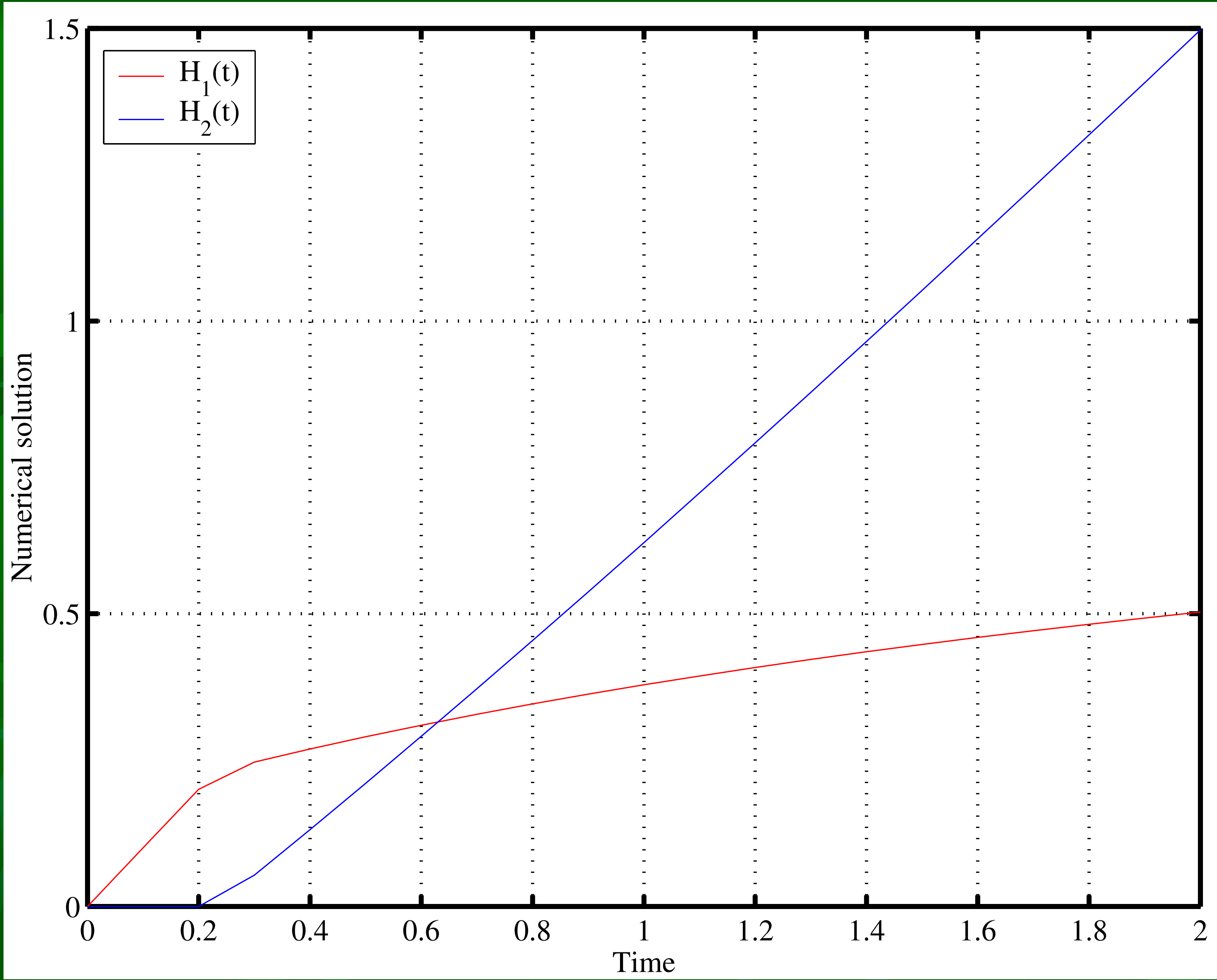
$$F_2(t)=t-F_1(t)\ \Rightarrow\ \tilde{\mathbf{K}}=\left\{F_1(t)\in L^2\left([0,2],\mathbf{R}\right):0\leq F_1(t)\leq t\right\}$$

$$\left[3(2+t)H_1(t)-t-\frac{1}{2}\right]\left[F_1(t)-H_1(t)\right]\geq 0\ \ \forall F_1(t)\in \tilde{\mathbf{K}}$$

$$H_1(t)=\begin{cases} t & \text{if} \quad 0\le t\le \dfrac{\sqrt{6}}{2}-1 \\ \dfrac{2t+1}{2t+6} & \text{if} \quad \dfrac{\sqrt{6}}{2}-1<t\le 2 \end{cases}$$

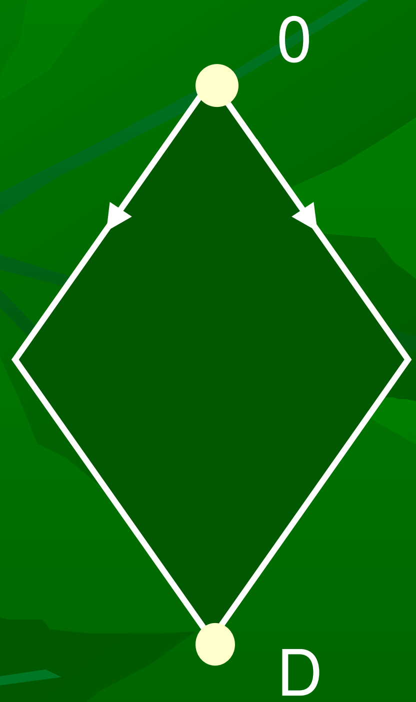
$$H_2(t)=\begin{cases} 0 & \text{if} \quad 0\le t\le \dfrac{\sqrt{6}}{2}-1 \\ \dfrac{2t^2+4t-1}{2t+6} & \text{if} \quad \dfrac{\sqrt{6}}{2}-1<t\le 2 \end{cases}$$

$$\left(H_1^n(t),H_2^n(t)\right)\to\left(H_1(t),H_2(t)\right)$$



1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time
Numerical solution
$H_1(t)$
$H_2(t)$

# Traffic Network




Cost functions:

$$C_1(H(t)) = H_1(t) + 1$$

$$C_2(H(t)) = H_2(t) + 2$$

$$\mathbf{K} = \Big\{ F(t) \in L^2([0,2], \mathbf{R}^2) : 0 \le F_1(t) \le t, 0 \le F_2(t) \le \frac{3}{2}t \text{ a.e.in } [0,2];$$

$$F_1(t) + F_2(t) = t \text{ a.e. in } [0,2] \Big\}$$

Vector field:

$$F: L^2\left([0,2],\mathbf{R}^2\right)\to L^2\left([0,2],\mathbf{R}^2\right)$$

$$\left(F_1\left(H\left(t\right)\right),F_2\left(H\left(t\right)\right)\right)=\left(H_1\left(t\right)+1,H_2\left(t\right)+2\right).$$

$$t_0\in\left\{\frac{k}{4}:k=0,\ldots,8\right\}\Rightarrow \text{sequence of PDS defined by :}$$

$$-F\left(H_1\left(t_0\right),H_2\left(t_0\right)\right)=\left(-H_1\left(t\right)-1,-H_2\left(t\right)-2\right) \text{ on}$$

$$\mathbf{K}_{t_0}=\left\{\left\{[0,t_0]\times\left[0,\frac{3}{2}t_0\right]\right\}\cap\left\{x+y=t_0\right\}\right\}$$

Unique equilibrium at $t_0$:

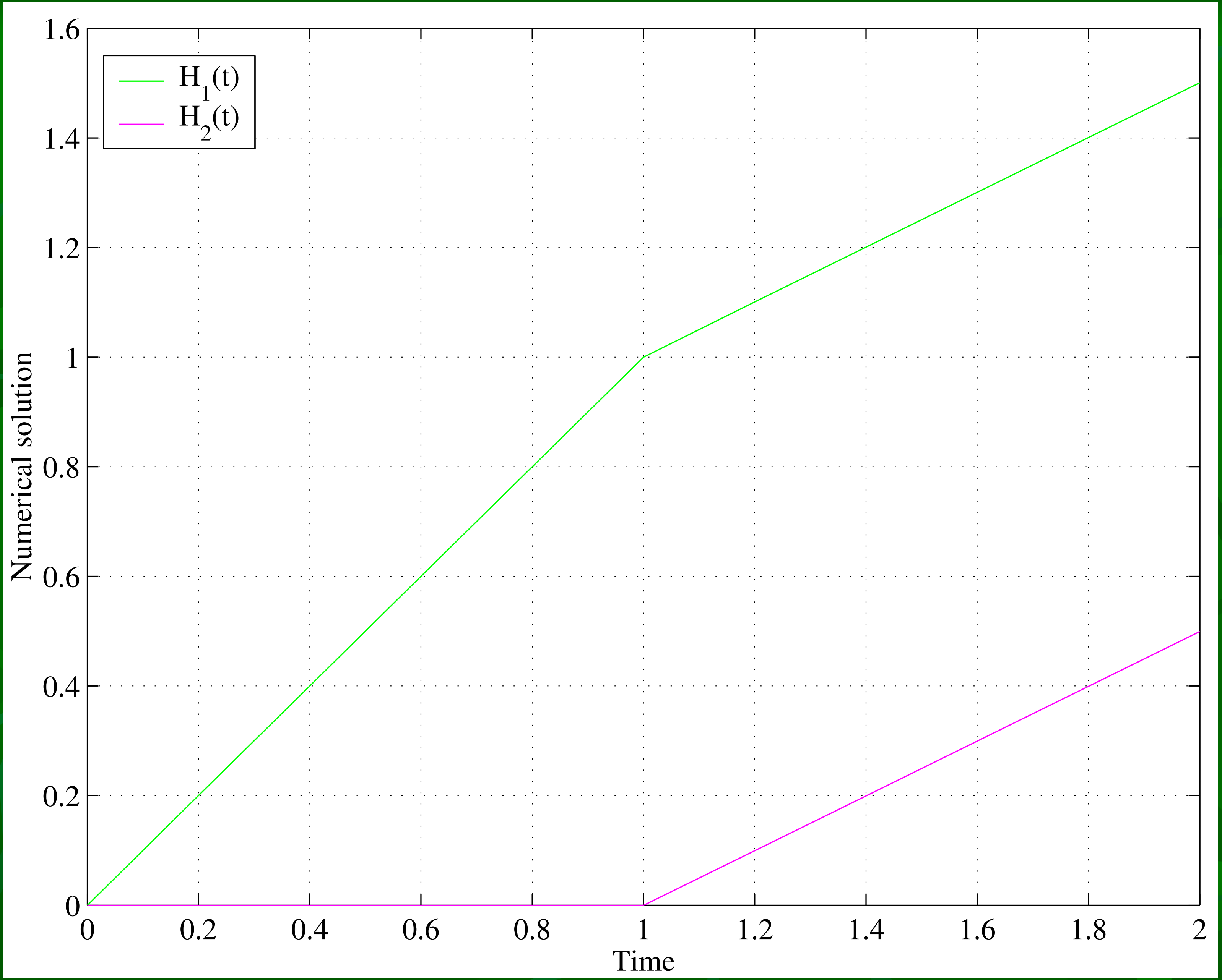
$\left(H_1\left(t_0\right), H_2\left(t_0\right)\right) \in \mathbf{R}^2$ :

$-F\left(H_1\left(t_0\right), H_2\left(t_0\right)\right) \in N_{\mathbf{K}_{t_0}}\left(H_1\left(t_0\right), H_2\left(t_0\right)\right)$

Explicit formulas:

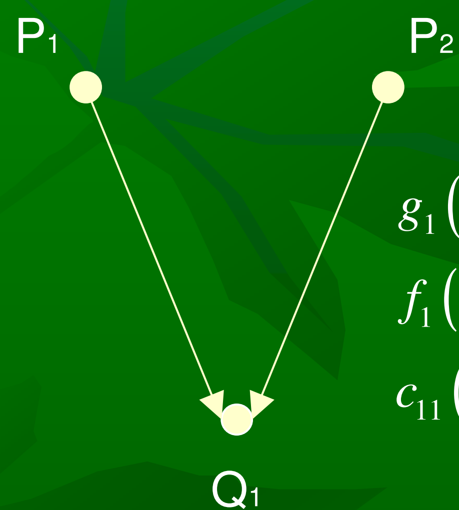
$$\begin{cases} H_1(t) = \begin{cases} t & \text{if } 0 \le t \le 1 \\ \dfrac{t+1}{2} & \text{if } 1 \le t \le 2 \end{cases} \\ H_2(t) = \begin{cases} 0 & \text{if } 0 \le t \le 1 \\ \dfrac{t-1}{2} & \text{if } 1 \le t \le 2 \end{cases} \end{cases}$$

| $t_0$ | $H_1(t_0)$ | $H_2(t_0)$ |
|---|---|---|
| 0 | 0 | 0 |
| 1/4 | 1/4 | 0 |
| 1/2 | 1/2 | 0 |
| 3/4 | 3/4 | 0 |
| 1 | 1 | 0 |
| 5/4 | 9/8 | 1/8 |
| 3/2 | 5/4 | 1/4 |
| 7/4 | 11/8 | 3/8 |
| 2 | 3/2 | 1/2 |



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Numerical solution
$H_1(t)$
$H_2(t)$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Time

# Economic Markets




$$g_1\left(p(t)\right)=p_1(t)+h(t);\, g_2\left(p(t)\right)=p_2(t)+k(t);$$

$$f_1\left(q(t)\right)=k(t)-q_1(t);$$

$$c_{11}\left(x(t)\right)=x_{11}(t);\, c_{21}\left(x(t)\right)=x_{21}(t)+1;$$

$$\mathbf{K}=\left\{u(t)=\left(p(t),q(t),x(t)\right)\in L^2\left(\left[0,\frac{1}{2}\right],\mathbf{R}^2\right)\times L^2\left(\left[0,\frac{1}{2}\right],\mathbf{R}\right)\times L^2\left(\left[0,\frac{1}{2}\right],\mathbf{R}^2\right):\right.$$

$$0\le p_1(t)\le h(t)+k(t)+1, 0\le p_2(t)\le h(t)+k(t)+1, 0\le q_1(t)\le h(t)+k(t)+1,$$

$$\left.0\le x_{11}(t)\le h(t)+k(t)+1, 0\le x_{21}(t)\le h(t)+k(t)+1\right\}$$

# Direct Method

$$\begin{cases} v(u(t))=0 \\ u(t)\in \mathbf{K} \end{cases} \Rightarrow \begin{cases} p_1(t)-x_{11}(t)+h(t)=0 \\ p_2(t)-x_{21}(t)+k(t)=0 \\ q_1(t)+x_{11}(t)+x_{21}(t)-k(t)=0 \\ p_1(t)+x_{11}(t)-q_1(t)=0 \\ p_2(t)+x_{21}(t)-q_1(t)+1=0 \\ u(t)\in \mathbf{K} \end{cases} \Rightarrow \varnothing$$

$$\mathbf{K}\cap\{p_2(t)=0\}: \begin{cases} -3h(t)+k(t)+1\ge 0 \\ -h(t)+2k(t)-3\ge 0 \end{cases}$$

$$p_1(t)=\frac{-3h(t)+k(t)+1}{5};\ p_2(t)=0; q_1(t)=\frac{-h(t)+2k(t)+2}{5};$$

$$x_{11}(t)=\frac{2h(t)+k(t)+1}{5}; x_{21}(t)=\frac{-h(t)+2k(t)-3}{5};$$

$$g_1(p(t))=\frac{2h(t)+k(t)+1}{5}; g_2(p(t))=k(t); f_1(q(t))=\frac{h(t)+3k(t)-2}{5};$$

$$c_{11}(x(t))=\frac{2h(t)+k(t)+1}{5}; c_{21}(x(t))=\frac{-h(t)+2k(t)+2}{5}; s_2(t)=\frac{h(t)+3k(t)+3}{5}.$$

# Discretization Procedure

$$h(t)=\frac{t}{2},\ k(t)=t+\frac{8}{7}$$

$$\int_0^{\frac{1}{2}}\left\langle A(t)u^*(t)+b(t),u(t)-u^*(t)\right\rangle dt\geq 0,\quad \forall u(t)\in \mathbf{K}$$

$$A=\begin{bmatrix}1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & -1\\ 0 & 0 & 1 & 1 & 1\\ 1 & 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 0 & 1\end{bmatrix},\quad b(t)=\begin{bmatrix}h(t)\\ k(t)\\ -k(t)\\ 0\\ 1\end{bmatrix}$$

$$\left[0,\frac{1}{2}\right]:0<\frac{1}{2n}<\ldots<\frac{j-1}{2n}<\frac{j}{2n}<\ldots<\frac{1}{2}$$

$$\mathbf{K}_j^n=\Big\{u_j^n=\left(p_{j1}^n,p_{j2}^n,q_{j1}^n,x_{j11}^n,x_{j21}^n\right):0\le p_{j1}^n\le\frac{6j-3}{8n}+\frac{15}{7},0\le p_{j2}^n\le\frac{6j-3}{8n}+\frac{15}{7},$$
$$0\le q_{j1}^n\le\frac{6j-3}{8n}+\frac{15}{7},\ 0\le x_{j11}^n\le\frac{6j-3}{8n}+\frac{15}{7},0\le x_{j21}^n\le\frac{6j-3}{8n}+\frac{15}{7}\Big\}$$

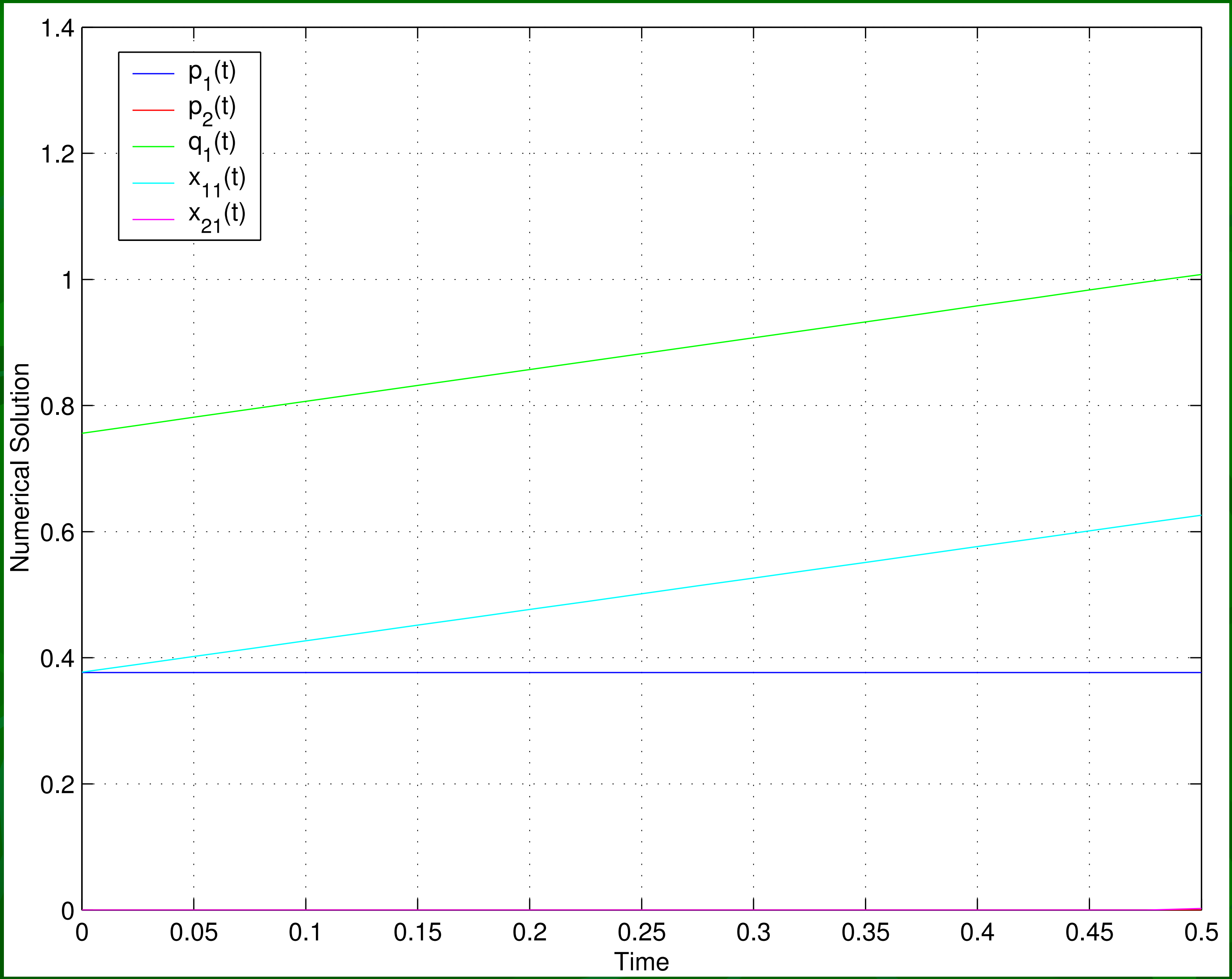
$$\left\langle A\left(u_j^n\right)^*+b_j^n,u_j^n-\left(u_j^n\right)^*\right\rangle\ge 0\ \ \forall u_j^n\in\mathbf{K}_j^n \quad \text{where} \quad b_j^n=\begin{bmatrix}\dfrac{2j-1}{8n}\\ \dfrac{2j-1}{4n}+\dfrac{8}{7}\\ -\dfrac{2j-1}{4n}-\dfrac{8}{7}\\ 0\\ 1\end{bmatrix}$$

$$\begin{cases} Au_j^n + b_j^n = 0 \\ u_j^n \in \mathbf{K}_j^n \end{cases} \Rightarrow \varnothing; \quad \mathbf{K}_j^n \cap \{p_{j2}^n = 0\} : \begin{cases} Au_j^n + b_j^n = 0 \\ p_{j2}^n = 0 \end{cases} \Rightarrow$$

$$\begin{cases} p_{j1}^n = \dfrac{-2j+1}{40n} + \dfrac{3}{7}; \; p_{j2}^n = 0; \; q_{j1}^n = \dfrac{6j-3}{40n} + \dfrac{6}{7}; \\ x_{j11}^n = \dfrac{2j-1}{10n} + \dfrac{3}{7}; \quad x_{j21}^n = \dfrac{6j-3}{40n} - \dfrac{1}{7}; \\ g_{j1}^n(p) = \dfrac{2j-1}{10n} + \dfrac{3}{7}; \; g_{j2}^n(p) = \dfrac{2j-1}{4n} + \dfrac{8}{7}; \; f_{j1}^n(q) = \dfrac{14j-7}{40n} + \dfrac{2}{7}; \\ c_{j11}^n(x) = \dfrac{2j-1}{10n} + \dfrac{3}{7}; \; c_{j21}^n(x) = \dfrac{6j-3}{40n} + \dfrac{6}{7}; \end{cases}$$

Excess: $s_{j2}^n = \dfrac{14j-7}{40n} + \dfrac{6}{7};$

Solution on $\mathbf{K}^n = \cap \mathbf{K}_j^n$: $u_n(t) = \sum_{j=1}^{n} \chi\left(\dfrac{j-1}{2n}, \dfrac{j}{2n}\right) u_j^n.$



1.4
1.2
1
0.8
0.6
0.4
0.2
0
Numerical Solution
p1(t)
p2(t)
q1(t)
x11(t)
x21(t)
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Time

# Financial Markets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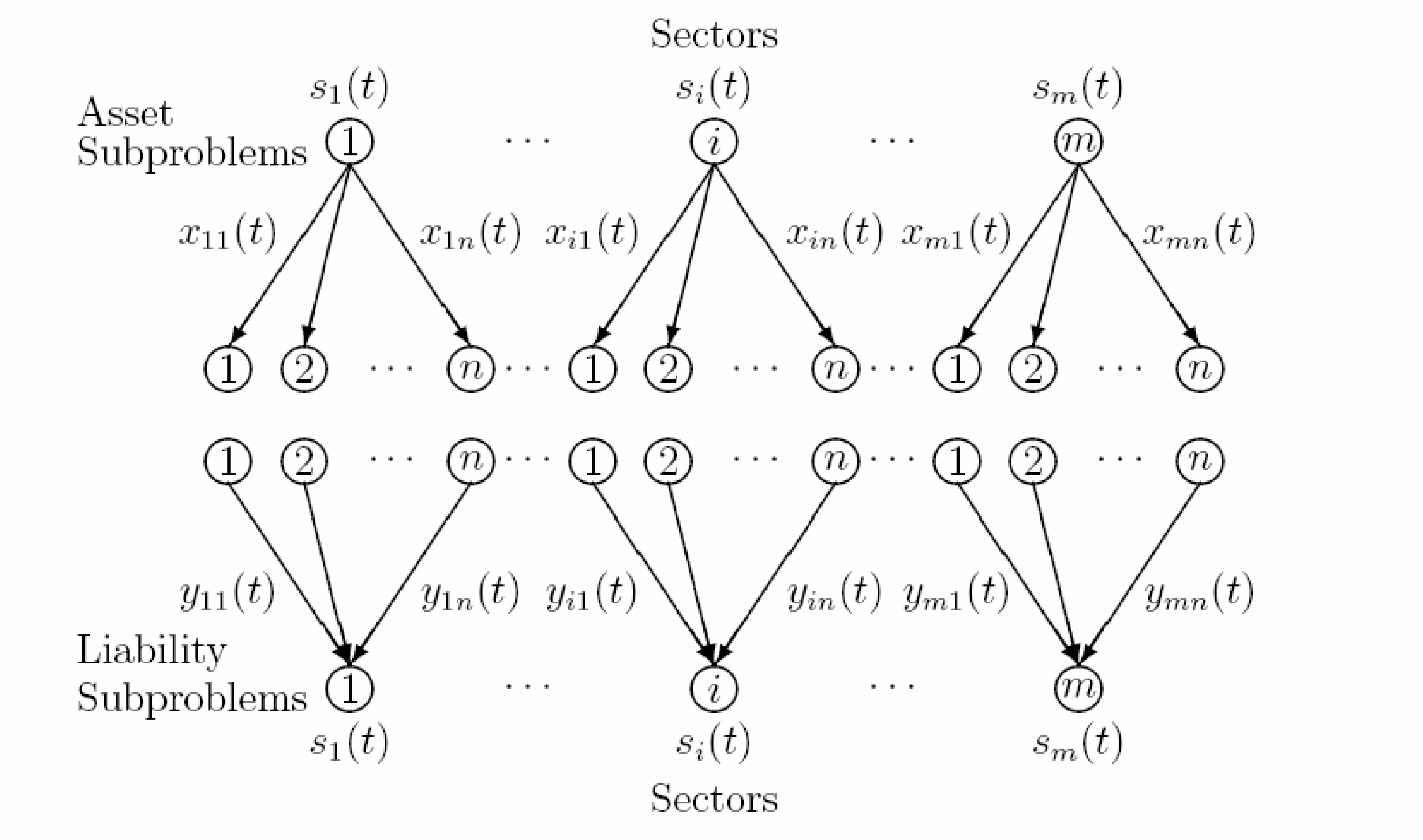
- $m$ **sectors**: households, domestic business, banks, financial institutions, state and local governments, …
- $n$ **financial instruments**: mortgages, mutual funds, savings deposits, money market funds, …
- $s_i(t)$: **total financial volume** theld by sector $i$ at time $t$
- $x_{ij}(t)$: amount of instrument $j$ held as an **asset** in sector $i$'s portfolio
- $y_{ij}(t)$: amount of instrument $j$ held as a **liability** in sector $i$'s portfolio


Sectors
$s_1(t)$
$s_i(t)$
$s_m(t)$
Asset Subproblems
$x_{11}(t)$
$x_{1n}(t)$
$x_{i1}(t)$
$x_{in}(t)$
$x_{m1}(t)$
$x_{mn}(t)$
$y_{11}(t)$
$y_{1n}(t)$
$y_{i1}(t)$
$y_{in}(t)$
$y_{m1}(t)$
$y_{mn}(t)$
Liability Subproblems
$s_1(t)$
$s_i(t)$
$s_m(t)$
Sectors

$$Q^i(t) = \begin{bmatrix} Q^i_{11}(t) & Q^i_{12}(t) \\ Q^i_{21}(t) & Q^i_{22}(t) \end{bmatrix}:$$ **variance-covariance matrix**

$$\begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}^T Q^i(t) \begin{bmatrix} x_i(t) \\ y_i(t) \end{bmatrix}:$$ **aversion to the risk** at time *t*

$r_j(t)$: **price of instrument** *j* at time *t*

$$P_i = \left\{ \left( x_i(t), y_i(t) \right) \in L^2\left( [0,T], \mathbf{R}^{2n} \right) : \sum_{j=1}^{n} x_{ij}(t) = s_i(t), \right.$$
$$\left. \sum_{j=1}^{n} y_{ij}(t) = s_i(t),\ x_{ij}(t) \geq 0,\ y_{ij}(t) \geq 0, \text{a.e.in } [0,T] \right\}:$$

**set of feasible assets and liabilities**