Some mathematical problems in geophysical fluid dynamics

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Overview

- Waves in geophysical fluid dynamics.
- Some observed phenomena that result from wave interactions.
- Introduction to the equations of geophysical fluid dynamics.
- An example of a wave–mean flow interaction:

  Rossby waves on a beta plane.
- Some analytical ideas.
- A few numerical results.

Geophysical fluid dynamics (GFD)

- The flow of fluids (liquids and gases) is modelled by PDEs based on Newton's Laws: conservation of mass, momentum and energy.
- Independent variables include:
  - space, e.g., (x, y, z) in rectangular coordinates
  - time t.
- Dependent variables include:
  - velocity, e.g., $\vec{v} = (u, v, w)$.
  - pressure p, density ρ, temperature T.
- In **GFD**, we add extra terms to represent the effects of
  - the earth's **rotation** (the Coriolis force)
  - **density stratification** (lighter fluid above, heavier fluid below)

The “beta plane” approximation for rotating flows

Consider 2-D **incompressible flow** in rectangular coordinates (x, y):

- Newton’s 2nd law ($\vec{F} = m\vec{a}$) $\Rightarrow$ 2 **momentum equations**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + fv$$

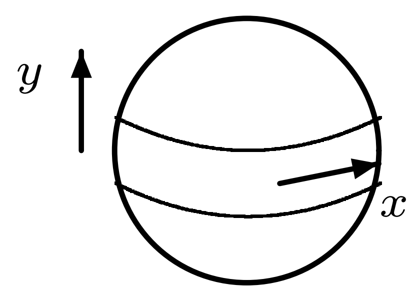
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - fu$$

time-dependence | advection | pressure gradient | friction | rotation

- Conservation of mass $\Rightarrow$ **continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- In the β-plane approximation, we assume that the Coriolis force $f \sim f_0 + \beta y$.



2-D fluid flow on a β-plane

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + fv \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - fu \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

Re-write in a form where we have only one dependent variable:

- Define streamfunction $\Psi(x, y, t)$ by $\frac{\partial \Psi}{\partial y} = -u, \quad \frac{\partial \Psi}{\partial x} = v$
- Differentiate equation (1) by y and equation (2) by x and subtract:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\nabla^2\Psi + \beta\frac{\partial \Psi}{\partial x} - \nu\nabla^4\Psi = 0$$

$\Rightarrow$

$$\nabla^2\Psi_t + \Psi_x\nabla^2\Psi_y - \Psi_y\nabla^2\Psi_x + \beta\Psi_x - \nu\nabla^4\Psi = 0$$

This is called the **barotropic vorticity equation**.

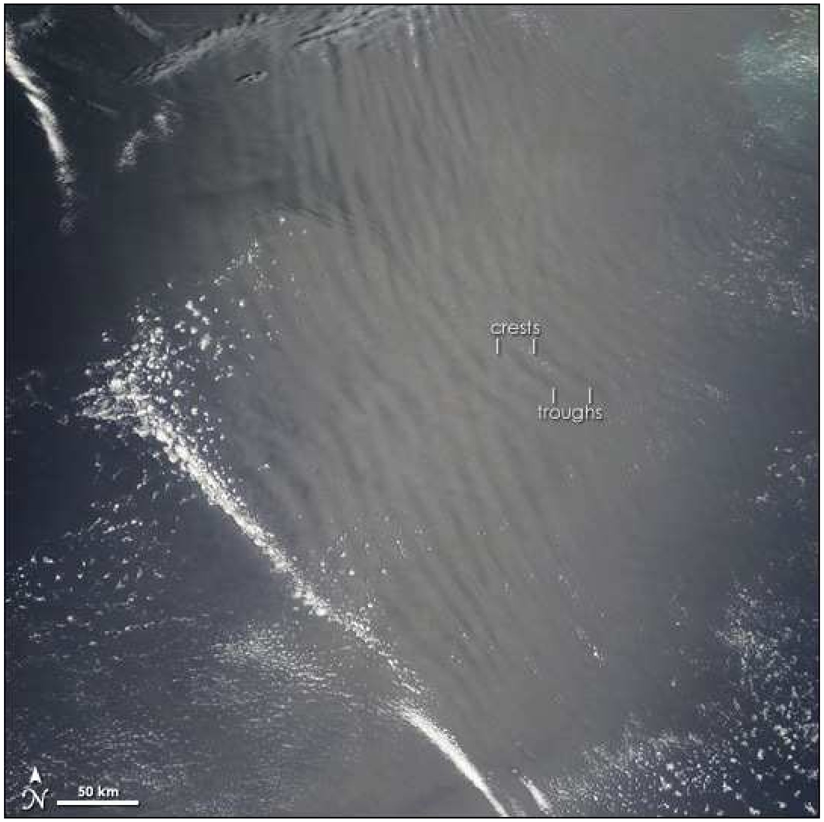
Waves in the atmosphere and ocean

There are various types of waves in geophysical flows, for example:

- **Rossby waves** (or planetary waves):
  - wavelengths $\sim 10,000 - 40,000$ km (almost the circumference of the earth)
  - result from earth's rotation
- Small-scale **gravity waves**:
  - wavelengths $\sim 10 - 1,000$ km
  - result from variation of density with height
  - sometimes forced by **topography** (mountains)
- Other planetary-scale waves:
  - Rossby-gravity waves
  - Kelvin waves

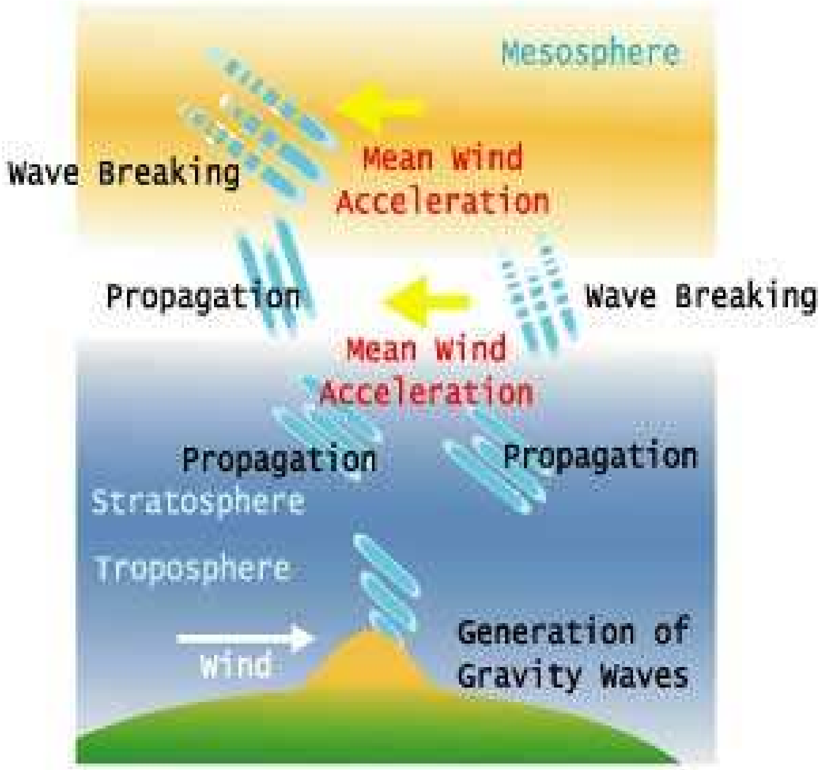
Atmospheric gravity waves between Edmonton and Edson (near Rocky Mountains). Taken by A. Mehta, 12/06/98 around 9am.

(From http://www.math.ualberta.ca/˜bruce/imagelinks/earth.html)



Atmospheric gravity waves over the Arabian Sea on May 23, 2005. The pattern seen is the "impression" of atmospheric gravity waves on the surface of the ocean.

(From http://earthobservatory.nasa.gov/Newsroom/NewImages/images.php3?img_id=16921)



Schematic diagram of topographic gravity waves.

(From http://sprg.ssl.berkeley.edu/atmos/gj_science.html)

Why is it important for us to understand geophysical waves?

Because waves interact with the general circulation of the atmosphere and cause various phenomena that affect us directly or indirectly.

For example:

- turbulence
- large-scale phenomena, such as the **quasi-biennial oscillation** and **stratospheric sudden warmings**

Interactions of waves with the general circulation are called **wave–mean-flow interactions**.

It is important to represent such interactions correctly in weather prediction and climate models.

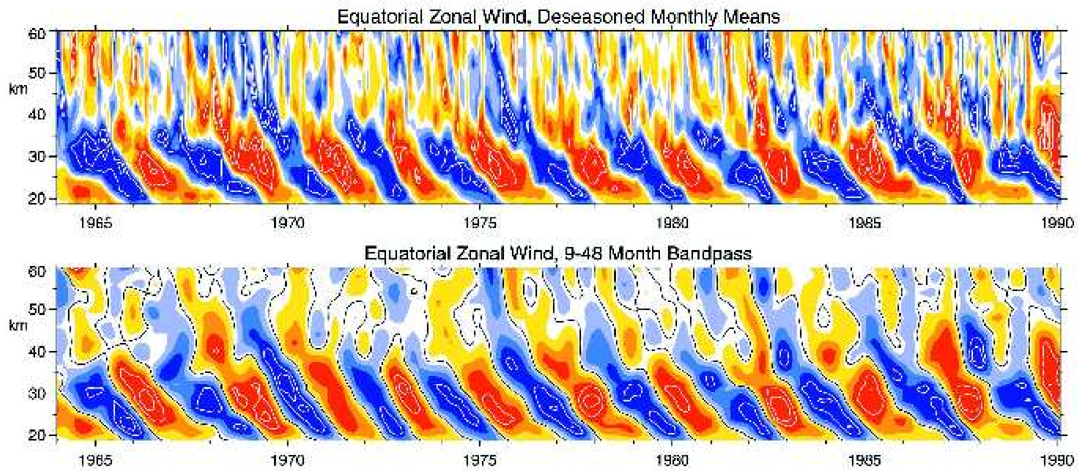


Plate 1. (top) Time-height section of the monthly-mean zonal wind component (m s^{-1}), with the seasonal cycle removed, for 1964–1990. Below 31 km, equatorial radiosonde data are used from Canton Island (2.8°N, January 1964 to August 1967), Gan/Maledive Islands (0.7°S, September 1967 to December 1975), and Singapore (1.4°N, January 1976 to February 1990). Above 31 km, rocketsonde data from Kwajalein (8.7°N) and Ascension Island (8.0°S) are shown. The contour interval is 6 m s^{-1}, with the band between −3 and +3 unshaded. Red represents positive (westerly) winds. After *Gray et al.* [2001]. In the bottom panel the data are band-pass filtered to retain periods between 9 and 48 months.

Contour plot of the east-west component of the wind in the middle atmosphere at the equator. Time is on the x-axis, and height above the ground is on the y-axis. Red = west-to-east winds, blue = east-to-west winds. Note that the wind changes direction every 26-28 months. This is called the **quasi-biennial oscillation**. It results from wave–mean fbw interactions in the middle atmosphere. (From Baldwin et al., Reviews of Geophysics, **39**, 2001)

How can we describe **wave–mean-flow interactions** in mathematical terms?

Assume wave is **sinusoidal**. For example, in 2-D we can write:

$$\psi(x,y,t) = \text{Re}\{Ae^{i(kx+ly-\omega t)}\}$$

or

$$\psi(x,y,t) = \text{Re}\{\phi(y)e^{i(kx-\omega t)}\}$$

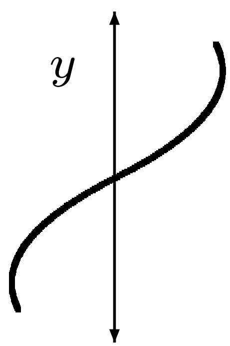
where k, $l =$ wavenumbers, $\omega =$ frequency.

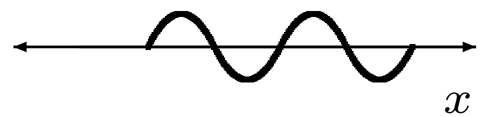
Let's illustrate this using the **barotropic vorticity equation**:

$$\nabla^2\Psi_t + \Psi_x\nabla^2\Psi_y - \Psi_y\nabla^2\Psi_x + \beta\Psi_x - \nu\nabla^4\Psi = 0$$

- Write $$\Psi(x,y,t) = \bar{\psi}(y) + \varepsilon\psi(x,y,t)$$

mean (shear flow)   perturbation (wave)





- Substitute into the BV equation and get a nonlinear equation for ψ. Since $\bar{\psi}(y)$ is known, we can solve for $\psi(x,y,t)$.

$$\nabla^2\psi_t + \bar{u}\nabla^2\psi_x + (\beta - \bar{u}_{yy})\psi_x + \varepsilon(\psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x) = 0.$$

- For $\varepsilon = 0$, equation is linear: easier (but not trivial!) to solve.

In fluid dynamics, we almost always have **nonlinear** equations to solve. We can linearize. But even the linear equations are usually not trivial to solve. For example, they may contain singularites....

Continue with our example: $\Psi(x, y, t) = \bar{\psi}(y) + \varepsilon\psi(x, y, t)$.

$\varepsilon = 0 \Rightarrow$ a linear equation for $\psi(x, y, t)$

Suppose waves are periodic in x and t:

$$\psi(x, y, t) = \text{Re}\{\phi(y)e^{ik(x-ct)}\}$$

wavenumber (k) phase speed (c)

$\Rightarrow$ an ODE for $\phi(y)$:

$$(\bar{u} - c)(\phi_{yy} - k^2\phi) + (\beta - \bar{u}_{yy})\phi = 0,$$

where $\bar{u}(y) = -\bar{\psi}_y$.

Rayleigh-Kuo equation:

$$\phi_{yy} + \left(-k^2 + \frac{\beta - \bar{u}_{yy}}{\bar{u} - c}\right)\phi = 0$$

But there is a complication:

The equation is singular if there is a point $y = y_c$ where $\bar{u}(y_c) = c$.

Question: How do we solve an ODE near a singular point?

Answer: **Use the method of Frobenius.**

We get 2 linearly independent series solutions about the singular point:

$$\phi_{\mathsf{A}}(y) = (y - y_c) + \frac{\beta - \bar{u}_c''}{2\bar{u}_c''}(y - y_c)^2 + \ldots,$$

$$\phi_{\mathsf{B}}(y) = 1 + \ldots + \frac{\beta - \bar{u}_c''}{2\bar{u}_c''}\phi_{\mathsf{A}} \log{(y - y_c)} + \ldots,$$

But what do we do with the log term?

The solution is undefined at $y = y_c$.

For $y < y_c$, we have to define $\log{(y - y_c)} = \log{|y - y_c|} + i\theta$, where $\theta = -\pi$. So the solution in discontinuous.

Also, the averaged momentum flux

$$F = -\frac{k}{2\pi}\int_0^{2\pi/k} uv dx = \frac{k}{2\pi}\int_0^{2\pi/k} \psi_x \psi_y dx$$

is discontinuous across $y = y_c$.

The reason we got the singularity was that

- we neglected the nonlinear terms
- we neglected viscosity (4th-order derivative)
- we assumed that $\psi(x, y, t) = \text{Re}\{\phi(y)e^{ik(x-ct)}\}$, i.e., that waves are periodic in time and x. So we got a 2nd-order ODE with no time-dependence.

Periodicity in x makes sense because x is the zonal coordinate:

- Rossby (planetary) waves: Assume that wavelength $2\pi/k =$ the circumference of the earth,
- smaller scale (topographic) waves: Assume mountains are periodic.

But there is less justification to assume periodicity in t.

So what happens if we don't assume periodicity in t. Can we get rid of the singularity then?

Let's write wave as

$$\psi(x, y, t) = \text{Re}\{\phi(y, t)e^{ikx}\}$$

and substitute in our linear equation

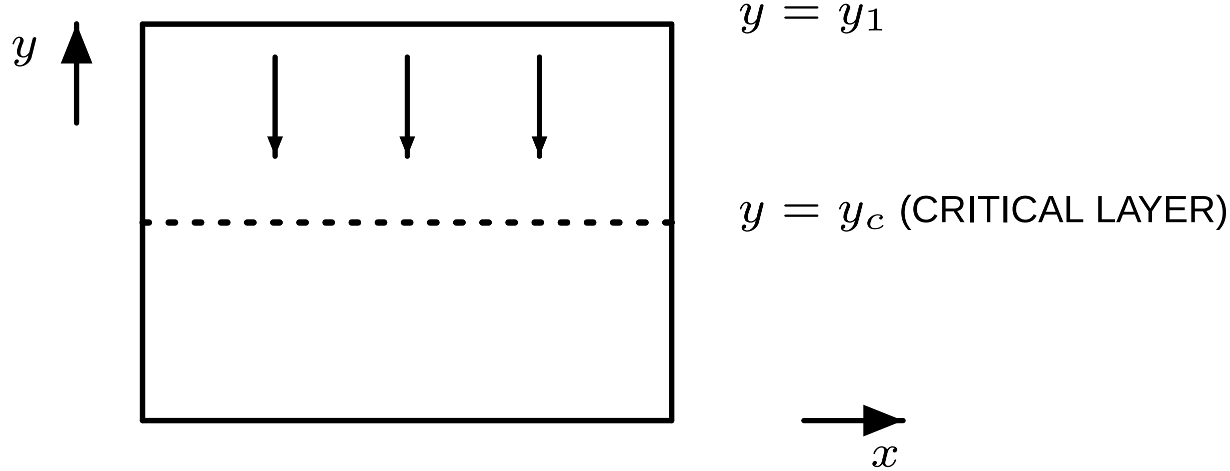
$$\nabla^2\psi_t + \bar{u}\nabla^2\psi_x + (\beta - \bar{u}_{yy})\psi_x = 0$$

Then we get a PDE for ϕ:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\phi_{yy} - k^2\phi\right) + (\beta - \bar{u}_{yy})\phi = 0$$

To solve the PDE, we need a boundary condition.

Consider the **boundary-value problem** in which the waves are forced at one boundary $y = y_1$ of a rectangular domain:



Boundary condition: $\psi(x, y_1, t) = e^{ikx}$, i.e., set $c = 0$ at the boundary.

Then the BVP is:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\left(\phi_{yy} - k^2\phi\right) + (\beta - \bar{u}_{yy})\phi = 0$$

with BC: $\phi(y_1, t) = 1.$

We shall see that things happen at the point y_c where $\bar{u}(y) = c = 0$. This is called the **critical layer**.

We solve the BVP by taking a Laplace transform in t, solving the transformed equation, and then inverting the transform.

Solution (for large t) is of the form:

$$\psi(x,y,t) = e^{ikx}\left\{\phi^{\infty}(y) + h_1(y)\frac{e^{-ikyt}}{k^2yt^2} + h_2(y)\frac{e^{-iky_1t}}{k^2yt^2}\right\},$$

where $\phi^{\infty}(y)$ is the solution of the steady problem we found earlier $(\phi^{\infty}(y) = a\phi_{\mathsf{A}}(y) + b\phi_{\mathsf{B}}(y))$

But there are problems:

(1) The solution of the linear equation is not valid as $y \longrightarrow 0$. (We thought we had gotten rid of the singularity, but it is still lurking in there!)

(2) How do we solve the nonlinear equation? We can't use Laplace transforms for that!

Answer to (1):

- Our solution is valid in the **outer region**, away from the point $y = y_c$.
- Near the point $y = y_c$, we have a **critical layer** where we need to find a different solution.
- It turns out that the thickness of the critical layer is $\varepsilon^{1/2}$.
- In the critical layer, we define a "stretched" variable $Y = \frac{y - y_c}{\varepsilon^{1/2}}$
$\Rightarrow$ a PDE in terms of Y which is valid in the inner region.
- We solve this **inner equation** and **"match"** the inner solution with the outer solution that we found already. We want:

$$\text{the inner solution} \longrightarrow \text{the outer solution, as } Y \to \infty$$
$$\text{the outer solution} \longrightarrow \text{the inner solution, as } (y - y_c) \to 0.$$

This procedure is called **the method of matched asymptotic expansions**.

Answer to (2):

To solve the nonlinear equation:

- We write the solution in powers of the parameter ε:
  $\psi(x,y,t) = \psi^{(0)}(x,y,t) + \varepsilon\psi^{(1)}(x,y,t) + \varepsilon^2\psi^{(2)}(x,y,t) + ...$
- The first term $\psi^{(0)}$ is the solution of the linear equation that we have found already.
- We substitute this series into our nonlinear equation and obtain equations for $\psi^{(0)}$, $\psi^{(1)}$, $\psi^{(2)}$,....
  At $O(1)$: $\nabla^2\psi_t^{(0)} + \bar{u}\nabla^2\psi_x^{(0)} + (\beta - \bar{u}_{yy})\psi_x^{(0)} = 0$
  At $O(\varepsilon)$: $\nabla^2\psi_t^{(1)} + \bar{u}\nabla^2\psi_x^{(1)} + (\beta - \bar{u}_{yy})\psi_x^{(1)}$
  $= -(\psi_x^{(0)}\nabla^2\psi_y^{(0)} - \psi_y^{(0)}\nabla^2\psi_x^{(0)})$
- Since we know $\psi^{(0)}$, we can find $\psi^{(1)}$, and then find $\psi^{(2)}$, and so on.
- We must do this in both the outer and inner regions and make sure that the solutions "match".

At the end of all this, we end up with approximate solutions that are valid in the different regions.

In our example problem, the solutions tell us that:

- The wave amplitude goes to zero at the critical layer.
- The momentum flux F is discontinuous across the critical layer.
- The wave doesn't just disappear at the critical layer. It is "absorbed" by the mean flow (momentum is transferred from the wave to the mean flow).
- The mean flow changes with time, according to:

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial F}{\partial y}.$$

- At later time, the wave is "reflected", i.e., momemtum is transferred from the mean flow to the wave.
- The approximate solution we have found describes a "wave–mean-flow interaction".

What does all this look like physically?

Let's look at some results of numerical simulations:

We solve our nonlinear time-dependent equation

$$\nabla^2\psi_t + \bar{u}\nabla^2\psi_x + (\beta - \bar{u}_{yy})\psi_x + \varepsilon(\psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x) = 0$$

using numerical methods:

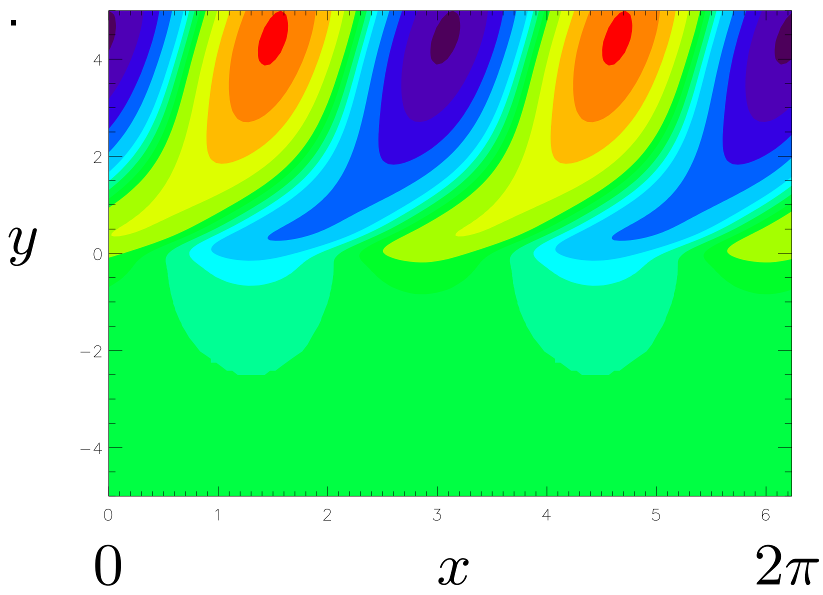
- fi nite differences in y and t
- a spectral method (Fourier series approximation) in x.

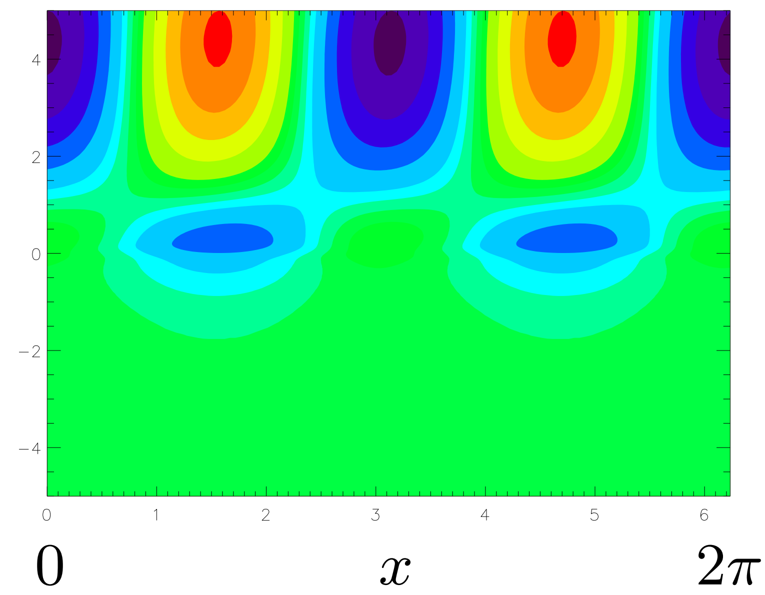
(Remember that our solution is periodic in x, so it makes sense to use a Fourier series representation in x.)

Some numerical results:

Apply boundary condition at $y = y_1$:

$$\psi(x, y_1, t) = \cos kx, \ \text{with} \ k = 2$$





Early time:
Linear propagation
Wave absorbed at critical layer

Late time:
Effects of nonlinearity
Wave reflections at critical layer

Conclusions

- Wave–mean-flow interactions can be decribed mathematically by PDEs.
- Approximate solutions can be found using analytical techniques or using numerical methods.
- In our example, the governing linear equation is singular if there is a place in the flow where the mean velocity equals the wave phase speed.
- In this region, the wave is absorbed/reflected by the mean flow.
- Momentum is transferred between the wave and the mean flow.
- Such wave–mean-flow interactions drive the general circulation of the atmosphere and ocean.