University of Waterloo

Fields Institute Seminar

New NLP problems for Power System Analysis and Operation in Competitive Electricity Markets

Kankar Bhattacharya
and
Claudio Cañizares
Department of Electrical & Computer Engineering
University of Waterloo, Waterloo, Ontario
kankar@ece.uwaterloo.ca
ccanizar@uwaterloo.ca
www.power.uwaterloo.ca

Outline

- Introduction and motivation
- Previous work in NLP problems for electricity markets:
  - Security-constrained optimal power flow (SC-OPF)
  - SC-OPF power market models
- New NLP problems and applications:
  - Multi-objective voltage-stability-constrained OPF (VSC-OPF)
  - Singular value VSC-OPF
  - OPF-based reactive-power market models:
    - Mixed integer-NLP problems
    - Elimination of integer variables
- Conclusions

Introduction and Motivation

- Optimization models have traditionally played a very important role in power system operations and planning studies
- With introduction of competitive markets some of these models have undergone a shift in paradigm
- Many new issues have emerged and so have new models to address these issues requiring faster computation methods and ability to handle large systems

Introduction and Motivation

- "Typical" optimization problems in power system operation:
  - Unit Commitment (UC) problem:
    - Scheduling generators over 1-week to 1-day horizon to minimize costs subject to a set of linear constraints
    - Mixed Integer-LP problem
  - Optimal Power Flow (OPF) problem:
    - Short-term generation schedules to minimize cost / loss subject to nonlinear constraints and limits
    - Nonlinear programming (NLP) problem

Introduction and Motivations

- Electricity market problems:
  - Obtain generation and load schedules to maximize "social welfare"
  - Viewed as an OPF problem
  - In some markets, nodal-prices are also obtained

- These problems have been successfully solved using a variety of well-known optimization techniques for large systems (e.g. Interior Point methods applied to NLP OPF problems).

Introduction and Motivations

- Deregulation has increased the need to seek optimal operating conditions while meeting system security constraints
- New optimization problems motivated by electricity market issues:
  - Stability-constrained OPF:
    - more secure and "cheaper" operation schedules
    - Some "security" constraints are replaced, in some cases, by implicit constraints.
  - Reactive power problems:
    - Obtain reactive power schedules to maximize "social welfare" while maintaining system security
    - Mixed Integer-NLP problem

Security Constrained OPF (SCOPF)

- "Standard" SCOPF problem:

$$
\begin{aligned}
\text{Min.} \quad & \sum_{i=1}^{N_G} a_i P_{G_i}^2 + b_i P_{G_i} + c_i \\
\text{s.t.} \quad & F_{pf}(\delta, V, Q_G, P_G) = 0 \\
& P_{G_{\min}} \leq P_G \leq P_{G_{\max}} \\
& P_T(\delta, V) \leq P_{T_{\max}} \\
& I_T(\delta, V) \leq I_{T_{\max}} \\
& Q_{G_{\min}} \leq Q_G \leq Q_{G_{\max}} \\
& V_{\min} \leq V \leq V_{\max}
\end{aligned}
$$

SCOPF... contd.

- The nonlinear power flow equations Fpf(δ,V,QG,PG) have the general form (2 equations per bus i = 1,...,N):

$$P_i - \sum_{k=1}^{N} V_i V_k [G_{ik} \cos(\delta_i - \delta_k) + B_{ik} \sin(\delta_i - \delta_k)] = 0$$

$$Q_i - \sum_{k=1}^{N} V_i V_k [G_{ik} \sin(\delta_i - \delta_k) - B_{ik} \cos(\delta_i - \delta_k)] = 0$$

$$P_i = \begin{cases} P_{G_i} & \forall i \in \text{generators} \\ -P_{L_i} & \forall i \in \text{loads} \end{cases}$$

$$Q_i = \begin{cases} Q_{G_i} & \forall i \in \text{generators} \\ -Q_{L_i} & \forall i \in \text{loads} \end{cases}$$

$$G_{ik}, B_{ik}, P_{L_i}, Q_{L_i} \rightarrow \text{constants}$$

SCOPF... contd.

- Objective is to minimize generation costs
- Grid "security" is represented by:
  - Line power flows P_T , typically computed off-line using an N-1 contingency criterion
  - Current thermal limits I_T
  - Bus voltage limits V
- These problems have been solved successfully for large systems (thousands of constraints) using Interior Point methods

SCOPF in Markets

- In electricity markets, the "typical" NLP SCOPF for a double auction market is:

$$
\begin{aligned}
\text{Max.} \quad & S_b = C_d^T P_d - C_s^T P_s \\
\text{s.t.} \quad & F_{pf}(\delta, V, Q_G, P_s, P_d) = 0 \\
& 0 \le P_s \le P_{s\text{max}} \\
& 0 \le P_d \le P_{d\text{max}} \\
& P_T(\delta, V) \le P_{T\text{max}} \\
& I_T(\delta, V) \le I_{T\text{max}} \\
& Q_{G\text{min}} \le Q_G \le Q_{G\text{max}} \\
& V_{\text{min}} \le V \le V_{\text{max}}
\end{aligned}
$$

SCOPF in Markets... contd.

where

$$P_i = \begin{cases} P_{Go_i} + P_{s_i} & \forall i \in \text{generators} \\ -P_{Lo_i} - P_{d_i} & \forall i \in \text{loads} \end{cases}$$

$$Q_i = \begin{cases} Q_{G_i} & \forall i \in \text{generators} \\ -Q_{L_i} & \forall i \in \text{loads} \end{cases}$$

$$G_{ik}, B_{ik}, P_{Go_i}, P_{Lo_i}, Q_{L_i} \rightarrow \text{constants}$$

SCOPF in Markets... contd.

- This model is basically an NLP SCOPF that maximizes social welfare
- Locational marginal prices (LMP) are byproducts of this optimization problem (the Lagrange multipliers of F_{pf}).
- In practice, market models based on LP models are more commonly used by system operators.

Linear SCOPF in Markets

- For example, the following multi-period LP model may be used to clear these markets (e.g. Ontario):

$$\text{Max.} \quad S_b = \sum_{t=1}^{N} C_{d_t}^T P_{d_t} - C_{s_t}^T P_{s_t}$$

$$\text{s.t.} \quad P_{d_{i_t}} - P_{s_{i_t}} - \sum_{j=1, j\neq i}^{n} P_{loss_{ij_t}} - \sum_{j=1, j\neq i}^{n} P_{ij_t} = 0 \quad \forall i, t$$

$$P_{ij_t} - b_{ij}(\delta_{i_t} - \delta_{j_t}) = 0 \quad \forall i, j, i \neq j, t$$

$$P_{T_t} = [P_{ij_t}] \leq P_{T_t \max} \quad \forall t$$

$$0 \leq P_{s_t} \leq P_{s_t \max} \quad \forall t$$

$$0 \leq P_{d_t} \leq P_{d_t \max} \quad \forall t$$

$$P_{s_t} - P_{s_{t+1}} \leq Rd_{s_t} \quad \forall t$$

$$P_{s_{t+1}} - P_{s_t} \leq Ru_{s_t} \quad \forall t$$

Linear SCOPF in Markets... contd.

- *Rd* and *Ru* represent the ramp-down and ramp-up constraints, respectively
- Other constraints are also included in these models (e.g. operating reserve in Ontario)
- The line power flow limits vary with the solution of the auction; thus, fixed limits are not representative of system conditions, negatively affecting prices and system security.
- This has led to the development of stability constrained OPF models.

Multi-objective Voltage Stability Constrained OPF (VSC-OPF)

- The objective is to maximize both social welfare and system "loadability", i.e. voltage stability margins (VSM):

$$
\begin{aligned}
\text{Max.} \quad & (1-w)\underbrace{(C_d^T P_d - C_s^T P_s)}_{S_b} + w\lambda_c \\
\text{s.t.} \quad & F_{PF}(\delta, V, Q_G, P_s, P_d) = 0 \\
& F_{PF_c}(\delta_c, V_c, Q_{Gc}, P_s, P_d, \lambda_c) = 0 \\
& 0 \le P_s \le P_{s\max} \\
& 0 \le P_d \le P_{d\max} \\
& \lambda_{c\min} \le \lambda_c \le \lambda_{c\max} \\
& I_T(\delta, V) \le I_{T\max} \\
& Q_{G\min} \le Q_G \le Q_{G\max} \\
& V_{\min} \le V \le V_{\max} \\
& I_{T_c}(\delta_c, V_c) \le I_{T_c\max} \\
& Q_{Gc\min} \le Q_{Gc} \le Q_{Gc\max} \\
& V_{c\min} \le V_c \le V_{c\max}
\end{aligned}
$$

Multi-objective VSC-OPF

- l_c represents the VSM and all c constraints correspond to the system at its "critical" point (max. VSM)
- By varying the weight w ($0 < w < 1$), more or less stress can be put on security
- In practice, w should be very small to avoid "undesirable" effects on the market power levels and prices

Fields Institute Seminar:
New NLP problems for Power System Analysis and
Operation in Competitive Electricity Markets

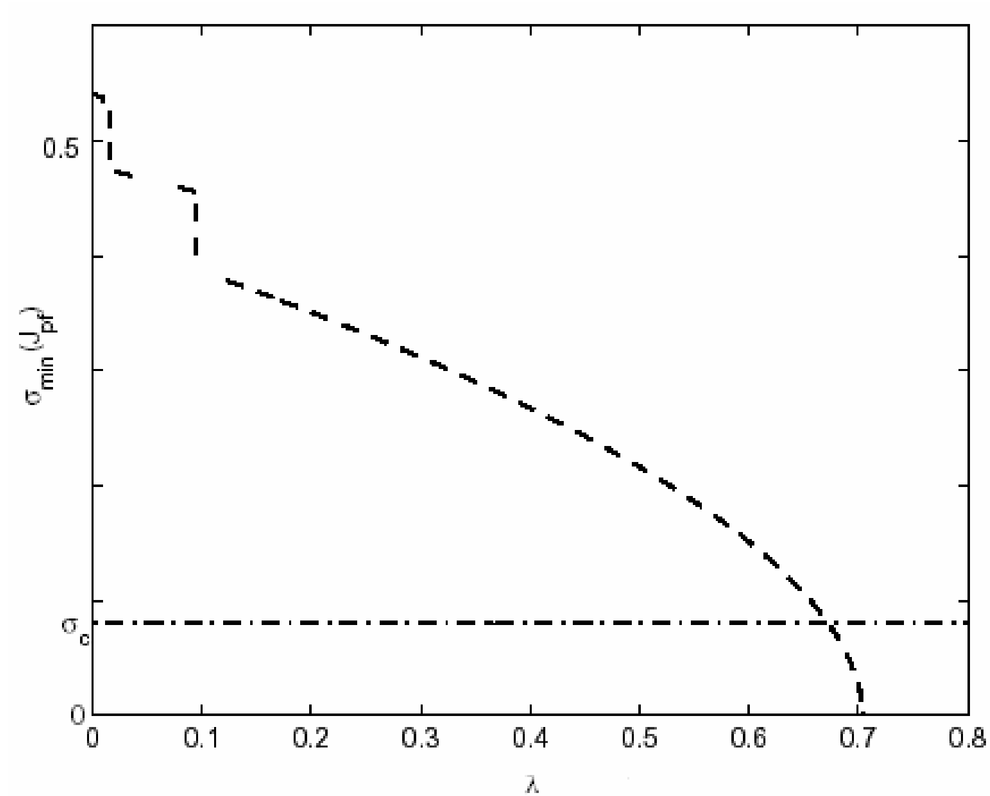
Singular Value VSC-OPF

- The objective again is to maximize social benefit while guaranteeing a min. VSM:

$$
\begin{aligned}
\text{Max.} \quad & S_b = C_d^T P_d - C_s^T P_s \\
\text{s.t.} \quad & F_{pf}(\underbrace{\delta, V, Q_G, P_s, P_d}_{z,p}) = 0 \\
& \sigma_{min}(\underbrace{D_z F_{pf}|_o}_{J_{pf}}) \geq \sigma_c \\
& 0 \leq P_s \leq P_{s\max} \\
& 0 \leq P_d \leq P_{d\max} \\
& I_T(\delta, V) \leq I_{T\max} \\
& Q_{G\min} \leq Q_G \leq Q_{G\max} \\
& V_{\min} \leq V \leq V_{\max}
\end{aligned}
$$

Singular Value VSC-OPF

- $s_{min}(J_{pf})$ is a VS index that becomes zero at a singularity point of the power flow Jacobian:



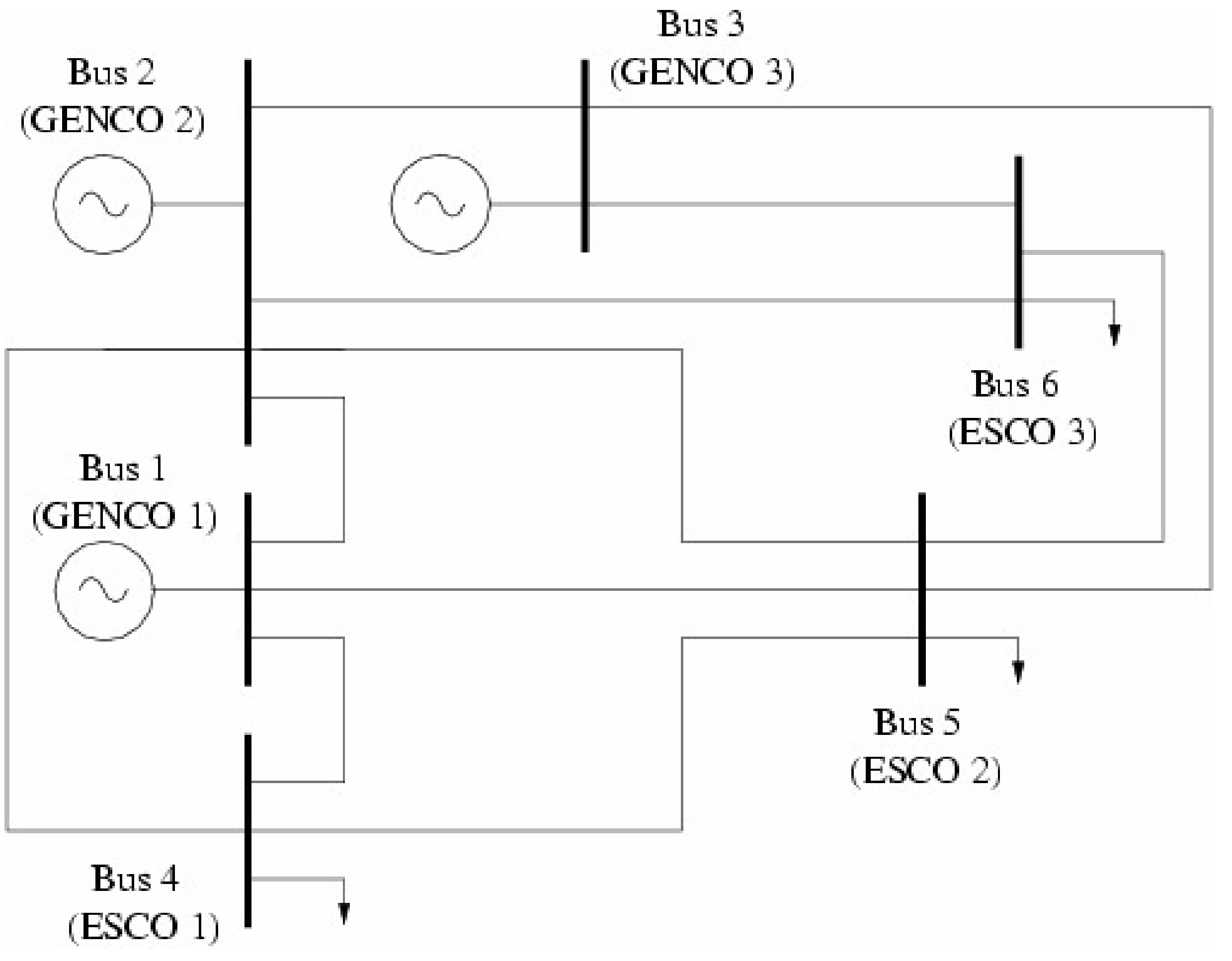
Singular Value VSC-OPF

- All these models only represent the system static behavior; however, stability and security is an inherently dynamic problem
- Some system dynamics can be represented in the VSC-OPF problem by replacing the $s_{min}(J_{pf})$ constraint with the singular value of a dynamic Jacobian

Fields Institute Seminar:
New NLP problems for Power System Analysis and Operation in Competitive Electricity Markets

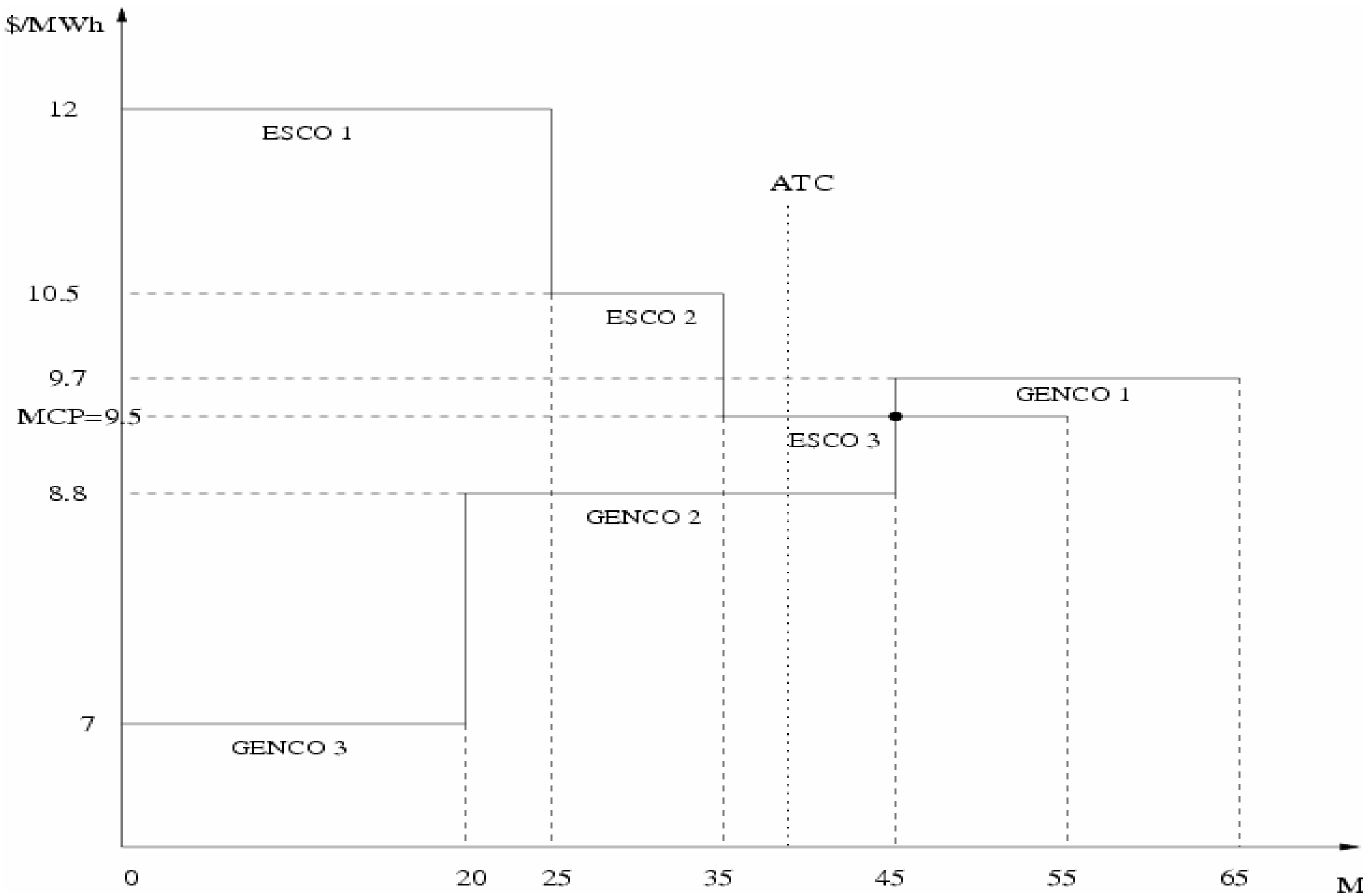
VSC-OPF

- Example:



VSC-OPF

– Unconstrained optimization:



Fields Institute Seminar:
New NLP problems for Power System Analysis and Operation in Competitive Electricity Markets

VSC-OPF

– SC-OPF with off-line line power flow limits:

Participant	V [p.u.]	ρ [$/MWh]	P_{BID} [MW]	P_0 [MW]	Pay [$/h]
GENCO 1	1.100	9.70	14.4	90	-1013
GENCO 2	1.100	8.80	2.4	140	-1253
GENCO 3	1.084	8.28	20.0	60	-663
ESCO 1	1.028	11.64	15.6	90	1229
ESCO 2	1.013	10.83	0.0	100	1083
ESCO 3	1.023	9.13	20.0	90	1005
TOTALS	$T = 315.6$ MW Losses $= 11.2$ MW		$\text{Pay}_{\text{IMO}} = 388$ $/h VSM $= 520$ MW		

VSC-OPF

- VSC-OPF (both techniques yield practically the same results for a "small" *w*):

Participant	V [p.u.]	ρ [$/MWh]	P_{BID} [MW]	P_0 [MW]	Pay [$/h]
GENCO 1	1.100	8.94	0.0	90	-805
GENCO 2	1.100	8.91	25.0	140	-1470
GENCO 3	1.100	9.07	20.0	60	-726
ESCO 1	1.021	9.49	25.0	90	1091
ESCO 2	1.013	9.57	10.0	100	1053
ESCO 3	1.039	9.35	8.0	90	916
TOTALS	$T = 323$ MW Losses $= 12$ MW		$\text{Pay}_{\text{IMO}} = 59$ $/h VSM $= 539$ MW		

VSC-OPF

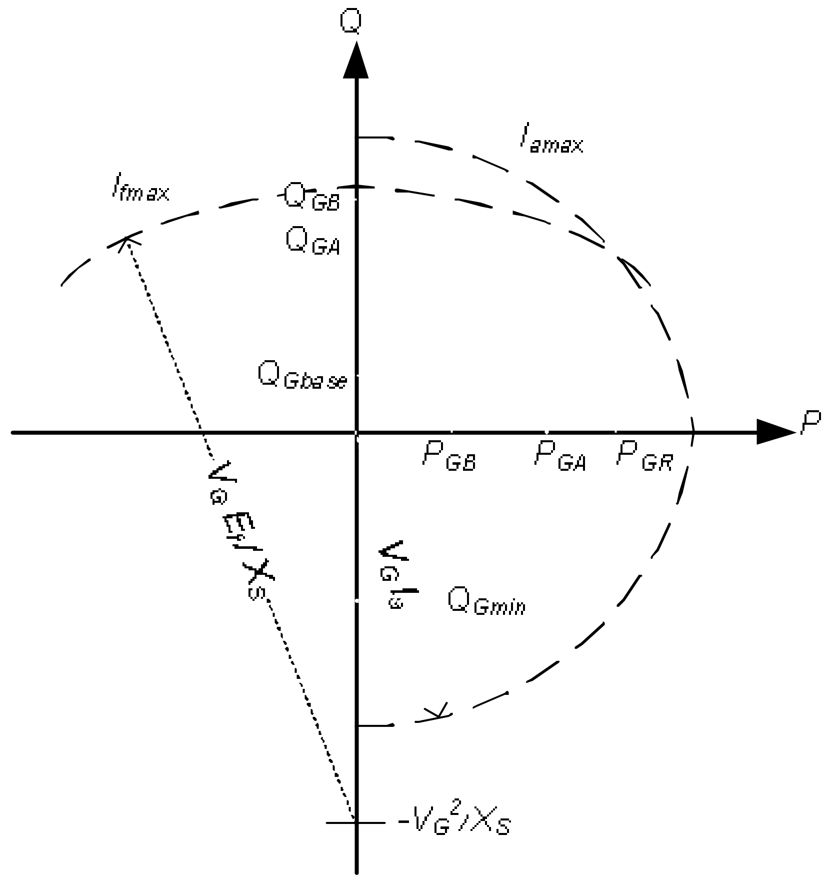
- Better operating conditions:
  - Higher voltages
  - Lower losses
  - Higher VSM
- Better market conditions:
  - Lower nodal-prices
  - Higher transaction levels

Reactive-Power OPF Auction

- Reactive power is very important to keep power systems secure
- It has an effect on pricing, so there is a need to “optimally” dispatch it (max. security at min. price)
- Need to provide financial incentives to produce and consume reactive power
- There is lack of transparency in the procurement of reactive power services

Reactive-Power OPF Auction

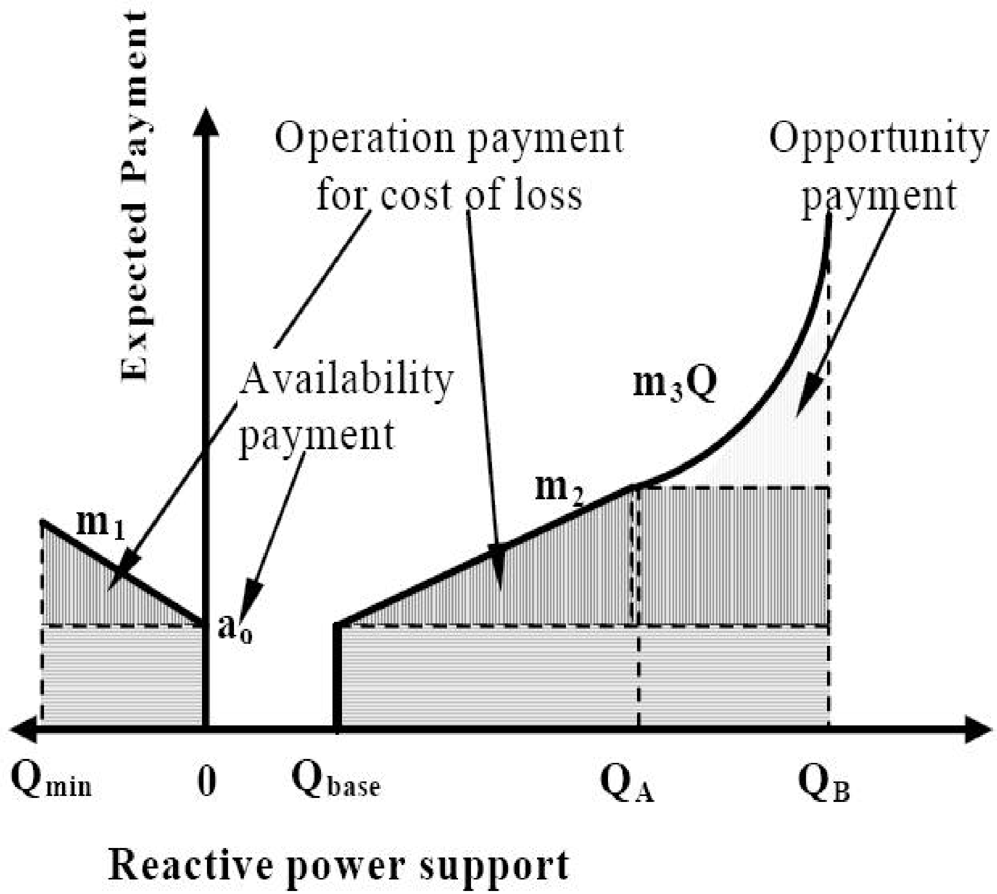
- Generator capability curve:



Fields Institute Seminar:
New NLP problems for Power System Analysis and Operation in Competitive Electricity Markets

Reactive-Power OPF Auction

- Reactive power prices:



Reactive-Power OPF Auction

$$\begin{aligned}
\text{Max.} \quad & S_w = \sum_i (C_{Li}\lambda_{Gi} - b_{Gi_o})Q_{Gi_2} + \\
& \sum_i \left((C_{Li}\gamma_{Gi} - b_{Gi_o})Q_{Gi_3} - \frac{1}{2} m_{Gi_o} (Q_{Gi_3} - Q_{Gi_A})^2 \right) \\
\text{s.t.} \quad & F_{pf}(\delta, V, Q_G, P_{Go}, P_{Lo}) = 0 \\
& w_{Gi_1} Q_{Gi_{\min}} \le Q_{Gi_1} \le w_{Gi_1} Q_{Gi_{blag}} \\
& w_{Gi_2} Q_{Gi_{blead}} \le Q_{Gi_2} \le w_{Gi_2} Q_{Gi_A} \\
& w_{Gi_3} Q_{Gi_A} \le Q_{Gi_3} \le w_{Gi_3} Q_{Gi_B} \\
& w_{Gi_1} + w_{Gi_2} + w_{Gi_3} = 1 \\
& Q_{Gi} = Q_{Gi_1} + Q_{Gi_2} + Q_{Gi_3} \\
& V_{\min} \le V \le V_{\max}
\end{aligned}$$

where

$$Q_{Gi_A} = \begin{cases} \sqrt{\left(\frac{V_{Gio} E_{f_{Gio}}}{X_{s_{Gi}}}\right)^2 - P_{Gio}^2} - \frac{V_{Gio}^2}{X_{s_{Gi}}} & \forall P_{Gio} < P_{Gi_R} \\ \sqrt{(V_{Gio} I_{a_{Gio}})^2 - P_{Gio}^2} & \forall P_{Gio} > P_{Gi_R} \end{cases}$$

$$Q_{Gi_B} = \sqrt{\left(\frac{V_{Gio} E_{f_{Gio}}}{X_{s_{Gi}}}\right)^2 - P_{Gi_{\min}}^2} - \frac{V_{Gio}^2}{X_{s_{Gi}}}$$

Reactive-Power OPF Auction

- b_{Gio}: Cost of loss price offer for operating in the region $Q_{base}<Q<Q_A$ (\$/MVARh)
- m_{Gio}: opportunity price offer for operating in the region $Q_A<Q<Q_B$ (\$/MVARh). In Ontario, IESO recognizes the MCP for energy scheduled, but not delivered.
- ?: dual (Lagrange multiplier) of the nodal reactive power balance constraint and denotes the sensitivity of the system loss parameter to change in reactive power injection at a bus (MW/MVAR)

Reactive-Power OPF Auction

- ?: dual (Lagrange multiplier) of the generator reactive power constraint; indicates by how much the system loss will change for a unit change in reactive power capability (MW/MVAR)
- This is a mixed integer-NLP (MINLP) problem
- It's computationally expensive

Eliminating Integer Constraints

- Objective is to develop a heuristic scheme to eliminate integer constraints and reduce the high computational burden of MINLP

- This is achieved through the following three steps:
  - Step 1:
    - Obtain $?_i$, $?_i$ and Q_{Gi} by solving an OPF to minimize losses, i.e. solve:

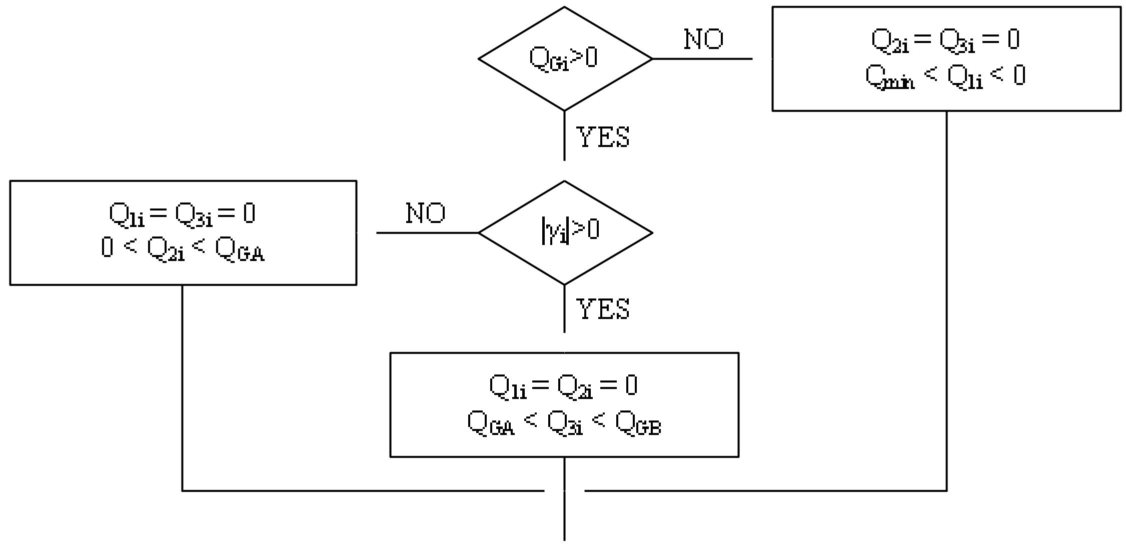
Eliminating Integer Constraints

$$\begin{aligned} \text{Min.} \quad & \sum_i P_{G_i} \\ \text{s.t.} \quad & F_{pf}(\delta, V, Q_G, P_G, P_{Lo}, LF) = 0 \\ & Q_{G_{\text{min}}} \le Q_G \le Q_{G\text{max}} \\ & V_{\text{min}} \le V \le V_{\text{max}} \end{aligned}$$

- Identify the region in which Q_{Gi} lies according to the value of $?_i$ and Q_{Gi}:

Fields Institute Seminar:
New NLP problems for Power System Analysis and Operation in Competitive Electricity Markets

Eliminating Integer Constraints



- Step 2:
  - Solve an OPF to maximize the Social Welfare (S_w) and obtain Q_{Gi}, i.e. solve:

Eliminating Integer Constraints

$$\text{Max.} \quad S_w = \sum_i (C_{Li}\lambda_{Gi} - b_{Gi_o})Q_{Gi_2} + \sum_i \left((C_{Li}\gamma_{Gi} - b_{Gi_o})Q_{Gi_3} - \frac{1}{2} m_{Gi_o}(Q_{Gi_3} - Q_{Gi_A})^2 \right)$$

$$\text{s.t.} \quad F_{pf}(\delta, V, Q_G, P_{Go}, P_{Lo}) = 0$$

$$Q_{Gi} = Q_{Gi_1} \text{ or } Q_{Gi_2} \text{ or } Q_{Gi_3}$$

$$\begin{cases} Q_{Gi_{\min}} \le Q_{Gi} \le Q_{Gi_{blag}} & \text{or} \\ Q_{Gi_{blead}} \le Q_{Gi} \le Q_{Gi_A} & \text{or} \\ Q_{Gi_A} \le Q_{Gi} \le Q_{Gi_B} \end{cases}$$

$$I_T(\delta, V) \le I_{T_{\max}}$$

$$V_{\min} \le V \le V_{\max}$$

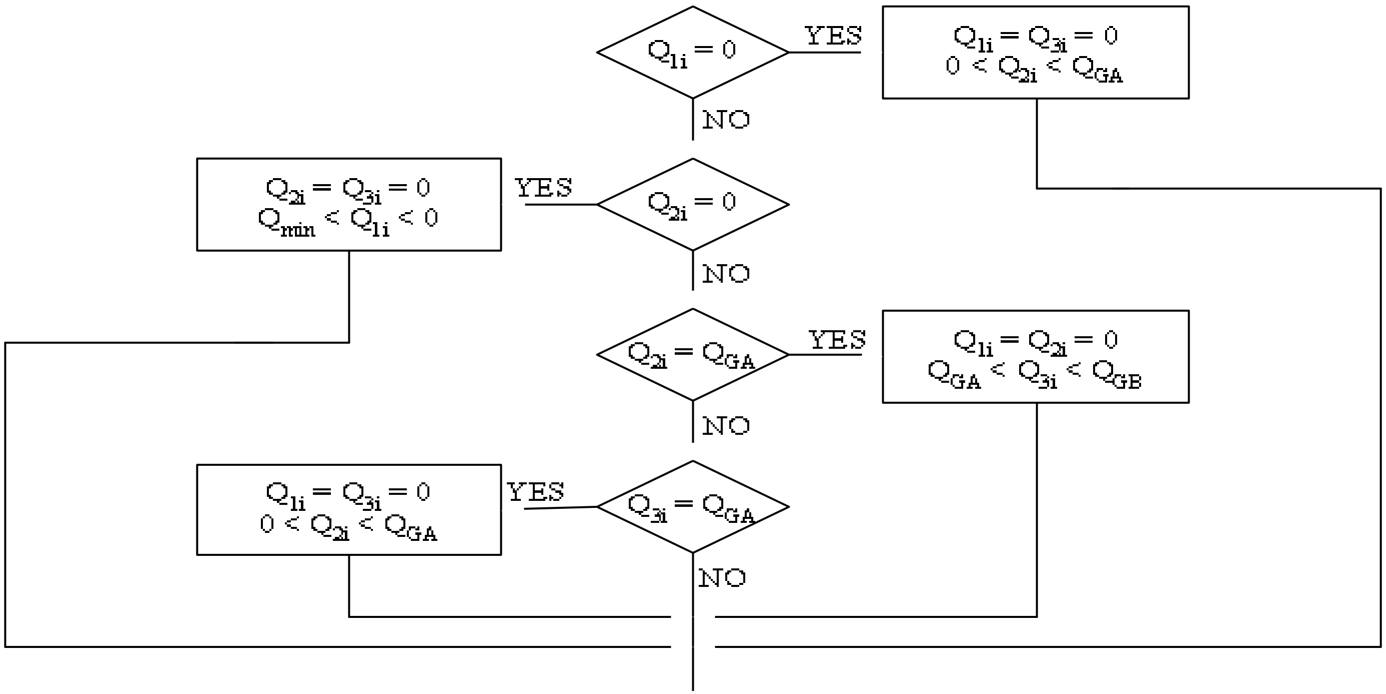
where

$$Q_{Gi_A} = \begin{cases} \sqrt{\left(\frac{V_{Gio}E_{f_{Gio}}}{X_{s_{Gi}}}\right)^2 - P_{Gio}^2} - \frac{V_{Gio}^2}{X_{s_{Gi}}} & \forall P_{Gio} < P_{Gi_R} \\ \sqrt{(V_{Gio}I_{a_{Gio}})^2 - P_{Gio}^2} & \forall P_{Gio} > P_{Gi_R} \end{cases}$$

$$Q_{Gi_B} = \sqrt{\left(\frac{V_{Gio}E_{f_{Gio}}}{X_{s_{Gi}}}\right)^2 - P_{Gi_{\min}}^2} - \frac{V_{Gio}^2}{X_{s_{Gi}}}$$

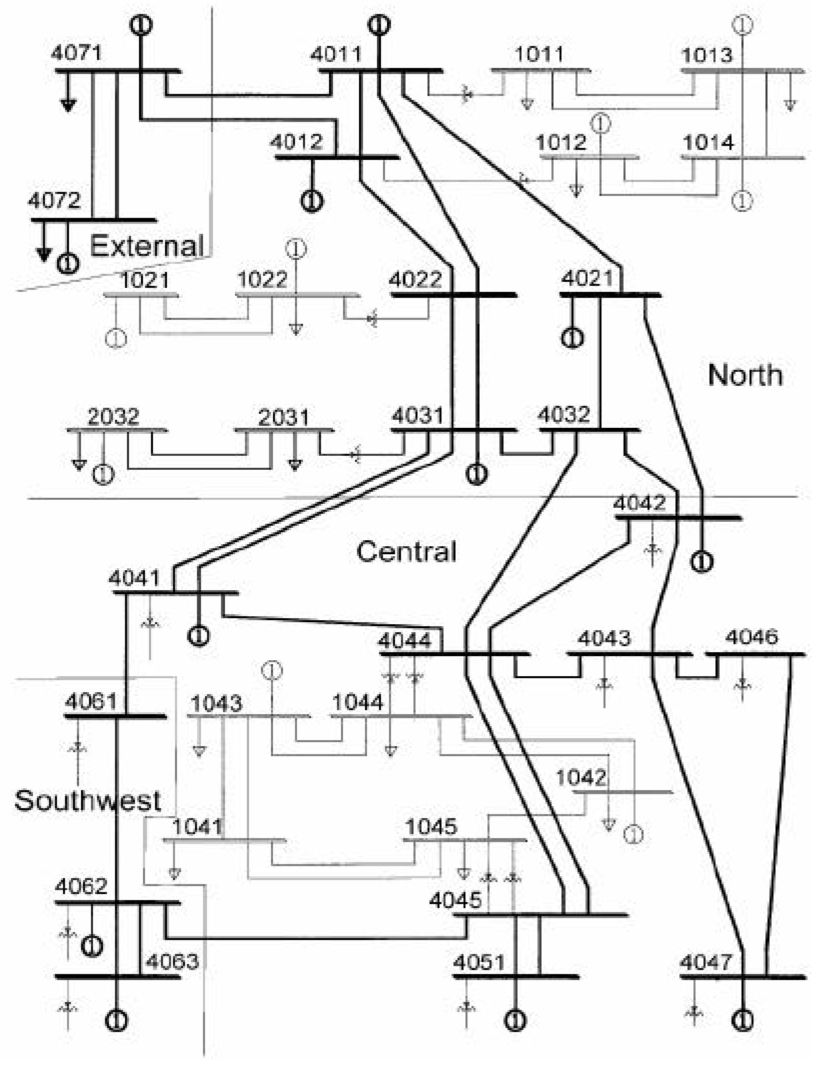
Eliminating Integer Constraints

- Step 3:
  - Update solution if Q_{Gi} in any region hits limits



Eliminating Integer Constraints

- CIGRE 32-bus system:
  - Known for voltage stability problems
  - There are 20 generators



Fields Institute Seminar:
New NLP problems for Power System Analysis and Operation in Competitive Electricity Markets

Eliminating Integer Constraints

– Step 1:

Bus	λ	γ	Q_G	Q_1	Q_2	Q_3
4072	0	0	4.05320		4.05320	
4071	0	0	0.86323		0.86323	
4011	0.00595	0	-1.0000	-1.0000		
4012	0.00185	0	-1.6000	-1.6000		
4021	0.21796	0	-0.3000	-0.3000		
4031	0.04059	0	-0.4000	-0.4000		
4042	0.06379	0	0	0		
4041	0.02638	0	-2.0000	-2.0000		
4062	0.00485	0	0	0		
4063	0	0	1.75463		1.75463	
4051	0	0	1.15767		1.15767	
4047	0.04254	0	0	0		
2032	0	0	0.73650		0.73650	
1013	0	0	1.04556		1.04556	
1012	0	0	-0.3644	-0.3644		
1014	0	0	0.41924		0.41924	
1022	0.00014	0.00014	0.90533			0.90533
1021	0.02960	0.02960	2.03846			2.03846
1043	0.06571	0.06571	0.97956			0.97956
1042	0	0	-0.3265	-0.3265		

Eliminating Integer Constraints

– Step 2:

Bus	Q_1	Q_2	Q_3	Q_G	Q_{1min}	Q_{1max}	Q_{2min}	Q_{2max}	Q_{3min}	Q_{3max}
4072		2.61574		2.61574	-3.00	0	0	42.981	42.981	68.769
4071	0			0	-0.50	0	0	1.7059	1.7059	2.7294
4011	0			0	-1.00	0	0	8.9927	8.9927	14.388
4012	0			0	-1.60	0	0	5.2844	5.2844	8.4550
4021	-0.30			-0.3000	-0.30	0	0	1.0235	1.0235	1.6376
4031	-0.40			-0.4000	-0.40	0	0	1.1941	1.1941	1.9106
4042	0			0	0	0	0	2.3882	2.3882	3.8212
4041	-2.00			-2.0000	-2.00	0	0	1.0235	1.0235	1.6376
4062	0			0	0	0	0	2.0471	2.0471	3.2753
4063		1.60328		1.60328	0	0	0	4.0941	4.0941	6.5505
4051		0.95654		0.95654	0	0	0	2.3882	2.3882	3.8212
4047	0			0	0	0	0	9.2069	9.2069	14.731
2032		0.43483		0.43483	-0.80	0	0	2.8100	2.8100	4.6400
1013		1.67565		1.67565	-0.50	0	0	2.5489	2.5489	4.0783
1012	0			0	-0.80	0	0	2.7294	2.7294	4.3670
1014	0			0	-1.00	0	0	5.5921	5.5921	8.9474
1022			**0.85294**	**0.85294**	**-0.25**	**0**	**0**	**0.8529**	**0.8529**	**1.3647**
1021			2.06510	2.06510	-1.60	0	0	2.0471	2.0471	3.2753
1043			1.09176	1.09176	-0.20	0	0	0.6824	0.6824	1.0918
1042	0			0	-0.40	0	0	1.3647	1.3647	2.1835

Social Welfare = 1,835.820 \$/hr.

Eliminating Integer Constraints

- Step 3:
  - Update the solution as $Q_3(1022) = Q_{3min}(1022)$
  - Hence, $Q_3(1022) = 0$ and $Q_{2min}(1022) = Q_2(1022) = Q_{2max}(1022)$.
  - When we update, the solution the value of the new S_w = 1,919.899 $/h.
  - This value is higher than the previous one, and hence the solution needs to be updated.
  - The new solution is:

Eliminating Integer Constraints

Bus	Q_1	Q_2	Q_3	Q_G	Q_{1min}	Q_{1max}	Q_{2min}	Q_{2max}	Q_{3min}	Q_{3max}
4072		2.73685		2.73685	-3.00	0	0	42.981	42.981	68.769
4071	0			0	-0.50	0	0	1.7059	1.7059	2.7294
4011	0			0	-1.00	0	0	8.9927	8.9927	14.388
4012	0			0	-1.60	0	0	5.2844	5.2844	8.4550
4021	-0.30			-0.3000	-0.30	0	0	1.0235	1.0235	1.6376
4031	-0.40			-0.4000	-0.40	0	0	1.1941	1.1941	1.9106
4042	0			0	0	0	0	2.3882	2.3882	3.8212
4041	-2.00			-2.0000	-2.00	0	0	1.0235	1.0235	1.6376
4062	0			0	0	0	0	2.0471	2.0471	3.2753
4063		1.67274		1.67274	0	0	0	4.0941	4.0941	6.5505
4051		0.82260		0.82260	0	0	0	2.3882	2.3882	3.8212
4047	0			0	0	0	0	9.2069	9.2069	14.731
2032		0.46831		0.46831	-0.80	0	0	2.8100	2.8100	4.6400
1013		1.98463		1.98463	-0.50	0	0	2.5489	2.5489	4.0783
1012	0			0	-0.80	0	0	2.7294	2.7294	4.3670
1014	0			0	-1.00	0	0	5.5921	5.5921	8.9474
1022		0		0	-0.25	0	0	0.8529	0.8529	1.3647
1021			2.27948	2.27948	-1.60	0	0	2.0471	2.0471	3.2753
1043			1.09176	1.09176	-0.20	0	0	0.6824	0.6824	1.0918
1042	0			0	-0.40	0	0	1.3647	1.3647	2.1835

Social Welfare = 1,919.899 \$/hr.

Concluding Remarks

- All these problems are challenging, large (several thousand constraints and variables) NLP problems
- Ad-hoc solution techniques have been developed based on nature of the problem
- Convergence issues need further study; however, global optimum values are not a great concern
- For these models to be useful in practice, solutions should obtained in 1-2 mins.

Conclusions

- "Novel" NLP solution methods might prove useful to solve some of these OPF problems (e.g. we are looking at applying SDP in some cases)
- Other NLP models are being developed to address some shortcomings of the proposed problems, such as:
  - Inclusion of system security in reactive power markets
  - Proper representation of system security in LP-based models

Singular Value VSC-OPF

- This is an NLP problem with an implicit constraint:

$$
\begin{aligned}
\text{Min.} \quad & S(\chi) \\
\text{s.t.} \quad & F(\chi) = 0 \\
& \underline{H} \leq H(\chi) \leq \overline{H} \\
& \underline{\chi} \leq \chi \leq \overline{\chi}
\end{aligned}
$$

  - Interior point solution approach:

$$
\begin{aligned}
\text{Min.} \quad & S(\chi) - \mu_s \sum_i (\ln s_i + \ln q_i) \\
\text{s.t.} \quad & F(\chi) = 0 \\
& -s - q + \overline{H} - \underline{H} = 0 \\
& -H(\chi) - q + \overline{H} = 0 \\
& s \geq 0, \quad q \geq 0
\end{aligned}
$$

Singular Value VSC-OPF

– Lagrange-Newton method:

$$
\begin{aligned}
L_\mu(u) \;=\;& S(\chi) - \mu_s \sum_i (\ln s_i + \ln q_i) - \rho^T F(\chi) \\
& -\varsigma^T(-s - q + \overline{H} - \underline{H}) - \tau^T(-H(\chi) - q + \overline{H})
\end{aligned}
$$

$$
\Rightarrow \qquad \nabla_u L_\mu(u) = 0
$$

– The solution procedure requires finding the Hessian:

$$
\nabla^2_\chi L_\mu(u) \;=\; \nabla^2_\chi S(\chi) - \rho^T \nabla^2_\chi F(\chi) + \tau^T \nabla^2_\chi H(\chi)
$$

Singular Value VSC-OPF

– Since $H(\chi)$ has an implicit constraint, to obtain $\nabla^2_\chi H(\chi)$:

$$\left.\frac{\partial^2 \sigma_{min}(J_{pf})}{\partial \chi_i \chi_j}\right|_{\chi_* + \Delta\chi} \approx \frac{\left.\frac{\partial \sigma_{min}(J_{pf})}{\partial \chi_i}\right|_{\chi_*} - \left.\frac{\partial \sigma_{min}(J_{pf})}{\partial \chi_i}\right|_{\chi_* + \Delta\chi}}{\Delta \chi_j}$$

based on approximations that are obtained from the properties of the singular value:

$$\Delta\sigma_{min}(J_{pf}) \approx U_1^T \, D_z^2 F|_* \, \Delta z \, V_1$$

$$\Delta\sigma_{min}(J_{pf}) \approx -U_1^T \, D_z^2 F|_* \, D_z F|_*^{-1} \, D_p F|_* \, \Delta p \, V_1$$