Chaotic Maps, Neglected Dynamics, and Frobenius-Perron Analysis

Matt Davison
mdavison@uwo.ca

Department of Applied Mathematics
The University of Western Ontario

Thanks

- Organizers and Audience
  For inviting/listening to an amateur
- Chris Essex, John Shiner
  Ideas
- NSERC, Swiss NSF
  Funding
- Sharon Wang
  Slides

Outline

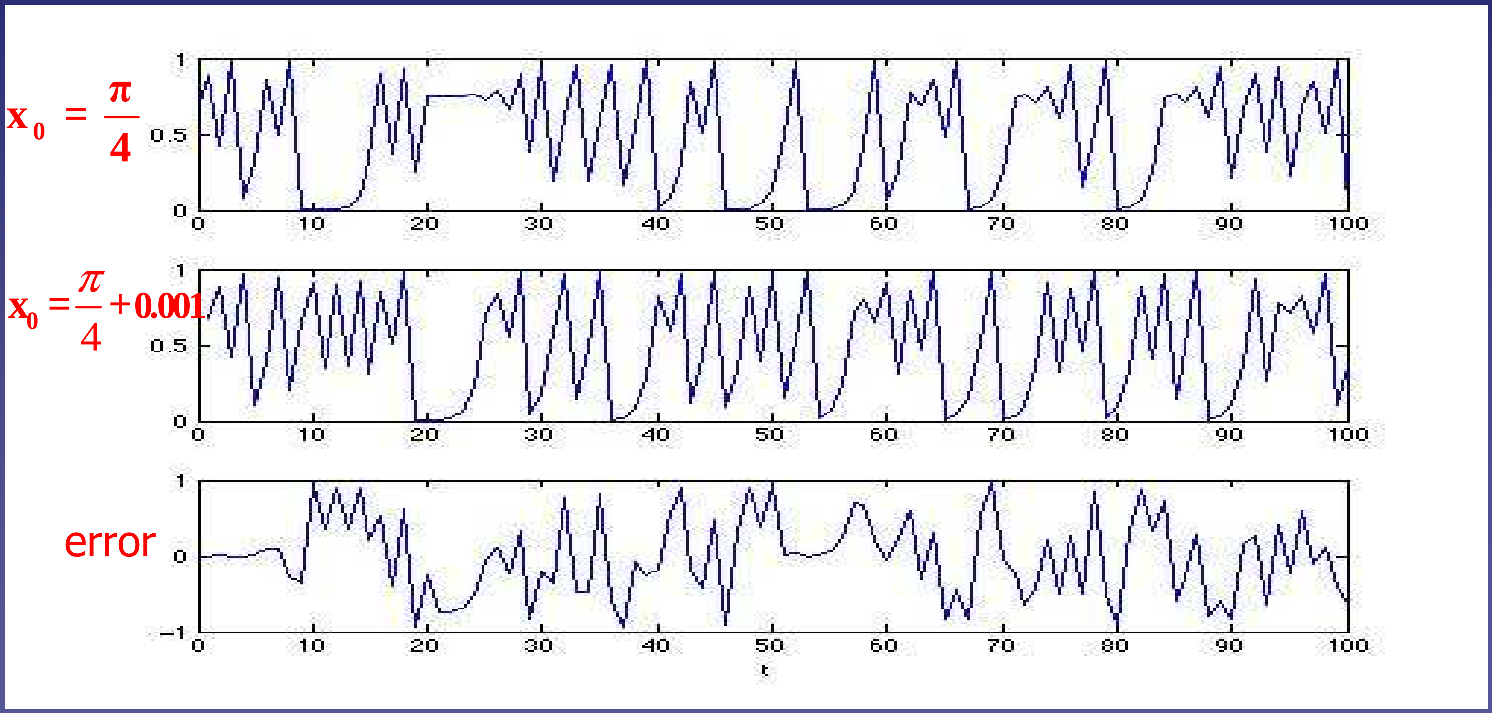
- The discrete logistic map with *r = 4*
  - Ø $X_{n+1} = 4X_n(1-X_n)$
  - Ø Pointwise unpredictable
  - Ø Very predictable on a population basis
  - Ø Frobenius-Perron Analysis
- Modelling issues

  Sensitive dependence on neglected dynamics?
- Weakly diffusively coupled binary shift maps

  Analytic Progress and Insights

Unpredictable Dynamics

- $X_{n+1} = 4X_n(1-X_n)$ time series



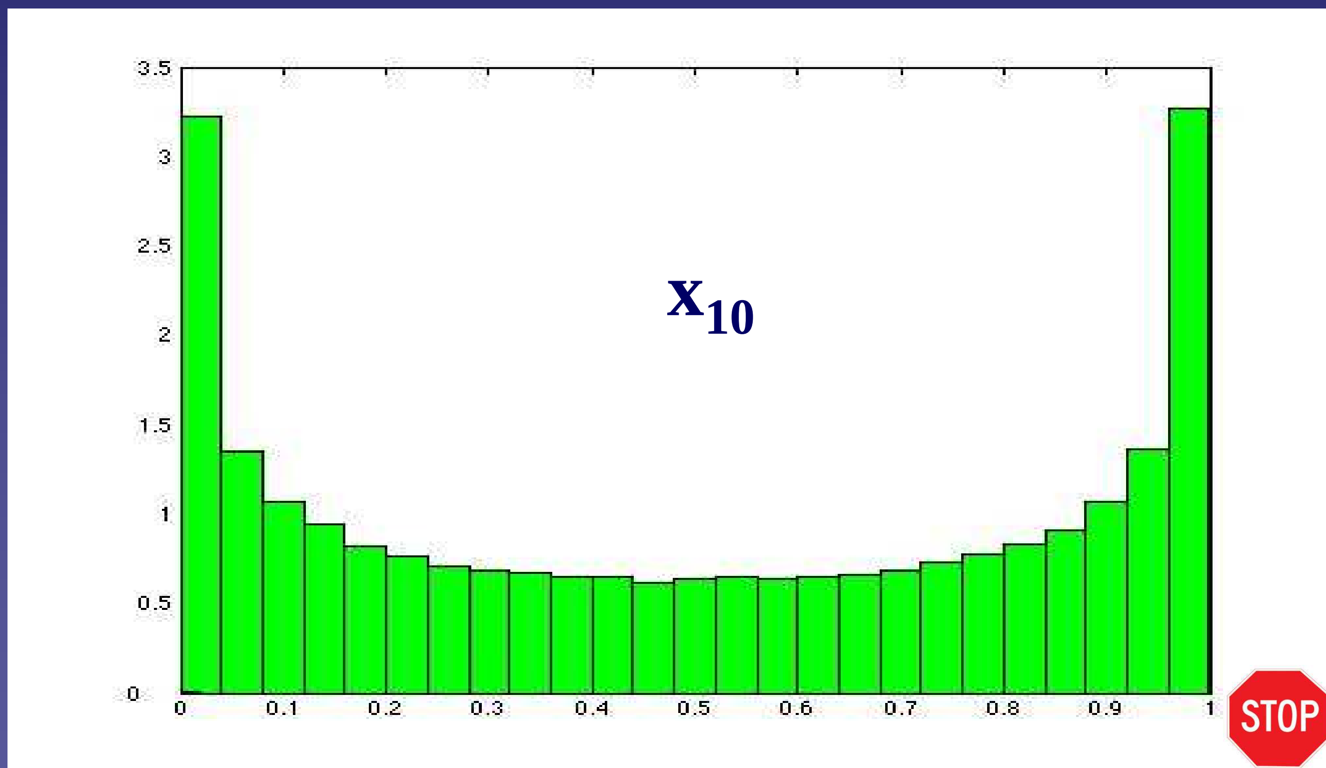
Sensitive to initial conditions

Look For Statistical Regularities

Start with 100,000 random points in [0, 1]

Evolve by logistic Map

$$X_{n+1} = 4X_n(1 - X_n)$$



Histogram

Probability Mathematics

- What do we see?
  - Ø much more regular
  - Ø seems to approach "steady state"!
- Can we understand this mathematically?
- Reference:

Chaos, Fractals and Noise Lasota and Mackey

Some Notations

- $x_{n+1} = S(x_n)$ -- a dynamical system on set X
- $f_n(x)$ -- a family of probability densities on X, i.e.,

  $\int_X f_n(y)dy = 1$ *How do these densities evolve?*
- A -- a subset of X
- Under some mild conditions about S to obtain

$$\int_{S^{-1}(A)} f_n(y)dy = \int_A f_{n+1}(y)dy$$

Density Function in R

- Specialize to X∈ **R** to get

$$\int_0^x f_{n+1}(y)dy = \int_{S^{-1}[0.1]} f_n(y)dy$$

- Differentiate both sides

$$f_{n+1}(x) = Pf_n(x) = \frac{d}{dx}\int_{S^{-1}[0,1]} f_n(y)dy$$

Frobenius-Perron operator (P)

Density Function of Logistic Map

- Further specialize to S(x)=4x(1-x)

$$y = 4x(1-x) \rightarrow x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-y}$$

$$S^{-1}[(0,x)] = (0, \frac{1}{2} + \frac{1}{2}\sqrt{1-x}) \cup (0, \frac{1}{2} - \frac{1}{2}\sqrt{1-x})$$

- Frobenius-Perron equation:

$$Pf(x) = \frac{d}{dx}\left[\int_0^{\frac{1}{2}-\frac{1}{2}\sqrt{1-x}} f(t)dt + \int_{\frac{1}{2}+\frac{1}{2}\sqrt{1-x}}^{0} f(t)dt\right]$$

$$= \frac{1}{4\sqrt{1-x}}\left[f(\frac{1}{2} - \frac{1}{2}\sqrt{1-x}) + f(\frac{1}{2} + \frac{1}{2}\sqrt{1-x})\right]$$

Density Function with $f_0(x)=1$

$$f_1(x) = \frac{1}{2\sqrt{1-x}},$$

$$f_2(x) = \frac{1}{4\sqrt{2}\sqrt{x(1-x)}}\left[\sqrt{1+\sqrt{1-x}} + \sqrt{1-\sqrt{1-x}}\right]$$

$$f_3(x) = \frac{1}{8\sqrt{2}\sqrt{x(1-x)}}\left[\sqrt{1+\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-x}}} + \sqrt{1+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-x}}} + \sqrt{1-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-x}}} + \sqrt{1-\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-x}}}\right]$$

$$\vdots$$

$$f_n(x) = \frac{g_n(x)}{\sqrt{x(1-x)}} \longrightarrow g_0 = \sqrt{x(1-x)}$$

$$g_{n+1}(x) = \frac{1}{2}\left[g_n\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right) + g_n\left(\frac{1}{2}-\frac{1}{2}\sqrt{1-x}\right)\right]$$

Simpler Functional Evolution Map g

$$g_1(x) = \frac{\sqrt{x}}{2},$$

$$g_2(x) = \frac{1}{4\sqrt{2}}\left[\sqrt{1+\sqrt{1-x}} + \sqrt{1-\sqrt{1-x}}\right],$$

$$g_3(x) = \frac{1}{8\sqrt{2}}\left[\sqrt{1+\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-x}}} + \sqrt{1+\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-x}}} + \sqrt{1-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-x}}} + \sqrt{1-\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-x}}}\right],$$

•

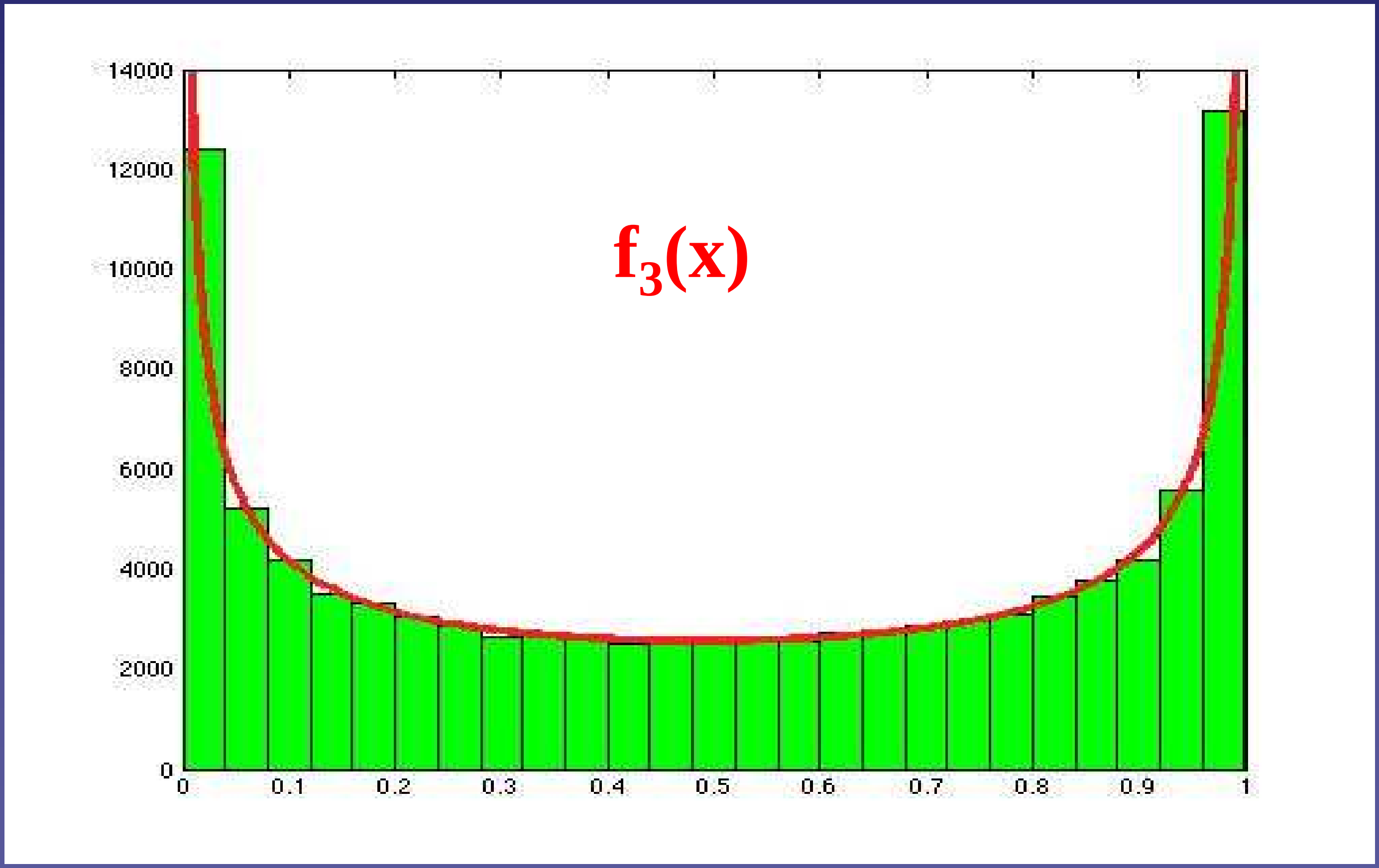
•

•

$$\lim_{n\to\infty} g_n(x) = \frac{1}{\pi}$$

$$\lim_{n\to\infty} \sum_{a_1=-1}^{1} \sum_{a_2=-1}^{1} \cdots \sum_{a_n=-1}^{1} \sqrt{1 + a_1\sqrt{\frac{1}{2} + \frac{a_2}{2}\sqrt{\frac{1}{2} + \cdots + \frac{a_n}{2}\sqrt{1-x}}}} = \frac{2^n\sqrt{2}}{\pi}.$$

Analytic Solution vs. Evolution



Natural Questions

- Properties of Perron-Frobenius operator P?

  $Pf_n(x) = f_{n+1}(x)$

- Does resulting map have a fixed point?

  It does! ⟶ $f^*(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$ [see Ulam '1948]

- What is the continuous-time analogue?

  -- Fokker Plank, Kolmogorov

Modelling Process

- Resemblance between model and reality?

  You have to ask the right questions here!
- Stochastic Stability?

  Chris Zeeman
- Sensitive dependence on neglected dynamics — if you are asking for trajectories
- What if you are asking for underlying probability densities?

Sensitive Dependence on Neglected Dynamics

- Test problem

$$x_{n+1} = (2x_n)\,\mathrm{mod}\,1$$ (Standard example for SDIC)

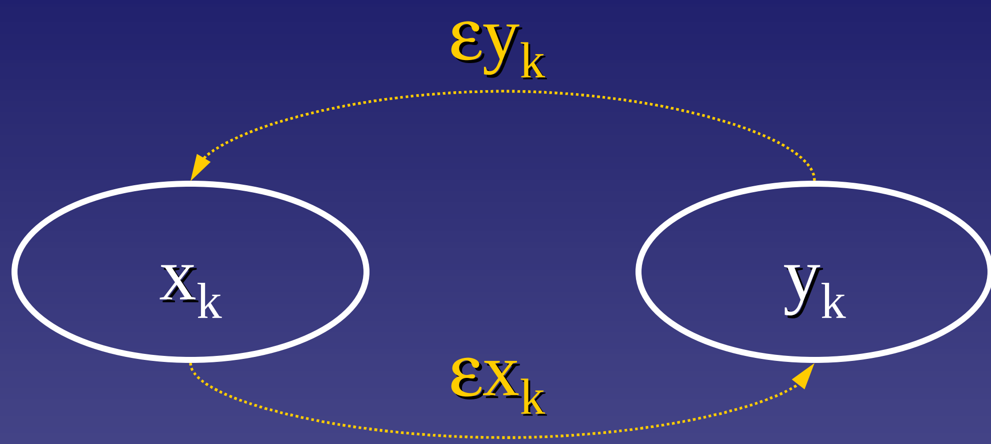
- Frobenius-Perron equation:

$$f_{n+1}(x) = Pf_n(x) = \frac{1}{2}\left[f_n\left(\frac{x}{2}\right) + f_n\left(\frac{x+1}{2}\right) \right]$$

- Very simple, with fixed point $f^*(x)=1$.

Very bad map pointwise, very nice map probabilistically

Sensitive Dependence on Neglected dynamics



$x_{k+1}=f(x_k)$ $y_{k+1}=f(y_k)$

$x_{k+1}=f(x_k) + \varepsilon(y_k - x_k)$ $y_{k+1}=f(y_k) + \varepsilon(x_k - y_k)$

Diffusive coupling

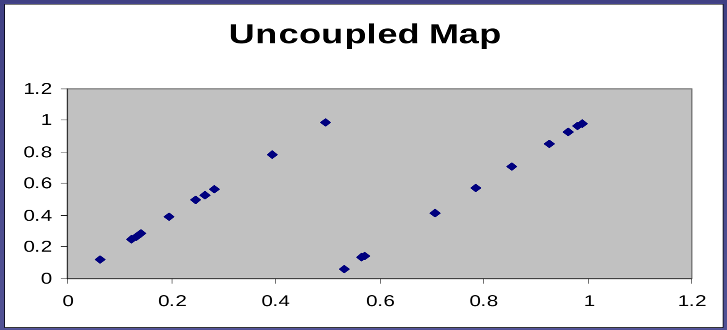
Lift into 2D

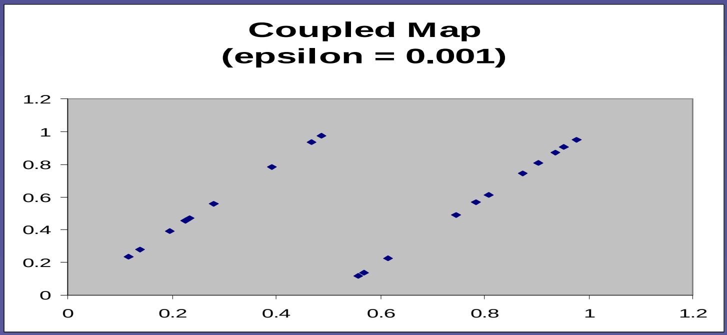
What if we throw some easily neglected dynamics into the mix?

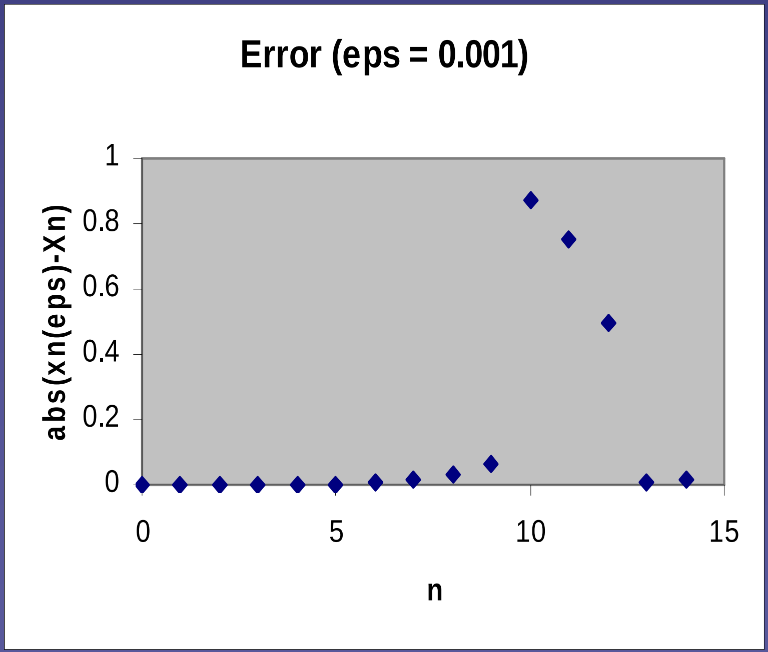
Diffusively coupled shift maps

$$x_{n+1} = [2x_n + \varepsilon(y_n - x_n)]\bmod 1$$
$$y_{n+1} = [2y_n + \varepsilon(x_n - y_n)]\bmod 1$$

$$[0,\ 1]\times[0,\ 1] \to [0,\ 1]\times[0,\ 1]$$







Coupling at Density Levels

- Diffusive coupled shift maps S = $S_1 \circ S_2$ $\quad [x, y] \longrightarrow [u, v]$

$$S_1: [0,\ 1]\times[0,\ 1] \rightarrow \left[0,\ \frac{2}{2-\varepsilon}\right]\times\left[0,\ \frac{2}{2-\varepsilon}\right],\ S_2: \left[0,\ \frac{2}{2-\varepsilon}\right]\times\left[0,\ \frac{2}{2-\varepsilon}\right] \rightarrow [0,\ 1]\times[0,\ 1]$$

- Inverse maps $S^{-1} = S_1^{-1} \circ S_2^{-1}$ $\quad [u, v] \longrightarrow [x, y]$

$$S^{-1}((x, y)) = \left\{\left[\frac{(2-\varepsilon)u}{4(1-\varepsilon)} - \frac{\varepsilon v}{4(1-\varepsilon)}, \frac{(2-\varepsilon)u}{4(1-\varepsilon)} - \frac{\varepsilon u}{4(1-\varepsilon)}\right]\right\}$$

- Functional equation operating on the densities

$$f_{n+1}(x, y) = Pf_n(x, y) =$$

$$= \frac{1}{4(1-\varepsilon)}\left[f\left(\frac{x}{2} + \frac{\varepsilon(x-y)}{4(1-\varepsilon)}, \frac{y}{2} + \frac{\varepsilon(y-x)}{4(1-\varepsilon)}\right) + f\left(\frac{x+1}{2} + \frac{\varepsilon(x-y+1)}{4(1-\varepsilon)}, \frac{y}{2} + \frac{\varepsilon(y-x+1)}{4(1-\varepsilon)}\right)\right.$$

$$\left. + f\left(\frac{x}{2} + \frac{\varepsilon(x-y+1)}{4(1-\varepsilon)}, \frac{y+1}{2} + \frac{\varepsilon(y-x+1)}{4(1-\varepsilon)}\right) + f\left(\frac{x+1}{2} + \frac{\varepsilon(x-y)}{4(1-\varepsilon)}, \frac{y+1}{2} + \frac{\varepsilon(y-x)}{4(1-\varepsilon)}\right)\right]$$

Does it Reduce Back? *Check it out*

- When $\varepsilon \to 0$,

$$f^*(x,y) = \frac{1}{4}\left[f^*(\frac{x}{2},\frac{y}{2}) + f^*(\frac{x+1}{2},\frac{y}{2}) + f^*(\frac{x}{2},\frac{y+1}{2}) + f^*(\frac{x+1}{2},\frac{y+1}{2}) \right]$$

- $f(x,y) = X(x)Y(y)$ reduces the system to

$$X(x) = \frac{1}{2}\left[X(\frac{x}{2}) + X(\frac{x+1}{2}) \right], Y(y) = \frac{1}{2}\left[Y(\frac{y}{2}) + Y(\frac{y+1}{2}) \right]$$

The decoupling ansatz restores the original dynamics

Solve the Original Functional Equation

- By symmetry: $f(x,y) = f(y, x)$
- Diffusion 'turns off' when $y=x$ suggests the ansatz $f(x,y)=g(y-x)$
- set $z= x-y$ to reduce the functional equation to

$$g(z) = \frac{1}{4(1-\varepsilon)}\left[2g(\frac{z}{2(1-\varepsilon)}) + g(\frac{z+1}{2(1-\varepsilon)}) + g(\frac{z-1}{2(1-\varepsilon)})\right]$$

- Note the symmetry $f(x,y)=f(y-x) \rightarrow g(z) = g(-z)$, then

$$g(z) = \frac{1}{2(1-\varepsilon)}\left[g(\frac{z}{2(1-\varepsilon)}) + g(\frac{z+1}{2(1-\varepsilon)})\right]$$

Reduce to the 1D shift map dynamics when $\varepsilon=0$

Solution

- First solution is $\delta(z)$, *i.e.* $f_\varepsilon^*(x, y) = \delta(x-y)$.
- Different from the original 1D solution $f^*(x, y) = 1$?
- Integrate out the "unknown" degree of freedom

$$Pf(x) = \int_{y=0}^{y=1} f^*(x, y)dy = \int_{y=0}^{y=1} \delta(x-y)dy = 1$$

- Another ansatz $f(x,y)=h(x+y)$ gives

$$h(z) = \frac{\alpha}{4}\left[h(\frac{z}{2}) + 2h(\frac{z+\alpha}{2}) + h(\frac{z+2}{2}) \right], \ (\alpha = \frac{1}{1-\varepsilon} > 1)$$

Conclusions and Future Work

- Conclusion
  - Ø Solve the functional equation
- Future Work
  - Ø Generalize this process to more maps.
- Question
  - Ø Is there ever Sensitive Dependence on Neglected Dynamics at the level of densities?

Thank You !