BODIL BRANNER

A HOLOMORPHIC TALE

GLIMPSES of HOLOMORPHIC DYNAMICS in the light of JOHN MILNOR's INFLUENCE on its DEVELOPMENT.

BOOK :

$\boxed{\text{DYNAMICS in ONE COMPLEX VARIABLE}}$

Vieweg 1999, 2000.
Princeton Univ. Press 2006.

PAPERS :

expositories

preprint $\longrightarrow$ publication

Renowned for :

- clarity
- elegance
- completeness
- beautiful computer pictures
- appropriate acknowledgement of others work

THREE THEMES:

- MODULI SPACES of cubic polynomials and quadratic rational maps suggesting non-local connectivity
- The combinatorial method for quadratic polynomials described by ORBIT PORTRAITS
- MATINGS and LATTÈS MAPS

CUBIC POLYNOMIALS

$P_{a,b}(z) = z^3 - 3a^2 z + b$

critical points: $\pm a$

PARAMETER SPACE: $\mathbb{C}^2$

$P_{a_1,b_1} \underset{\substack{\text{hol.}\\ \text{conj.}}}{\sim} P_{a,b} \iff (a_1, b_1) = (\pm a, \pm b)$

MODULI SPACE: $\mathcal{M}(3) = \mathbb{C}^2$

$(A, B) \qquad A = a^2, \; B = b^2$

one representative for each holomorphic conjugacy class.

REAL CUBIC POLYNOMIALS

$(A, B) \in \mathbb{R}^2$

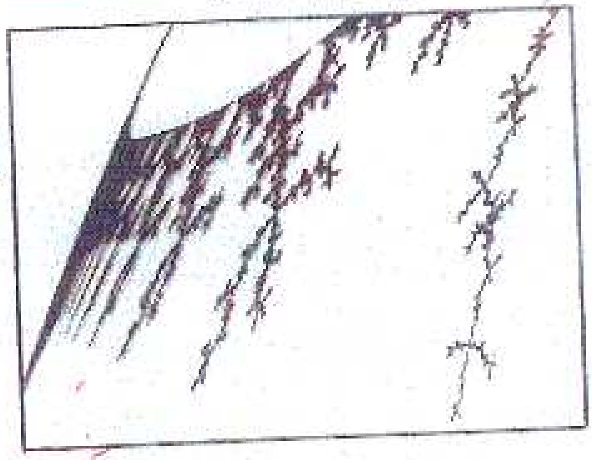
If $B<0$, a representative with real coefficients is

$$-x^3 - 3Ax + \sqrt{|B|}$$

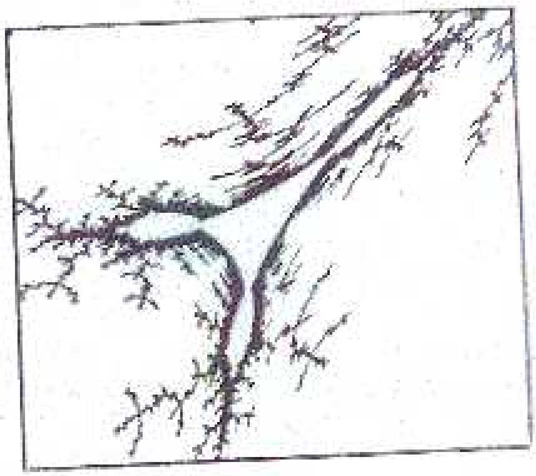
Hence, two complex conjugate crit. points with complex conjugate orbits.

$\boxed{\text{on iterated cubic maps.}}$ Experiment. Math. 1 (1992), 5-24.

of the
ectedness locus $\mathcal{C}(3)$
$\mathbb{R}^2$, in the quadrant
$A > 0$, $B < 0$

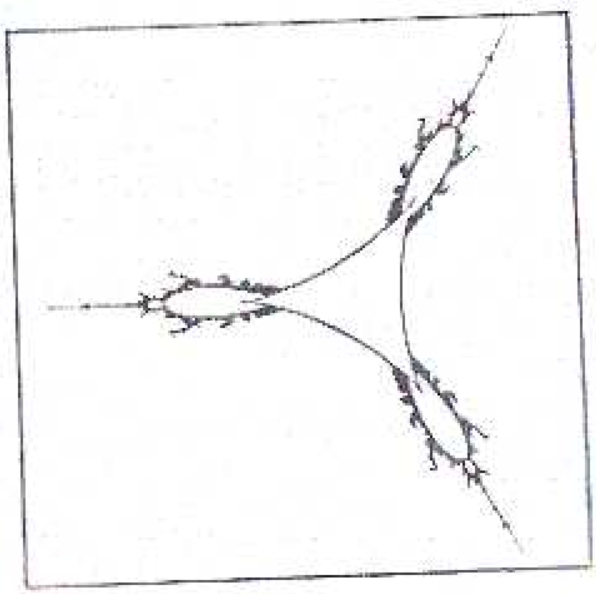


$(3) \subset \mathbb{C}^2$
connected and
n-locally connected



Enlargement

totype model for
transitive case:
connectedness locus,
tricorn, of the
-quadratic family
$z \mapsto (z^2+c)^2 + \bar{c}$



QUADRATIC RATIONAL MAPS Rat_2

Strong analogy between the theory of quadratic rational maps and cubic polynomials.

In both cases:

- * two free critical points
- • three fixed points
  (counted in $\hat{\mathbb{C}}$ for rat.
  in $\mathbb{C}$ for pol.)

The MODULI SPACE is $\mathbb{C}^2$.

There are many different normal forms, for instance

$$f(z) = a\left(z + \frac{1}{z}\right) + b$$

* ± 1 c.p., • ∞ fixed

$\boxed{\text{Geometry and dynamics of quadratic rational maps.}}$

Experiment. Math. 2 (1993), 37-83

ORBIT PORTRAITS of quadratic polynomials.

$\boxed{\text{Periodic orbits, external rays and the Mandelbrot set}}$
Géométrie complexe et systèmes dynamiques (Orsay, 1995), Astérisque, No. 261 (2000), 277-333.

$\boxed{\text{Fixed point portraits, with Lisa Goldberg}}$
Ann. Sci. Ecole Norm. Sup. 26 (1993), 51-98.

The main purpose is to give an alternative and simplified proof of the landing of parameter rays $R_M(t)$, for t periodic under doubling.

The proof is based on orbit portraits.

MOTIVATION

Suppose $\mathcal{O} = \{z_1, \cdots, z_p\}$ is a repelling or parabolic orbit of period p of some quadratic pol. $z \mapsto z^2 + c$. Let A_j denote the set of angles of the dynamic rays landing at z_j. Then $\{A_1, \cdots, A_p\}$ is called the orbit portrait $\mathcal{P} = \mathcal{P}(\mathcal{O})$ of $\mathcal{O}$

PROPERTIES and DEFINITION of an ORBIT PORTRAIT

PROPERTIES

(1) A_j is a finite subset of $\mathbb{Q}/\mathbb{Z}$.

(2) $A_j \xrightarrow{2\cdot} A_{j+1}$ is a bijection under doubling, preserving cyclic order.

(3) $\exists\, r \geq 1$ s.t. any $t \in \bigcup_{j=1}^{p} A_j$ has period rp under doubling.

(4) $A_1, \cdots, A_p$ are pairwise unlinked.

DEF

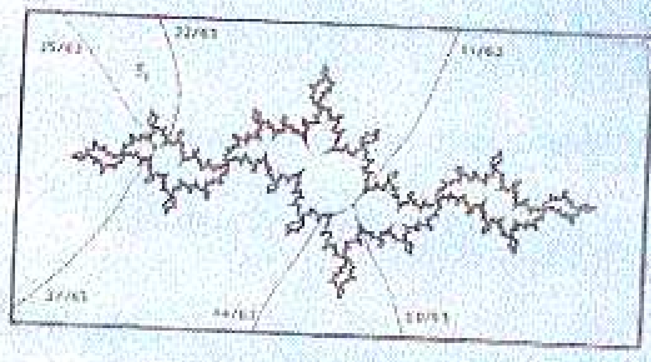
Any $\{A_1, \cdots, A_p\}$ satisfying the above is called an orbit portrait, $\mathcal{P}$.

$|A_j| = v$ is called valence. We have $r \leq v$.

REALIZATION

Any orbit portrait $\mathcal{P}$ is realized by some quadratic polynomial.

EXAMPLES of ORBIT PORTRAITS $\mathcal{P}$



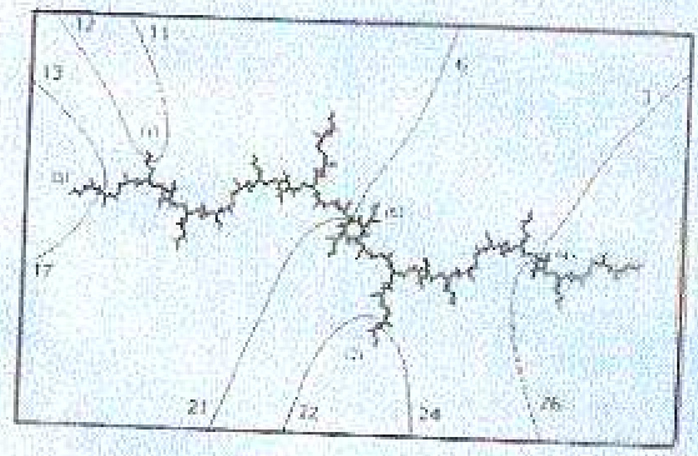
SATELLITE TYPE:

unit: $\frac{1}{63}$

$$\{\underbrace{\{22, 25, 37\}}_{A_1}, \underbrace{\{44, 50, 11\}}_{A_2}\}$$

$p = 2,\ v = 3,\ r = 3$

$I_{\mathcal{P}} =]\frac{22}{63}, \frac{25}{63}[$



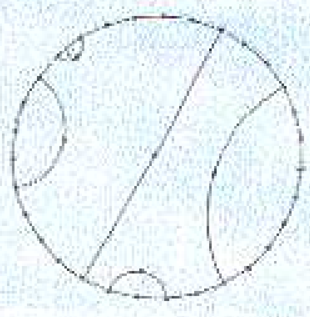
PRIMITIVE TYPE:

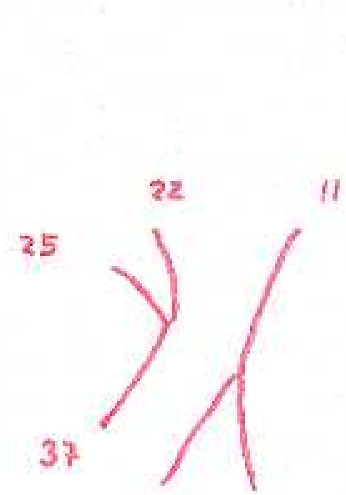
unit: $\frac{1}{31}$

$$\{\underbrace{\{11, 12\}}_{A_1}, \underbrace{\{22, 24\}}_{A_2}, \underbrace{\{13, 17\}}_{A_3}, \underbrace{\{26, 3\}}_{A_4}, \underbrace{\{21, 6\}}_{A_5}\}$$

$p = 5,\ v = 2,\ r = 1.$

$I_{\mathcal{P}} =]\frac{11}{31}, \frac{12}{31}[$




22
11
25
37
44
50

TWO POSSIBILITIES

① PRIMITIVE CASE:

If $r < v$, then $r = 1$, $v = 2$,
so $\mathcal{P}$ contains two orbits of angles.

② SATELLITE CASE:

$r = v$, so $\mathcal{P}$ contains one orbit of angles.

Each $\mathcal{P}$ has a unique smallest arc
$I_{\mathcal{P}} =]t_-(\mathcal{P}), t_+(\mathcal{P})[$
among the complementary arcs to $\bigcup_{j=1}^{\ell} A_j$,
the characteristic arc for $\mathcal{P}$.

MAIN THEOREM in PARAMETER PLANE

Given $\mathcal{P}$.

The two parameter rays $\mathcal{R}_M(t_\pm(\mathcal{P}))$ land at a common parabolic point $r_{\mathcal{P}} \in M$.

PREPARATION, for $c \notin M$.

- a rational dynamic ray $R_{K_c}(\theta)$ lands if it does not bounce at precritical points

- if $c \in \mathcal{R}_M(t(c))$ then

  $\mathcal{R}_{K_c}(\theta)$ bounces $\iff t(c) \in \{2\theta, 4\theta, 8\theta, \cdots\}$

KEY LEMMA for $c \notin M$

f_c admits a periodic orbit with portrait $\mathcal{P}$

$$\Updownarrow$$

$t(c) \in I_{\mathcal{P}} = \,]\,t_-(\mathcal{P}), t_+(\mathcal{P})\,[$

SKETCH of proof of MAIN THEOREM.

① For any $t \in \bigcup_{j=1}^{p} A_j$ let $c_0 \in \overline{\mathcal{R}_M(t)} \setminus \mathcal{R}_M(t)$.

Any neighborhood of c_0 contains c s.t. $\mathcal{R}_{K_c}(t)$ bounces.

Therefore c_0 must have the following property:

$\exists x_0 \in J_{c_0}$ s.t. $f_{c_0}^{\circ p}(x_0) = x_0$ and $(f_{c_0}^{\circ p})'(x_0) = +1$.

If not, all $t \in \bigcup_{j=1}^{p} A_j$ would land at repelling periodic points with a stable portrait. — A contradiction.

There are only finitely many c-values with this property.

Hence, $\mathcal{R}_M(t)$ lands at c_0.

②

$\mathbb{C} \setminus \bigcup_{t \in \cup A_j} \overline{\mathcal{R}_M(t)}$ is open.

If $\mathcal{R}_M(t_-)$ and $\mathcal{R}_M(t_+)$ would land at different points, this would violate the key lemma.

MATINGS.

Pasting together Julia sets:
a worked out example of matings

Experiment. Math. 13 (2004), 55-92.

Topological mating of two quadratic polynomials f_1 and f_2 with connected and locally connected Julia set.

Connect $\mathcal{R}_{K_1}(t)$ with $\mathcal{R}_{K_2}(-t)$.

Let $\underset{ray}{\sim}$ be the smallest equivalence relation so that $\overline{\mathcal{R}_{K_1}(t)}$ and $\overline{\mathcal{R}_{K_2}(-t)}$ lie in one equivalence class.

Thm of. R.L. Moore

$S^2/\underset{ray}{\sim}$ is homeomorphic to S^2

$\Updownarrow$

no equivalence class separates S^2 into two or more connected components.

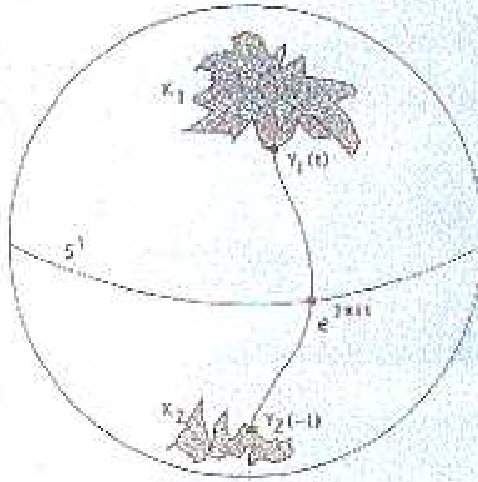
If so, we define $f_1 \amalg f_2 : S^2 \to S^2$.

$F \in \mathrm{Rat}_2$ is a geometric mating of f_1 and f_2

if $F \underset{top}{\sim} f_1 \amalg f_2$

Illustration of topological mating of f_1 and f_2

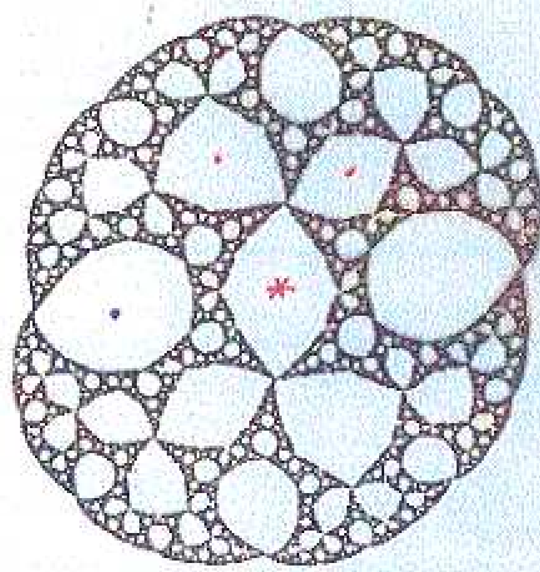
and $\underset{\text{ray}}{\sim}$

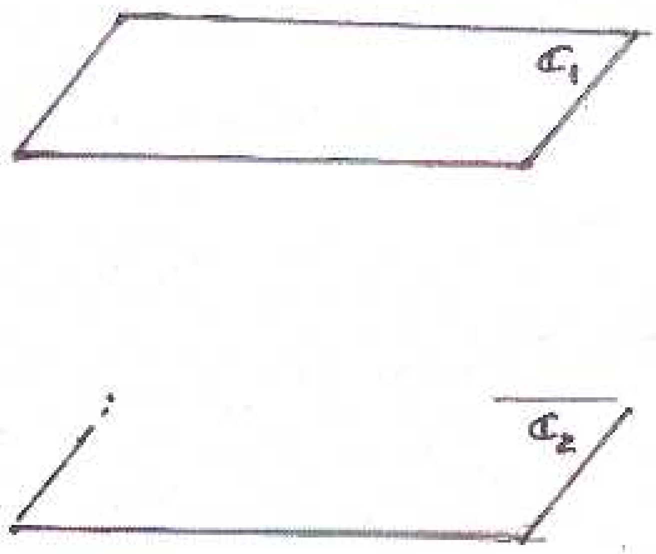


From the cover of the Stony Brook preprint series

mating of
the basilica
and the rabbit

∞




$\mathbb{C}_1$
$\mathbb{C}_2$

THE WORKED OUT EXAMPLE

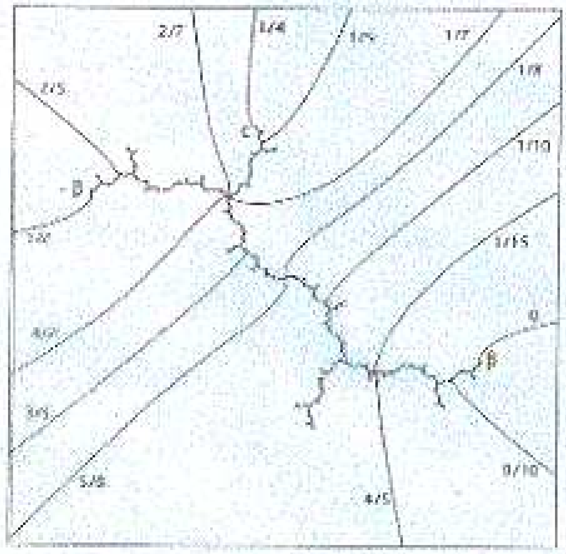
$f = f_{c_{\frac{1}{4}}} : z \mapsto z^2 + c_{\frac{1}{4}}$

where $c_{\frac{1}{4}}$ = landing point of $R_M(\frac{1}{4})$.

critical orbit:

$0 \mapsto c_{\frac{1}{4}} \mapsto -\beta \mapsto \beta \circlearrowleft$

$J_{c_{\frac{1}{4}}}$ is a dendrite

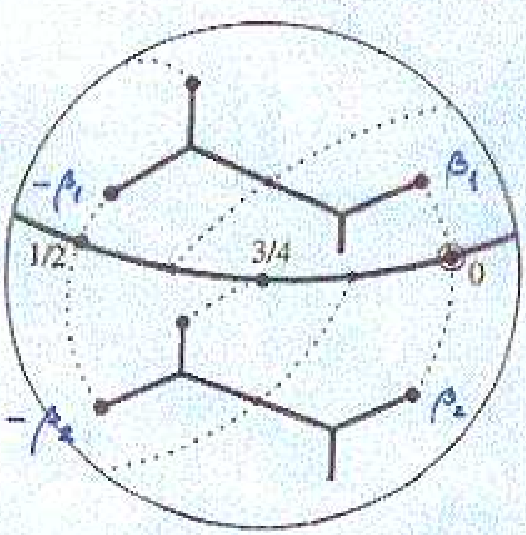


critical orbits of $f \amalg f$:

$$\begin{array}{l} 0_1 \mapsto c_1 \searrow \\ \qquad\qquad\quad -\beta \mapsto \beta \circlearrowleft \\ 0_2 \mapsto c_2 \nearrow \end{array}$$

The geometric mating exists:

$F \underset{\substack{top\\conj}}{\sim} f \amalg f$



$F(z) = \frac{i}{2}\left(z + \frac{1}{z}\right)$

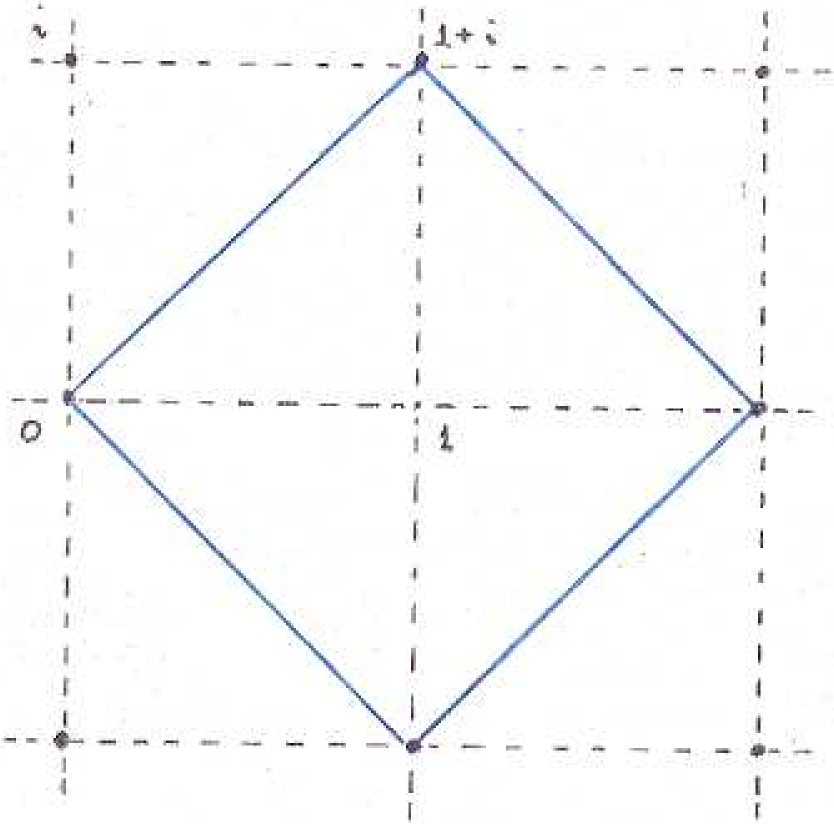
$$\begin{array}{l} +1 \mapsto +i \searrow \\ \qquad\qquad\quad 0 \mapsto \infty \circlearrowleft \\ -1 \mapsto -i \nearrow \end{array}$$

A LATTÈS MAP of deg 2

THE LATTÈS MAP.

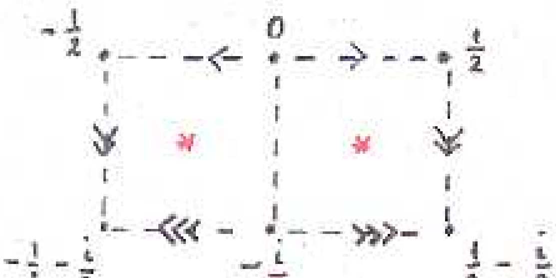
$$
\begin{tikzcd}
\mathbb{T} = \mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z} \arrow[r, "L"] \arrow[d, "p"'] & \mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z} \arrow[d, "p"] \\
\mathbb{T}/\pm = \hat{\mathbb{C}} \arrow[r, "F"] & \hat{\mathbb{C}}
\end{tikzcd}
$$

$L(w) = (1-i)w$



Lattice

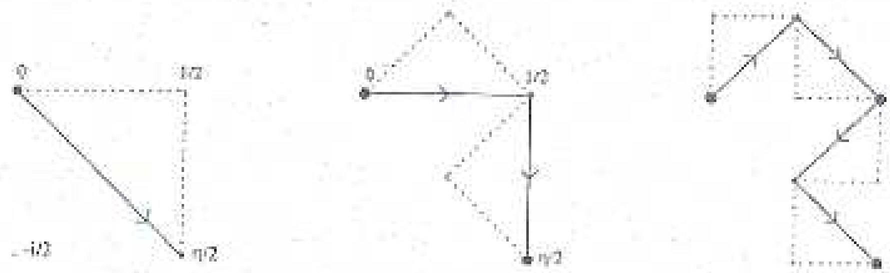
$\mathbb{Z} \oplus i\mathbb{Z}$

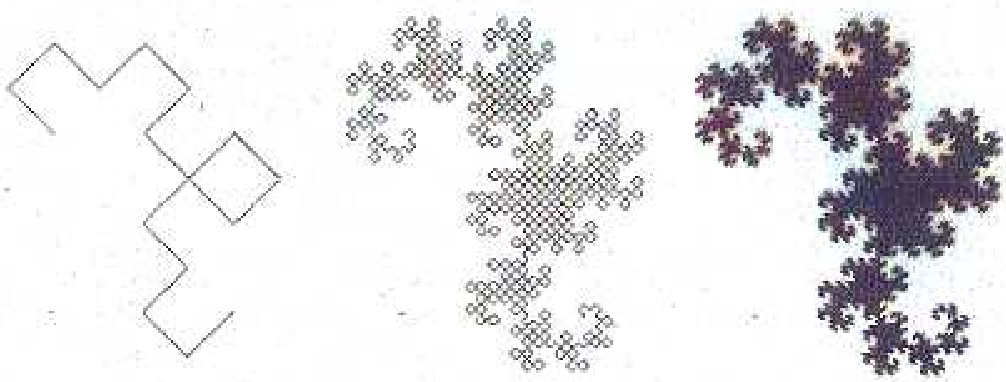


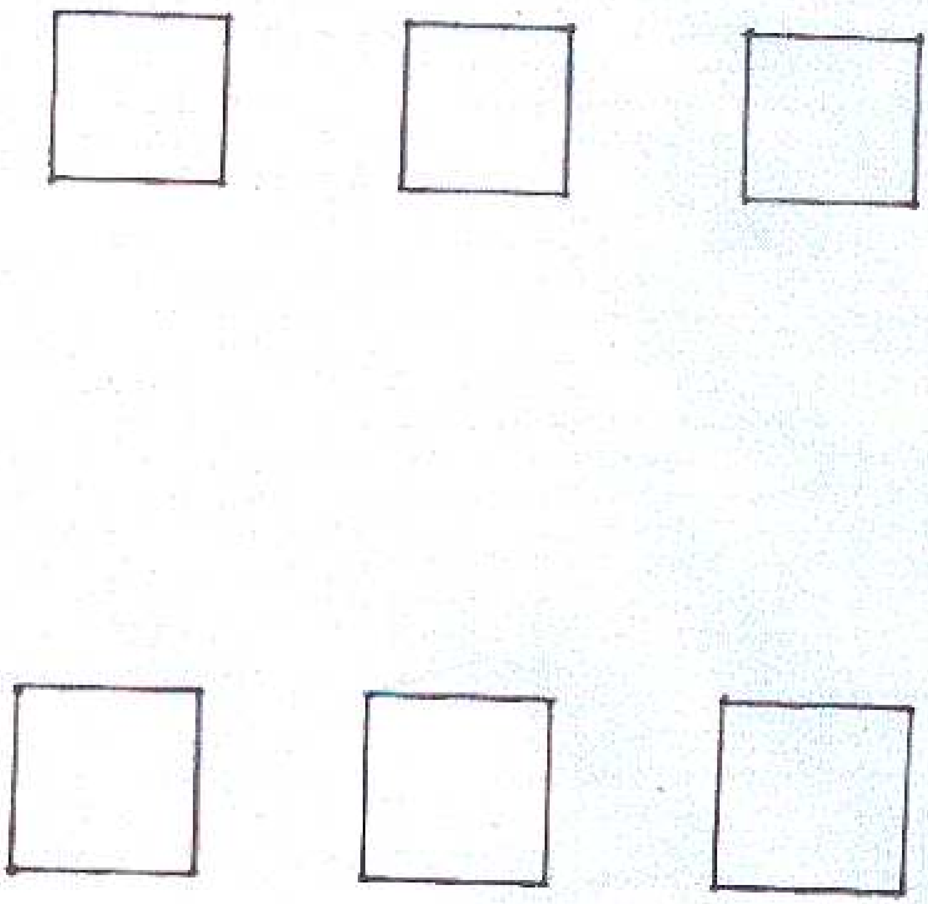
Pillow case

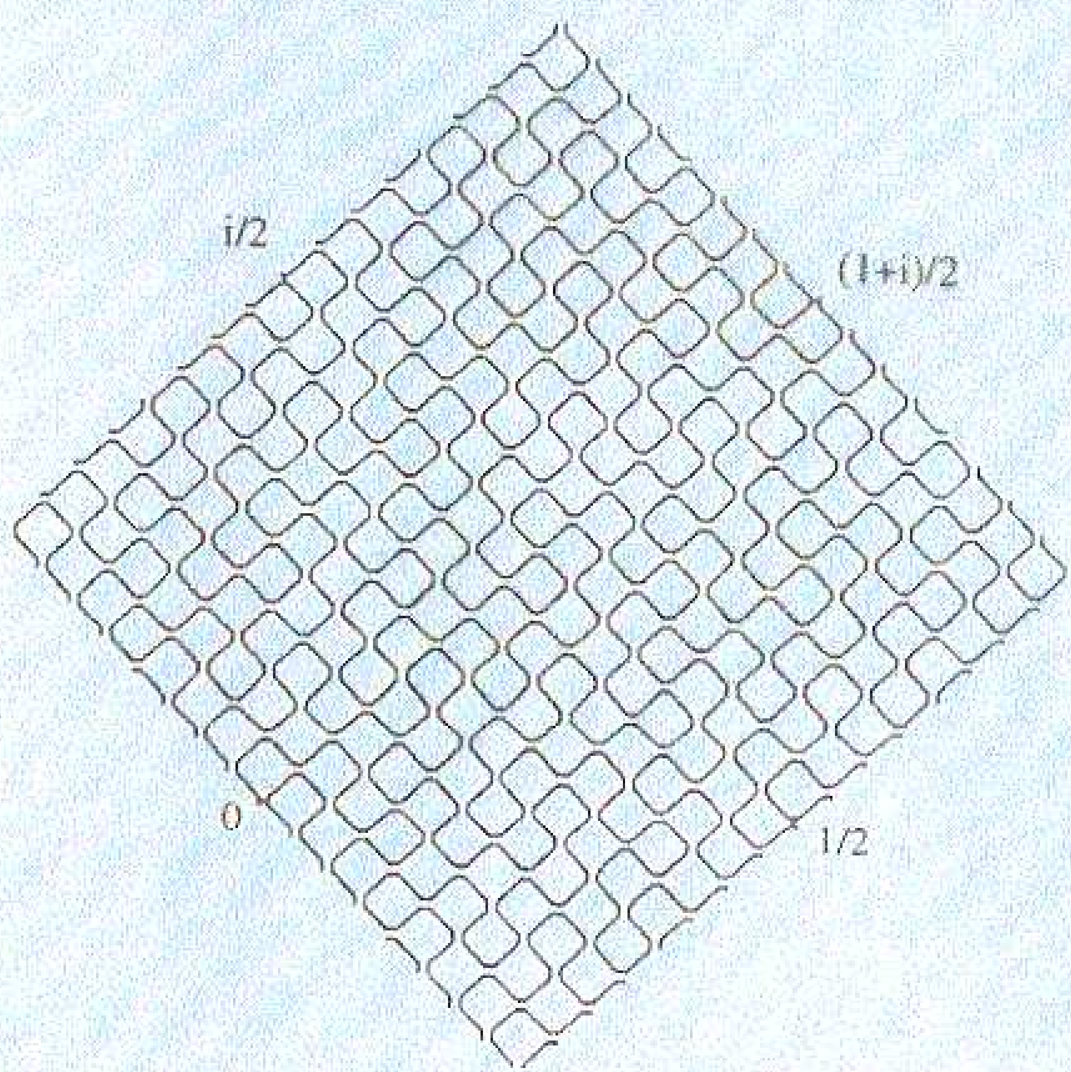
$\hat{\mathbb{C}}$

The Heighway Dragon construction



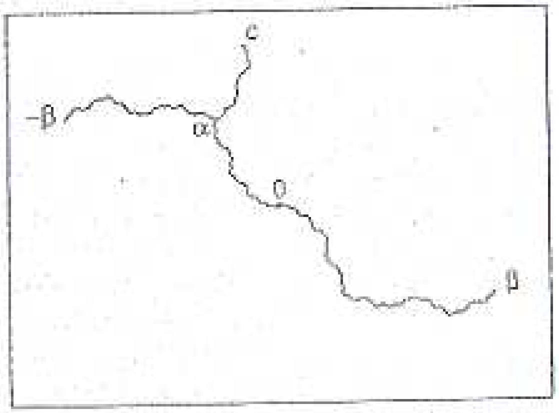




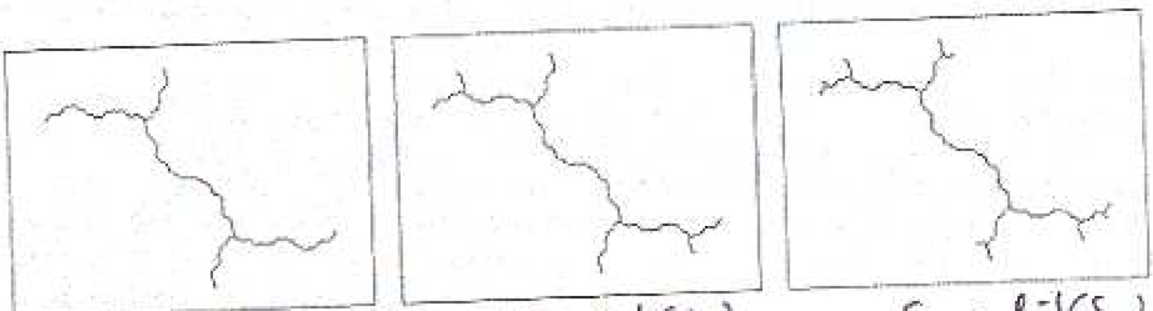


A fundamental domain of

$\mathbb{T}/\pm$

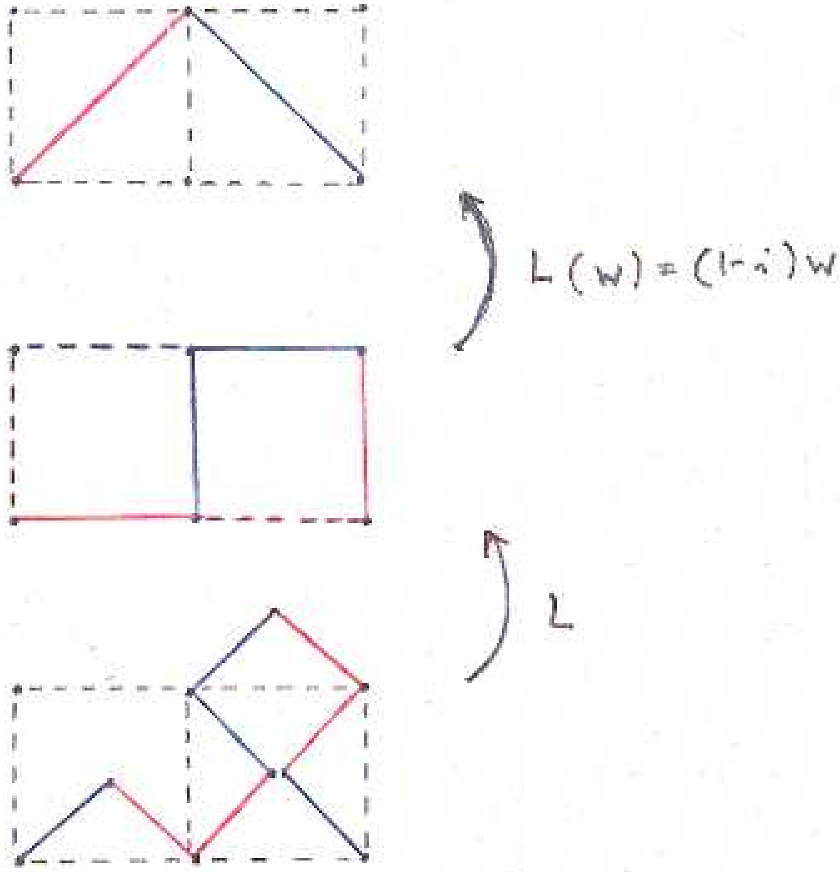


The Hubbard tree of f
the spine $[-\beta, \beta] = S_0$

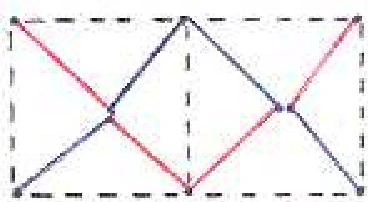


$S_1 = f^{-1}(S_0)$ $S_2 = f^{-1}(S_1)$ $S_3 = f^{-1}(S_2)$

Successive preimages of the spine,
the limit S_∞ is dense in $J(f)$.



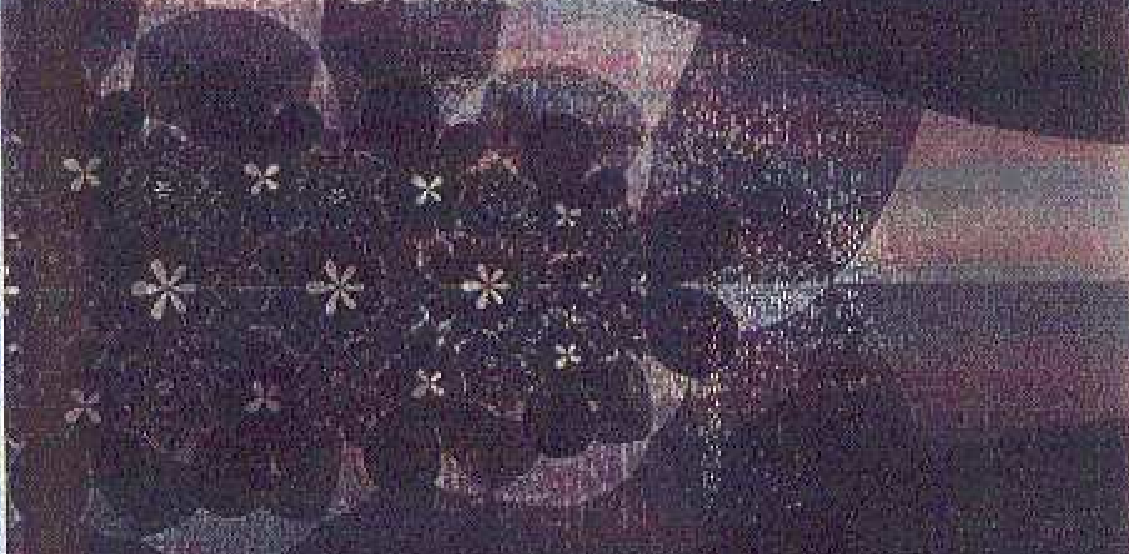
redrawn in
a fundamental
domain



On Lattès Maps
pp. 9-43
(2006)

Dynamics on the Riemann Sphere

A Bodil Branner Festschrift



Poul G. Hjorth
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Editors

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