DEGENERATING GEOMETRIC STRUCTURES VIA PROJECTIVE GEOMETRY

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WANT TO UNDERSTAND
OF HYPERBOLIC STRUCTURES THAT
DEGENERATE.

→ TRANSITION TO OTHER GEOMETRIES

EX: HYP. TRIANGLE

$\Sigma \theta_i < \pi$

DOUBLE ALONG EDGES ⟶

$(S^2, 3\text{ pts})$
CONE ANGLES
$\alpha_i = 2\theta_i$, $\Sigma \alpha_i < 2\pi$

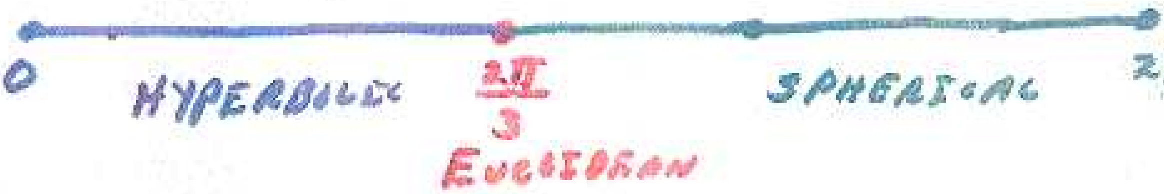
GAUSS-BONNET: K=CURVATURE

$$\boxed{K(\text{AREA}) + \sum 2\pi - \alpha_i = 2\pi \chi}$$

AREA → 0 AS $\Sigma \alpha_i \to 2\pi$
RESCALE SO AREA ≡ 1. K → 0
GET DOUBLED EUC. TRIANGLE
$\Sigma \alpha_i > 2\pi$ SPHERICAL TRIANGLE

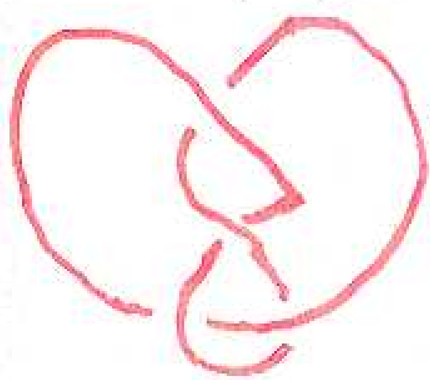
→ FAMILY OF CONE STRUCTURES ON $(S^3,$

$\alpha_1 = \alpha_2 = \alpha_3 \qquad 0 \leq \alpha_i \leq 2\pi$



Ex: $\qquad M^3 = S^3 - K$

K = FIGURE 8



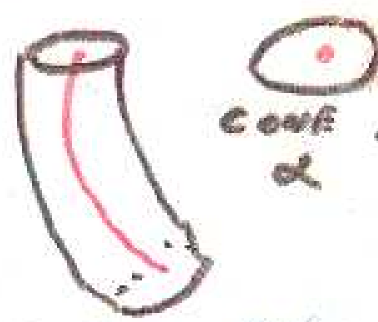
HYPERBOLIC CONE STRUCTURE

FOR $\boxed{0 \leq \alpha < 2\pi/3}$

parametrized by $\underline{\alpha}$

EUCLIDEAN: $\alpha = 2\pi/3$

SPHERICAL: $2\pi/3 < \alpha \leq \pi$ ← 2-FOLD BRANCHED COVER IS LENS SPA



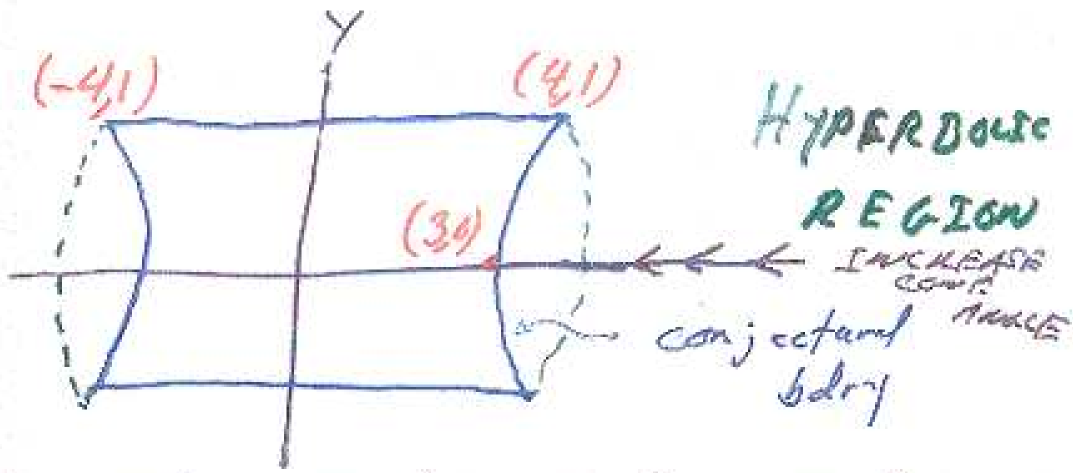
CONE ANG α

→ CAN'T GO ALL WAY TO 2π

CAN GO $\alpha = 4\pi/3$

(HILDEN, LOZANO, MONTESINOS)

HYPERBOLIC DEHN SURGERY SPACE
FIGURE 8 KNOT COMPLEMENT



$\{(x,y) \mid x\mu + y\lambda = 2\pi i\}$ μ, λ complex lengths of meridian and longitude

$(x,y) = g(p,q) \leftrightarrow$ cone angle $\frac{2\pi}{g}$ on (p,q)-filling of boundary torus

$M = S^3 - K$

Have $\rho_t : \pi_1 M \longrightarrow SO(3,1)$ ($\mathrm{Isom}^+ \mathbb{H}^3$)

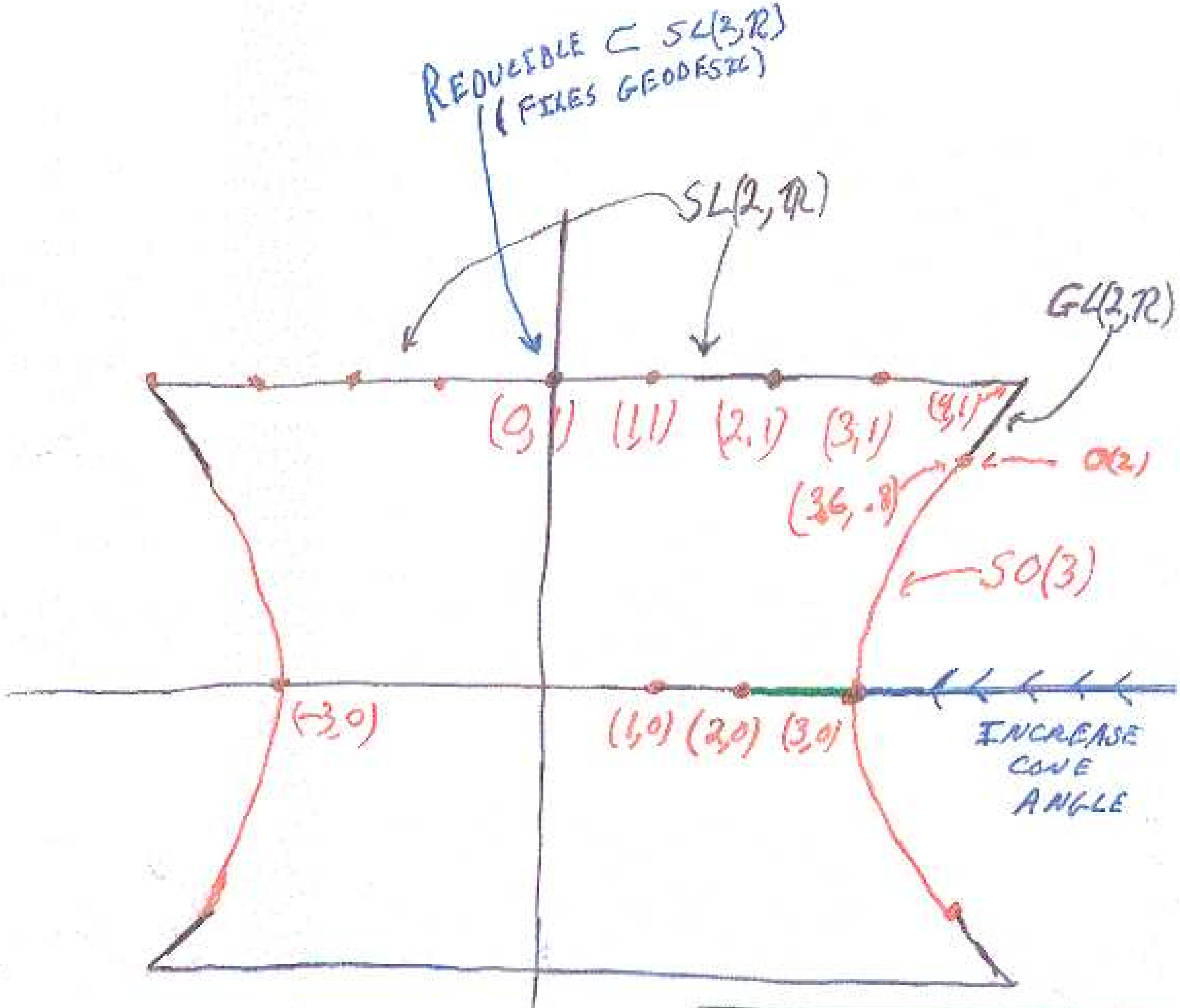
$\rho_t \to \rho_0 : \pi_1 M \longrightarrow SO(3) \leftarrow$ fixes a pt.

<u>WANT</u> <u>TO</u> <u>LIFT</u> <u>TO</u> <u>EUCLIDEAN</u> <u>STRUCTURE</u>

$$1 \to \mathbb{R}^3 \to \mathrm{Isom}^+(E^3) \to SO(3) \to 1$$

→ Connect various types of <u>degenerations</u> with <u>other</u> geometries

$$\boxed{E^3,\ S^3,\ \mathrm{SOL},\ \mathrm{NIL},\ \widetilde{SL}_2,\ \mathbb{H}^2\times\mathbb{R},\ S^2\times\mathbb{R}}$$



HODGSON : BDRY ⟷
0 VOL. REPS

⇒ LIFT BDRY REPS TO
3D GEOMETRY REPS?

⟶

$Isom^+(E^3) \longrightarrow SO(3)$ near (3,0) ✓

$Isom(SOL) \longrightarrow \mathbb{R}$ (0,1) ✓

$Isom(\widetilde{SL(2,R)}) \to Isom(H^2)$ $(k,1)$ $k=\pm1,2,3$ ✓

$= SL(2,\mathbb{R})$ $(GL(2,\mathbb{R}))$

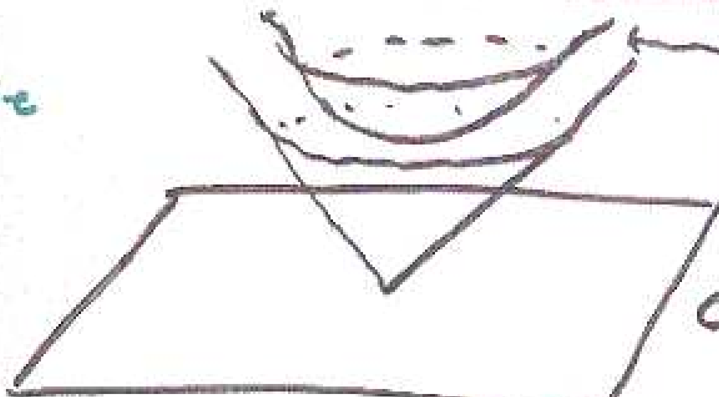
$Isom(NIL) \longrightarrow O(2)$? cone angle 5π on (4,2)

Projective Model for $\mathbb{H}^n$

Light cone

quadratic form

$$\sum_{i=1}^{n} x_i^2 - x_{n+1}^2$$



$\mathbb{H}^n$

$\{x \mid q(x) = -1\}$

$\subset \mathbb{R}^{n+1}$

Then $\mathbb{H}^n \subset \mathbb{RP}^n$ as ball

$\mathrm{Isom}^+(\mathbb{H}^n) = SO(n,1) \subset PGL(n+1)$
preserving quadratic form

Consider forms

$$q_s(x) = \sum_{i=1}^{n} s x_i^2 - x_{n+1}^2 \qquad 0 < s \le 1$$

Level sets $q_s(x) = -1 \subset \mathbb{RP}^n$
larger balls

Subgroups preserving form conjugate in Projective group

As $s \to 0$ $q_s(x) \to -x_{n+1}^2$

level set $q_s(x) = -1$ $\rightsquigarrow$ $x_{n+1} = 1$

group becomes $Aff(n) \subset PGL(n+1)$
subgroup preserving "∞"

→ limit actually in <u>$\mathrm{Isom}\, E^n$</u>

Let $s \in [-1, 0)$ $\quad -q_s(x) = -s \sum x_i^2 + x_{n+1}^2$

subgroups conjugate to $SO(n+1)$

$\boxed{\text{Spherical geometry}}$

preserves conic: $-q_s(x) = 1$

Algebraic version of metric rescaling
Get $\mathbb{H}^3, E^3, S^3$ in <u>1</u> "geometry"

<u>Advantage</u>: All representation varieties contained in one space.

PROJECTIVE REPRESENTATIONS AND 3-D GEOMETRIES

E. MOLNAR, THIEL, THURSTON

IMAGES IN $PGL(4,\mathbb{R})$

$\mathbb{H}^3$: $\begin{bmatrix} \underbrace{v_1 \; v_2 \; v_3}_{\text{tangent space}} & v_4 \end{bmatrix}$

$\|v_4\|^2 = -1$

$\|v_i\|^2 = 1 \quad i=1,2,3$

$\langle v_i, v_j \rangle = 0 \quad i \neq j$

w.r.t. hyperbolic quadratic form

v_4: image of $(0,0,0,1)$

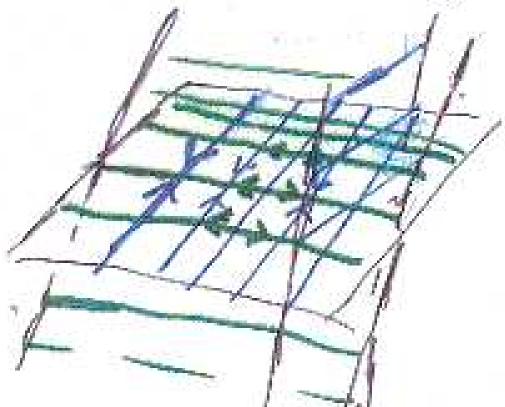
$$\longrightarrow \begin{bmatrix} SO(3) & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

SO(3) rep

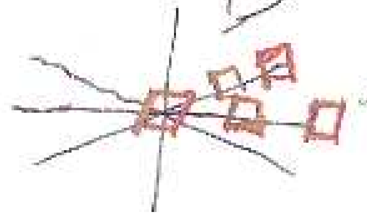
$$\begin{bmatrix} SO(3) & \begin{matrix} X \\ Y \\ Z \end{matrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix}$$

Euc rep.

SOL: $\begin{bmatrix} e^{z} & 0 & 0 & X \\ 0 & e^{-z} & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$



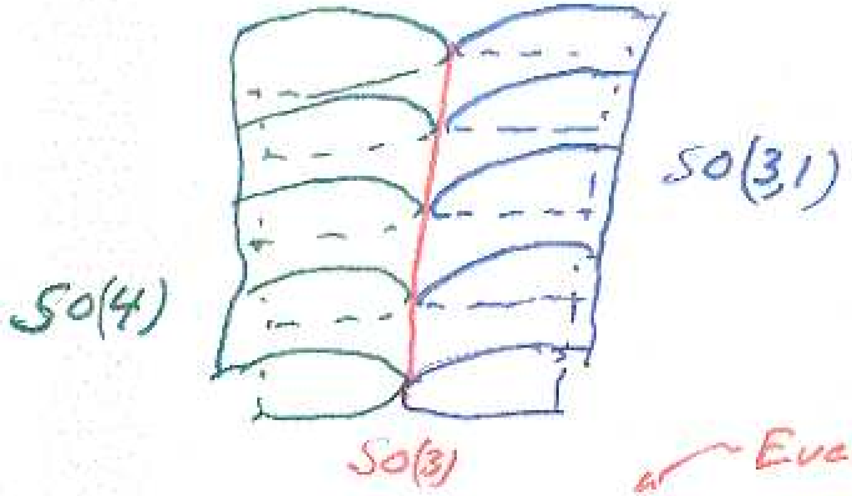
NIL: $\begin{bmatrix} \overbrace{\begin{matrix} 1 & \\ & 1 \end{matrix}}^{O(2)} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} X \\ Y \end{matrix} \\ -Y \;\; X & 1 & Z \\ 0 \;\; 0 & 0 & 1 \end{bmatrix}$

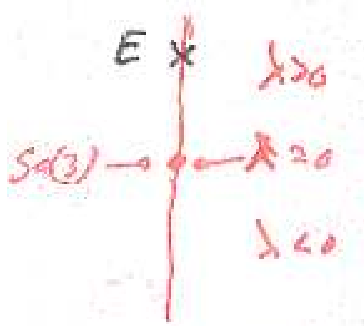


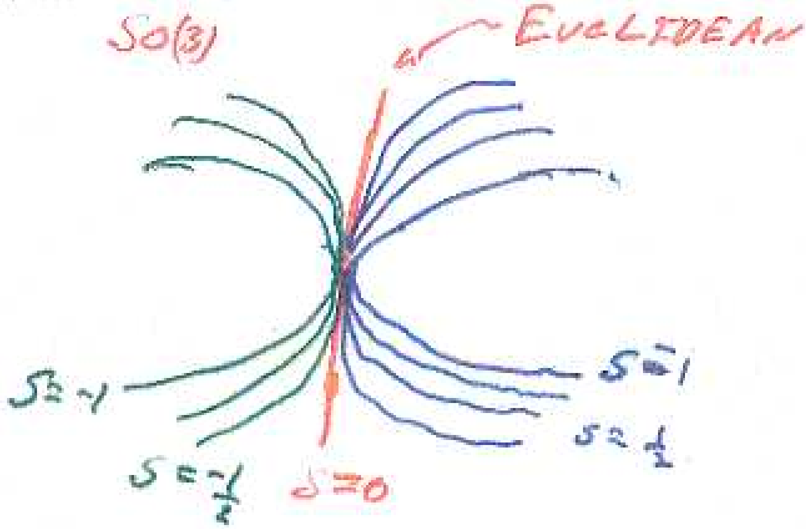
ROTATE AS MOVE OUT RADIALLY

LIFTS OF $\mathrm{Isom}(E^2)$ TO $\mathbb{R}^3$ PRESERVING CONTACT PLANE FIELD

REPRESENTATION VARIETIES AND DEHN SURGERY SPACE

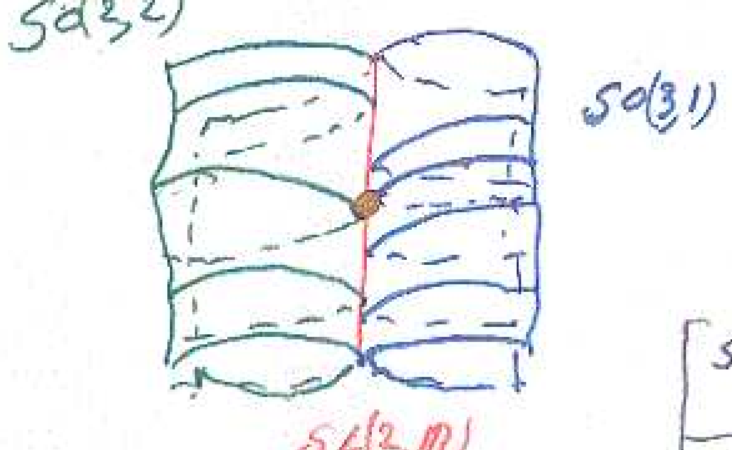






CAN SCALE EUCLIDEAN STRUCTURES

SIMILARITY NEAR (0,1) ON



SOL REP lies "above" Reducible rep

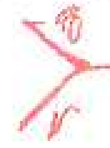
over rest of $SL(2,\mathbb{R})$ get

$$\begin{bmatrix} SO(3,1) & \begin{matrix} x \\ y \\ z \end{matrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} \overset{??}{\longleftrightarrow} \begin{pmatrix} \mathrm{Isom}(\widetilde{SL}_2) \\ \downarrow \\ U(1,1) \end{pmatrix}$$

APPLICATIONS AND QUESTION

→ ORBIFOLD Th^{m}: (Q^3, Σ) CPT., ORIENTABLE, IRREDUCIBLE, ATORIDAL ORBIFOLD, THEN (Q, Σ) IS GEOMETRIC IF $\Sigma \neq \emptyset$, 1-dim.

Thurston (1981)
BOILEAU, PORTI, LEEB
COOPER, HODGSON, —



(2,3,4)
(2,3,k)
k = 3,4,5

→ Spherical case arises through first degenerating to Euclidean

Original proof

RICCI FLOW

Complete

angle 0

Σ link

Now

Σ graph harder

orbifold angles

Porti - Weiss (local rigidity - analysis)
$\alpha \leq \pi$

Cooper - (no analysis)

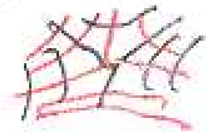
→ Constructing hyperbolic structures

EX: S_g genus g $\quad \psi : S \circlearrowleft$ pseudo-anosov

$M^3 = S \times I / (x,0) \sim (\psi x, 1)$ $\quad \psi$ preserves meas. foliations F_1, F_2

$\psi(F_1) = \lambda F_1 \qquad \psi(F_2) = \lambda^{-1} F_2 \qquad \lambda > 1$

M has singular sol structure



Cone angles $= k\pi \qquad k = \#$ prongs ≥ 3

Problem: Construct hyp. structure on M^3 by decreasing angles to 2π (1st go from Sol to H^3)

→ Understand when a (singular) structure from "other" geometries can be regenerated to $\mathbb{H}^3$ (HODGSON, PART I)

→ Families of examples

→ Which M^3 can be given a (singular) non-hyperbolic geometric structure? (All?)

→ When can it be deformed to give smooth geometric structure (geom. decomp.)
Typically: will change geometries
decrease angles