Quasi-Fuchsian surfaces in hyperbolic knot complements

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Let $M = \mathbb{H}^3/\Gamma$ be a finite-volume hyperbolic 3-manifold.

Let $H \subset \Gamma$ be isomorphic to the fundamental group of a closed surface of genus at least two. By results of Bonahon, Marden, and Thurston, either:

a. H is quasi-Fuchsian: $\Lambda(H) \cong S^1$, or

b. H is geometrically infinite: $\Lambda(H) = S^2_\infty$, or

c. H contains a parabolic element.

Questions: Does Γ contain a closed surface group?

If so...what about the geometry?

If M closed, there are no parabolics. Most known constructions give quasi-Fuchsian examples. Is there always a geometrically infinite surface subgroup? (virtual fiber conjecture.)

If M is non-compact, a closed surface group in Γ can never be geometrically infinite. Most known constructions give accidental parabolics. Is there always a quasi-Fuchsian surface subgroup?

From now on, suppose $M = \mathbb{H}^3/\Gamma$ is a *hyperbolic knot complement*. i.e. M is finite-volume, with a single cusp.

Cooper, Long and Reid showed that Γ contains a closed surface subgroup of genus at least two. However, the surface subgroup is constructed by a tubing operation, and thus always contains parabolic elements.

We show

Theorem 1. *(M-Zhang) If $M = \mathbb{H}^3/\Gamma$ is a hyperbolic knot complement, then Γ contains a closed, quasi-Fuchsian surface group.*

Topological application:

Corollary 2. *Every hyperbolic knot complement contains an essential, immersed surface which survives all but finitely many surgeries.*

This extends a result of Cooper-Long.

Approach:
Construct a "nice" hyperbolic manifold, Y, with: convex boundary; a π_1-injective, quasi-Fuchsian boundary component S; and a local isometry $f : Y \to M$. Then

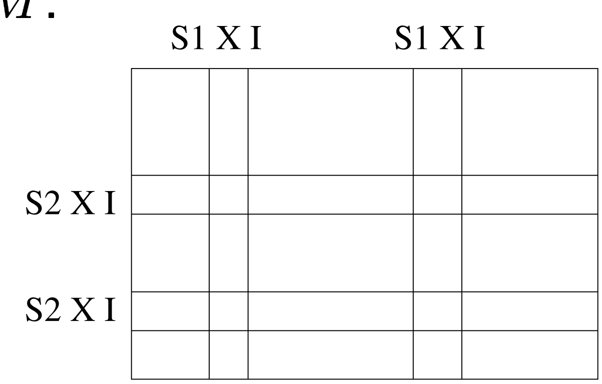
Lemma 3. *$f : S \to M$ is a closed, quasi-Fuchsian surface.*

Proof. Given $\gamma \in \pi_1 S \subset \pi_1 Y$.
Then γ is freely homotopic to a non-trivial, closed geodesic ℓ.
$f(\ell)$ is a closed geodesic in M.
Therefore $f_* \gamma$ is freely homotopic to a non-trivial, closed geodesic.
Therefore $f_* \gamma$ is non-zero and non-parabolic.
Therefore $f_* \pi_1 S$ is a quasi-Fuchsian surface group. $\diamondsuit$

Construction of Y: first try

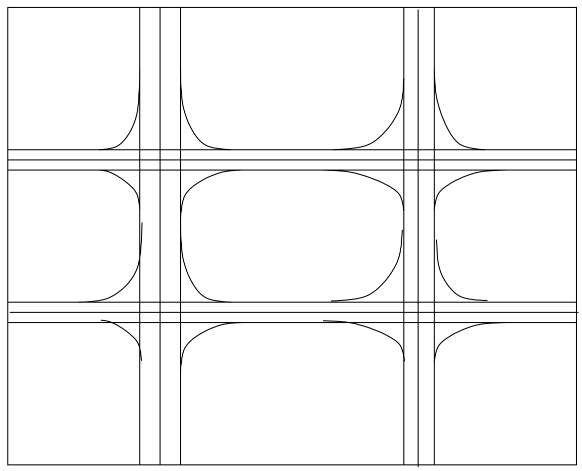
Let $M^- = M - (T \times (0, \infty))$. By Culler-Shalen, M^- contains a pair of incompressible surfaces S_1, S_2, representing distinct slopes in ∂M^-. Suppose S_1 and S_2 intersect transversely.

Let $Y = (S_1 \times I) \cup (S_2 \times I) \cup (T \times [0, \infty))$. Hyperbolic structure on M induces a hyperbolic structure on Y, so inclusion is a local isometry from Y to M.



PROBLEM: The structure on Y does not have convex boundary.

GOAL: Try to smooth out the corners.



The goal may be impossible for this particular choice of Y. Need more room to smooth. This is achieved by replacing the given surfaces S_i with certain carefully chosen finite covers $\widetilde{S}_i$. These covers are thickened, then glued together along certain sub-manifolds, and finally attached to the cusp, to produce Y.

Main issues:

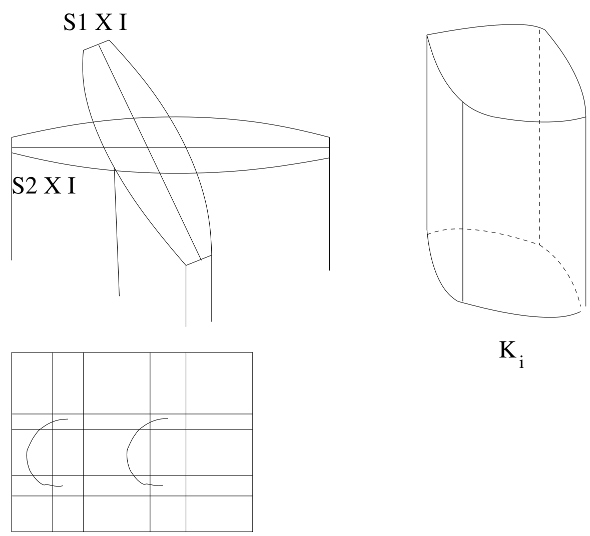
a. How to glue. Require topological picture of "gluing pieces". Need to choose where to insert them.

b. How to smooth. Require results on convex gluing of hyperbolic 3-manifolds.

c. How to find the covers. Requires algebraic results. In particular, must prove that free groups satisfy a stronger version of the LERF property.

Gluing pieces

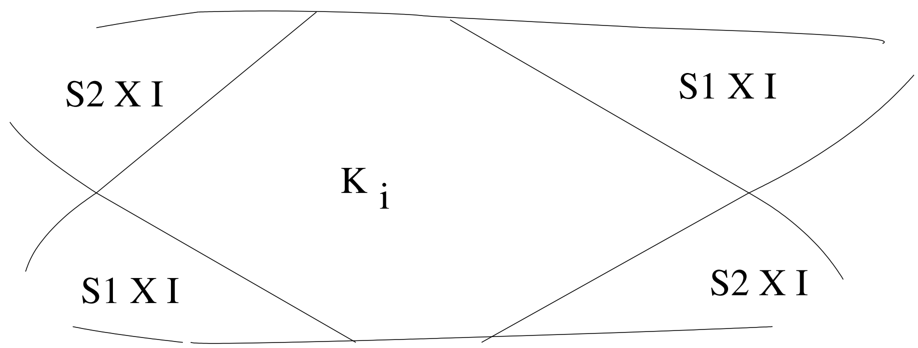
Easy case: Suppose S_1 and S_2 are totally geodesic.

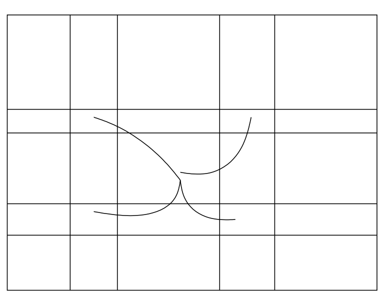


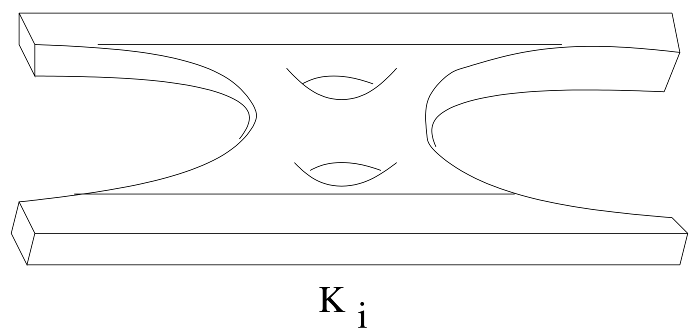
In general, may assume S_1 and S_2 are qF. Let $\Gamma_i = \pi_1 S_i \subset \Gamma$.

Then $\Gamma_1 \cap \Gamma_2$ is a finitely generated, quasi-Fuchsian group, corresponding to some (possibly immersed) sub-manifold K in $S_1 \times I$ and $S_2 \times I$.

K is handlebody with “spikes”.







May perform gluing: $(S_1 \times I) \cup_K (S_2 \times I)$.

Have a good picture of the parabolic region.

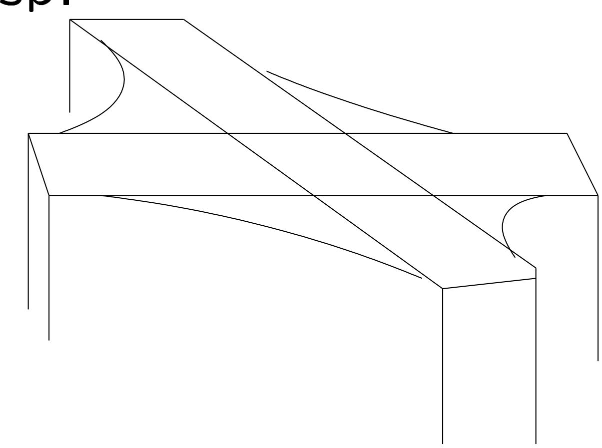
Convex gluing (c.f. Baker-Cooper)

Turns out, can perform **convex** gluing of $S_1 \times I$ and $S_2 \times I$ along K if:

a. K is embedded in $S_i \times I$, and

b. K has a large collar neighborhood in $S_i \times I$.

Moreover, can construct an explicit picture of the convex gluing, and control its intersection with the cusp.



Construction of Y (second try)

Wish to:

a. Replace S_i with certain covers $\widetilde{S}_i$.

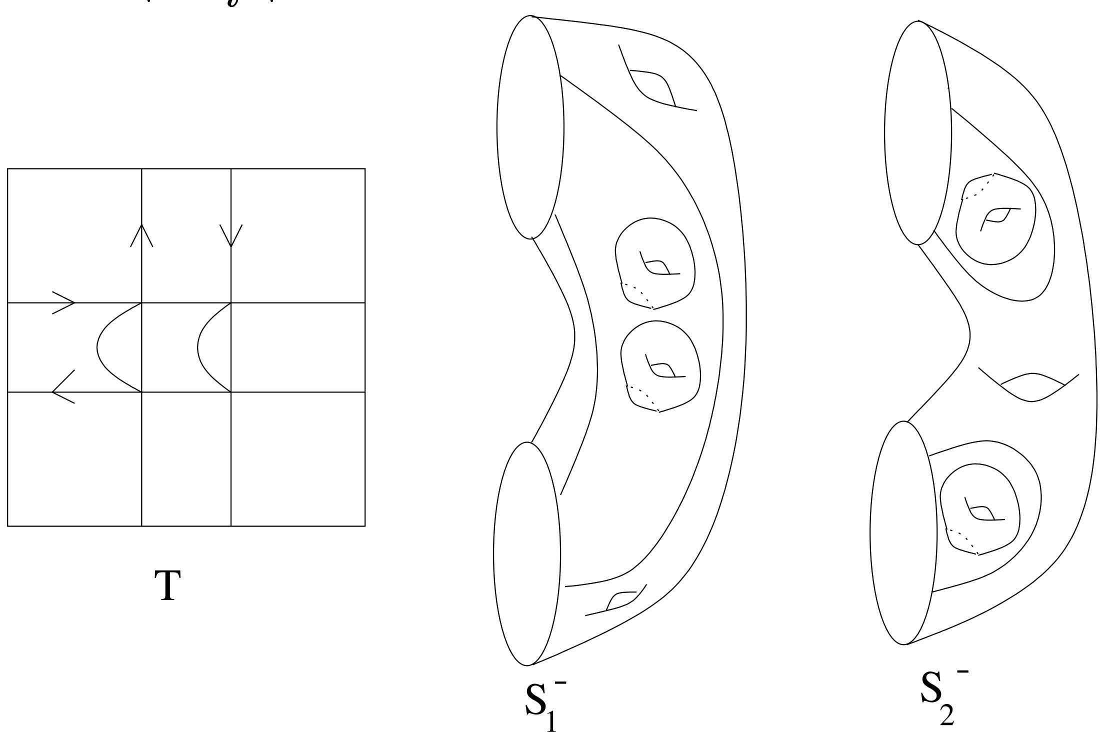
b. Select certain conjugates of $\Gamma_1 \cap \Gamma_2$, corresponding to certain 3-manifolds K_i, which will be the "gluing pieces".

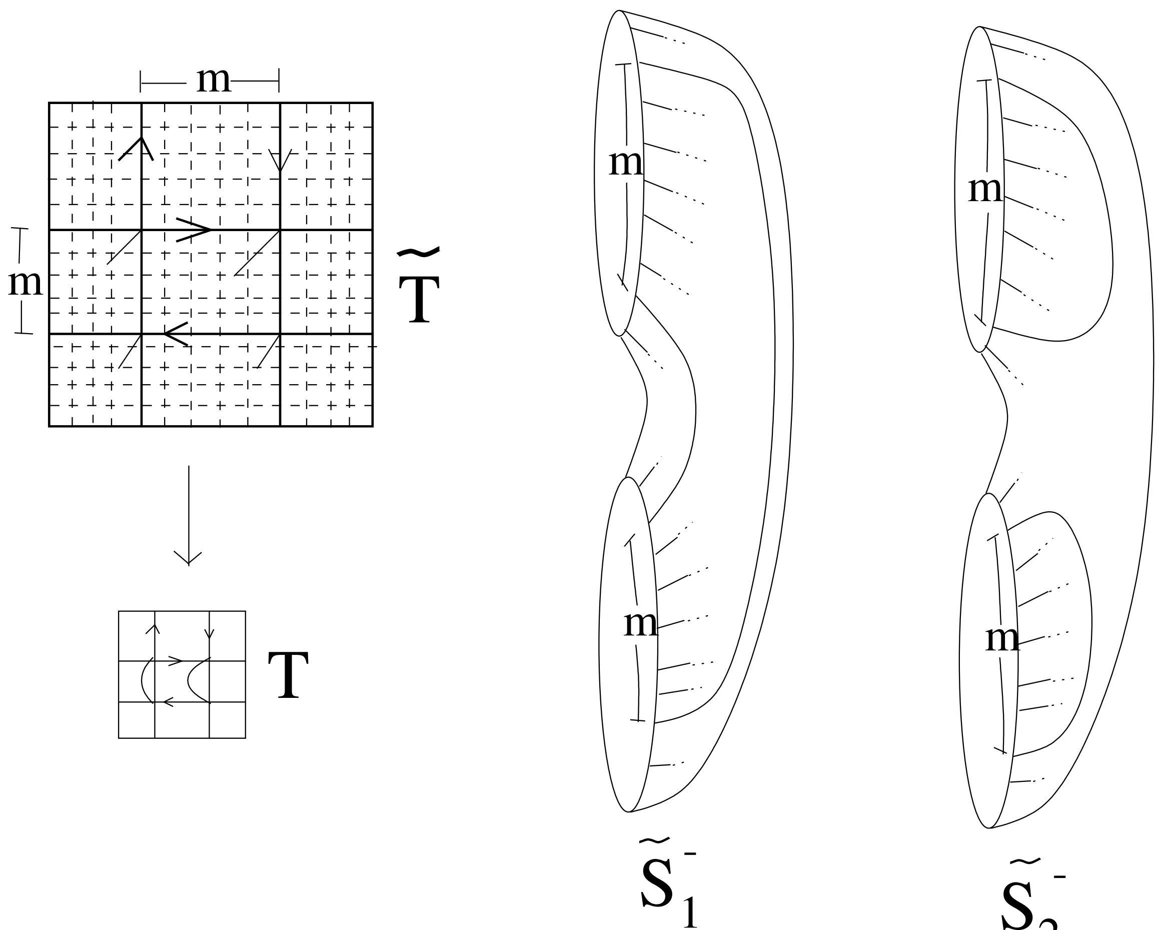
c. Glue $\widetilde{S}_1 \times I$ and $\widetilde{S}_2 \times I$, by inserting gluing pieces in certain places.

d. Cap off (convexly) with the cusp.

Illustration

Suppose $|\partial S_i^-| = 2$ and slopes are distance one.





$$Y = (\tilde{S}_1^- \times I) \cup_{K_1 \cup K_2} \cup (\tilde{S}_2^- \times I) \cup (\tilde{T} \times [0, \infty)).$$

To make this work, need:
a. Gluing pieces lift to embeddings in $\widetilde{S}_i$, with large collar neighborhoods. (To allow convex gluing.)
b. The resulting parabolic region consists of **evenly spaced** parallelograms, with **long sides**. (To allow capping off with cusp.)

In light of b., we further require:
c. The number of boundary components of $\widetilde{S}_i$ is the same as the number of boundary components of S_i.

Algebra
Translate desired properties into group theory.

Spacing issue becomes very technical. We shall focus on issues a and c. Part a is a standard application of LERF property of free groups. However, parts a and c together require:

Strong LERF Let F be the fundamental group of a connected compact orientable surface S^- with genus g and with $b > 0$ boundary components. We may choose a free basis of F:

$$a_1, b_1, a_2, b_2, ..., a_g, b_g, x_1, ..., x_{b-1}$$

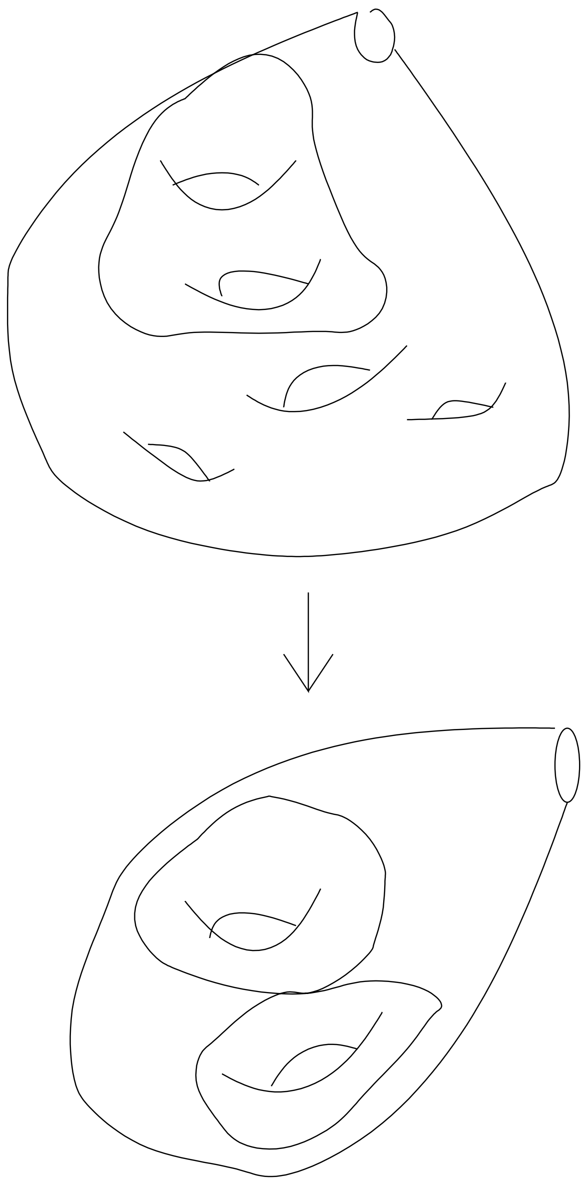
Boundary components of F are represented by

$$\begin{aligned} X &= \{x_1, ..., x_{b-1}, x_b\}, \text{ where} \\ x_b &= [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] x_1 x_2 \cdots x_{b-1} \end{aligned}$$

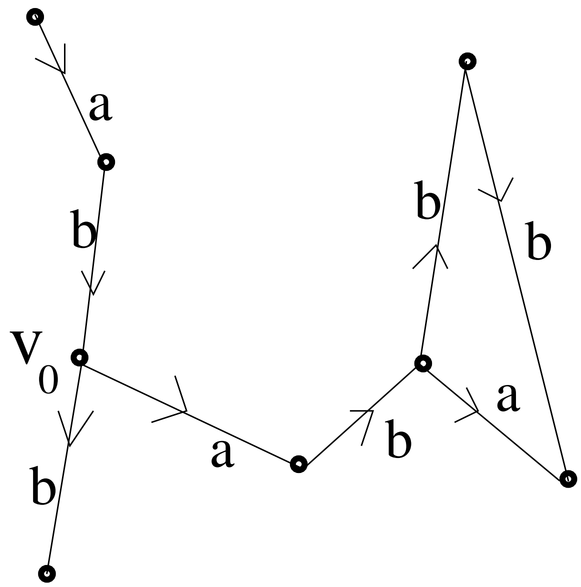
Theorem 4. *Let $H \subset F$ be a finitely generated subgroup containing no peripheral elements, and let $Y = \{y_1, ..., y_a\} \subset F - H$. Then, for some $n \in \mathbb{Z}^+$, there exists an index n subgroup J of F such that*

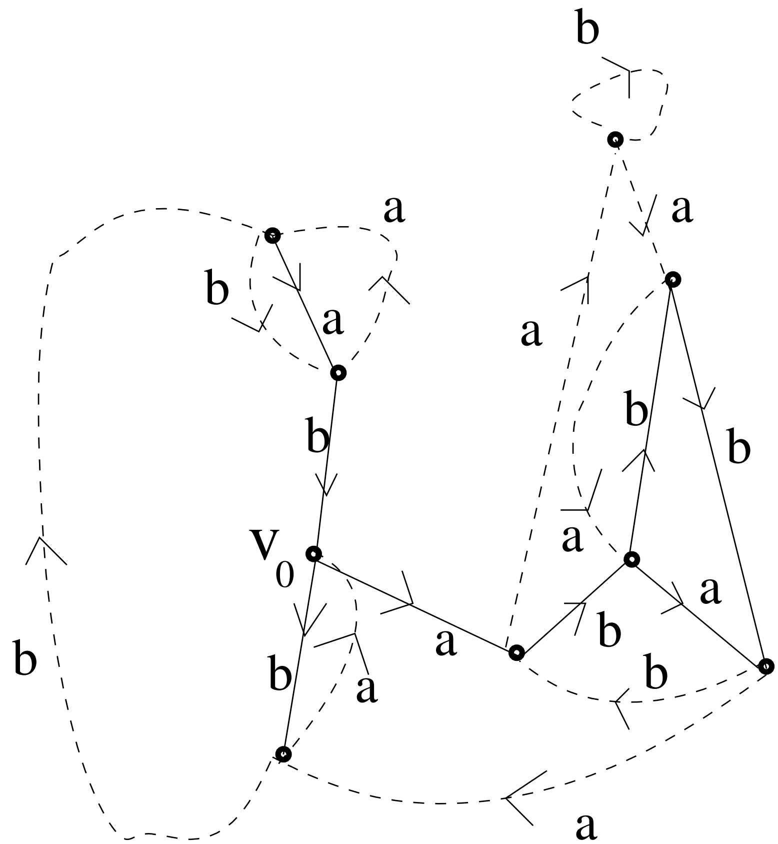
$$\begin{aligned} J &\supset H, \\ Y \cap J &= \emptyset \text{ , and} \\ x^i &\notin J \quad \forall x \in X, \ 1 \leq i < n. \end{aligned}$$

Corollary 5. *Let S be a hyperbolic surface with $b > 0$ boundary components, and let $f : S^1 \to S$ be an immersion of a geodesic. Then there is a finite cover $\tilde{S}$ of S,* **with b boundary components**, *such that f lifts to an embedding $\tilde{f} : S^1 \to \tilde{S}$.*



Proof of Theorem 4 is a combinatorial analysis of labeled graphs.

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