Spiral Anchoring Under Full Euclidean Symmetry-Breaking

A dynamical system approach

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April 29, 2006
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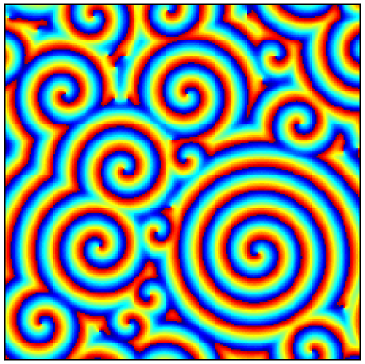
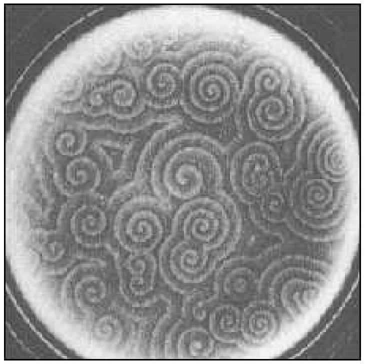
Historical Perspective

Spirals in Nature

Static occurrences: snail's shell, seeds in the sunflower, falcon's hunting path, etc.
Dynamic occurrences: hurricanes, galaxies, heart tissue, retina, chemical reactions, slime mold aggregates, flame fronts, etc.





(Hendrey, Ott and Antonsen:2000,Ball:1994).

Historical Perspective
Why study spirals?

Spiral waves have been linked to disruptions of the heart's normal electrical cycle (Winfree:1995, Witkowski *et.al.*:1998). Most such *arrhythmias* are harmless but if they are

> re-entrant in nature and [...] occur [in the ventricles] because of the spatial distribution of cardiac tissue (Keener and Sneyd:1998),

they can seriously hamper the pumping mechanism of the heart and lead to death.

Historical Perspective

Classification of spiral waves

Spiral propagation is classified according to its *tip path*, which is defined by following an arbitrary point on the wave front in time.

RW MRW (in) MTW MRW (out)

(Movies from Sandstede:2006)

Historical Perspective

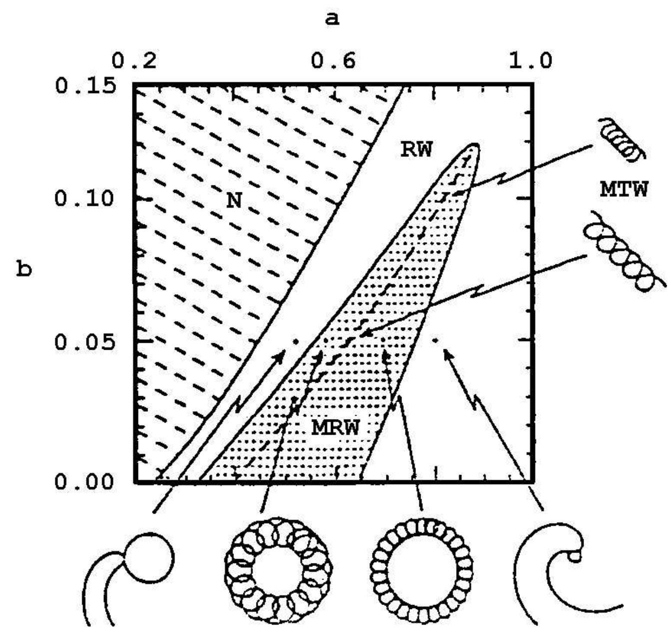
Barkley's RDS

The prototypical RDS

$$u_t = 50u(1-u)\left(u - \frac{v+b}{a}\right) + \Delta u$$
$$v_t = u - v,$$

where a, b are system parameters.

RDS can sustain rotating waves (RW), modulated rotating waves (MRW) and modulated traveling waves (MTW).



(Barkley:1994)

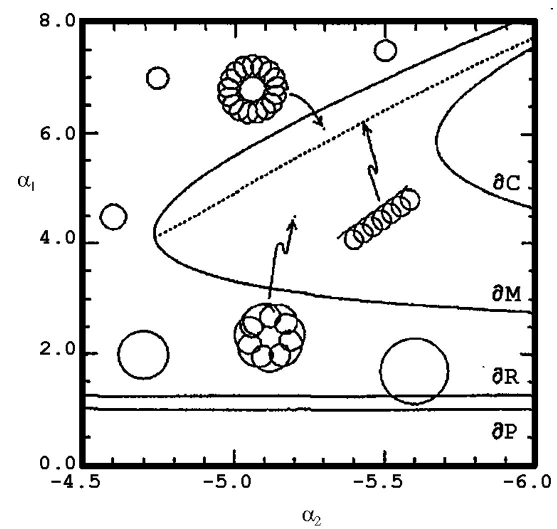
Historical Perspective

Barkley's ad hoc ODE system

The 5−dimensional system on $\mathbb{C} \times \mathbb{C} \times \mathbb{S}^1$:

$$\dot{p} = v$$
$$\dot{v} = v\left[f(|v|^2, w^2) + iwh(|v|^2, w^2)\right]$$
$$\dot{w} = wg(|v|^2, w^2)$$

where $f(\xi, \zeta) = -\frac{1}{4} + \alpha_1\xi + \alpha_2\zeta - \xi^2$, $g(\xi, \zeta) = \xi - \zeta - 1$ and $h(\xi, \zeta) = \gamma_0$.



(Barkley and Kevrekidis:1994)

The Dynamical System Approach

Equivariance of vector fields

Let Γ be a group acting linearly on a vector space X. A family of vector fields $f_\lambda : X \to X$ is $\Gamma-$*equivariant* if for all λ,

$$\gamma \cdot f_\lambda(x) = f_\lambda(\gamma \cdot x), \quad \forall \gamma \in \Gamma, x \in X.$$

$\Gamma-$equivariant ODE systems are such that

$x(t)$ is a solution $\iff$ $\gamma \cdot x(t)$ is a solution for all $\gamma \in \Gamma$.

The Dynamical System Approach

$\mathbb{SE}(2)-$equivariance of RDS

$\mathbb{SE}(2) \cong \mathbb{SO}(2)\dot{+}\mathbb{R}^2$, with

$$(R_1, S_1) \cdot (R_2, S_2) = (R_1 R_2, S_1 + R_2 S_2), \quad \forall (R_j, S_j) \in \mathbb{SE}(2),$$

acts on function spaces *via*

$$(\gamma \cdot v)(x) = ((R, S) \cdot v)(x) = v(R^{-1}(x - S)).$$

$\mathbb{SE}(2)-$equivariance of RDS:

$$(R, S) \cdot (f(\vec{u}) + \Delta \vec{u}) = f((R, S) \cdot \vec{u}) + \Delta((R, S) \cdot \vec{u}), \quad (R, S) \in \mathbb{SE}(2).$$

The Dynamical System Approach

Barkley's insight

In a RDS, the linearization at a RW at the onset of Hopf bifurcation has five critical eigenvalues:

1. $\lambda_R = 0$ (due to rotational symmetry)

2. $\lambda_T = \pm i\omega$ (due to translational symmetry)

3. $\lambda_B = \pm i\beta_0$ (responsible for the Hopf bifurcation from RW to MRW and *vice-versa*)

(Barkley:1992,1994)

The Dynamical System Approach

Essential dynamics for Hopf bifurcation from a 1−armed spiral

The dynamics are described by a 5−dimensional ODE system on the center bundle $V = \mathbb{SE}(2) \times \mathbb{C}$:

$$\begin{aligned} \dot{p} &= e^{i\varphi} F^p(q, \overline{q}) \\ \dot{\varphi} &= F^\varphi(q, \overline{q}) \\ \dot{q} &= F^q(q, \overline{q}), \end{aligned} \tag{1}$$

where $F^\varphi(0) = \omega_{\mathrm{rot}} \in \mathbb{R}$, $F^q(0) = 0$ and $DF^q(0) = i\omega_{\mathrm{per}} \in i\mathbb{R}$.

RW: $q = 0$
MRW: q−component has a 2π−periodic solution

(Golubitski, LeBlanc and Melbourne:1997)

The Dynamical System Approach

$\mathbb{SE}(2)-$equivariance of the center bundle equations

$\mathbb{SE}(2) \cong \mathbb{C} \dot{+} \mathbb{S}^1$, with

$$(p_1, \varphi_1) \cdot (p_2, \varphi_2) = (e^{i\varphi_1} p_2 + p_1, \varphi_1 + \varphi_2), \quad \forall (p_j, \varphi_j) \in \mathbb{SE}(2),$$

acts on the center bundle $\mathbb{SE}(2) \times \mathbb{C}$ *via*

$$(x, \theta) \cdot (p, \varphi, q) = (e^{i\theta} p + x, \varphi + \theta, q), \quad \forall (x, \theta) \in \mathbb{SE}(2).$$

The center bundle system (1) is $\mathbb{SE}(2)-$equivariant under this action.

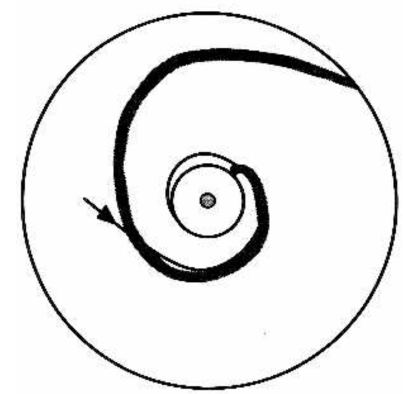
The Effects of Forced Euclidean Symmetry-Breaking

Non-Euclidean media

Physical experiments and Nature are not perfectly Euclidean. For instance, cardiac tissue is finite, *anisotropic* and distributed non-uniformly (Keener and Sneyd:1998).

'Far' from the inhomogeneities, the domain 'looks' Euclidean. Furthermore, if the anisotropy ratio is 'slight', the domain also 'looks' Euclidean.

This partly Euclidean structure translates mathematically *via forced Euclidean symmetry-breaking* (FESB).



(LeBlanc and Wulff:2000)

The Effects of Forced Euclidean Symmetry-Breaking

Center of anchoring/repelling

A $2\pi-$periodic solution $\tilde{p}$ of the $p-$component of (1) is called a *perturbed rotating wave of* (1).

It is characterized by its center

$$[\tilde{p}] = \frac{1}{2\pi}\int_0^{2\pi} \tilde{p}(t)\,dt.$$

If $\tilde{p}$ attracts all nearby solutions, $[\tilde{p}]$ is a *center of anchoring*. If $\tilde{p}$ repels all nearby solutions, $[\tilde{p}]$ is a *center of repelling*.

The Effects of Forced Euclidean Symmetry-Breaking

What symmetries are allowed?

Let $\xi \in \mathbb{C}$. In $\mathbb{SE}(2) = \mathbb{C} \dot{+} \mathbb{S}^1$, the group of rotations around ξ is

$$\mathbb{SO}(2)_\xi = \{(\xi, 0) \cdot (0, \theta) \cdot (-\xi, 0) \mid \theta \in \mathbb{S}^1\} < \mathbb{SE}(2)$$

The symmetry-breaking lattice in the TSB case:

$$\begin{array}{c} \mathbb{SE}(2) \\ | \\ \mathbb{SO}(2)_\xi \\ | \\ \{1\} \end{array}$$

The Effects of Forced Euclidean Symmetry-Breaking

Perturbations of the center bundle equations

Let Γ be an element of the symmetry-breaking lattice. A $\Gamma-$equivariant perturbation of (1) is a system of ODE of the form

$$\begin{aligned} \dot{p} &= e^{i\varphi}\Big[F^p(q,\overline{q}) + \varepsilon\mathcal{F}^p(p,\overline{p},\varphi,q,\overline{q},\varepsilon)\Big] \\ \dot{\varphi} &= F^{\varphi}(q,\overline{q}) + \varepsilon\mathcal{F}^{\varphi}(p,\overline{p},\varphi,q,\overline{q},\varepsilon) \\ \dot{q} &= F^q(q,\overline{q}) + \varepsilon\mathcal{F}^p(p,\overline{p},\varphi,q,\overline{q},\varepsilon), \end{aligned} \tag{2}$$

where $\mathcal{F}$ is sufficiently smooth, uniformly bounded and commutes with the $\Gamma-$restricted $\mathbb{SE}(2)-$action on the center bundle.

The perturbation $\mathcal{F}$ is not completely arbitrary: the symmetry group Γ plays a role.

The Effects of Forced Euclidean Symmetry-Breaking

1 TSB perturbation – RW

WLOG, assume $\Gamma = \mathbb{SO}(2)_0$. Them, (2) is equivalent to

$$\begin{aligned} \dot{p} &= e^{i\varphi}\Big[v + \lambda H(pe^{-i\varphi}, \overline{p}e^{i\varphi}, \lambda)\Big] \\ \dot{\varphi} &= 1 \end{aligned} \tag{3}$$

where $v \in \mathbb{C}$, $\lambda \in \mathbb{R}$ is small and H is sufficiently smooth and uniformly bounded in p, $\overline{p}$.

Theorem 1 (LeBlanc, Wulff:2000)
Set $a = \mathrm{Re}\Big[D_1 H(-iv, i\overline{v}, 0)\Big]$. If $a \neq 0$, then for all $\lambda \neq 0$ small enough, (3) has a unique family of perturbed rotating waves p_λ with $[p_\lambda] = 0$, whose stability is exactly determined by the sign $a\lambda$.

The Effects of Forced Euclidean Symmetry-Breaking

2 simultaneous TSB perturbations – RW

Let $0 = \xi_1 \neq \xi_2$ be the centers of the perturbations: the center bundle equations reduce to

$$\begin{aligned} \dot{p} &= e^{i\varphi}\Big[v + \lambda_1 H_1(pe^{-i\varphi}, \overline{p}e^{i\varphi}, \lambda_1) + \lambda_2 H_2((p-\xi_2)e^{-i\varphi}, (\overline{p}-\overline{\xi}_2)e^{i\varphi}, \lambda_2)\Big] \\ \dot{\varphi} &= 1 \end{aligned} \tag{4}$$

where $v \in \mathbb{C}$, $\lambda \in \mathbb{R}^2$ is small and H_j is sufficiently smooth and uniformly bounded in p, $\overline{p}$, $j = 1, 2$.

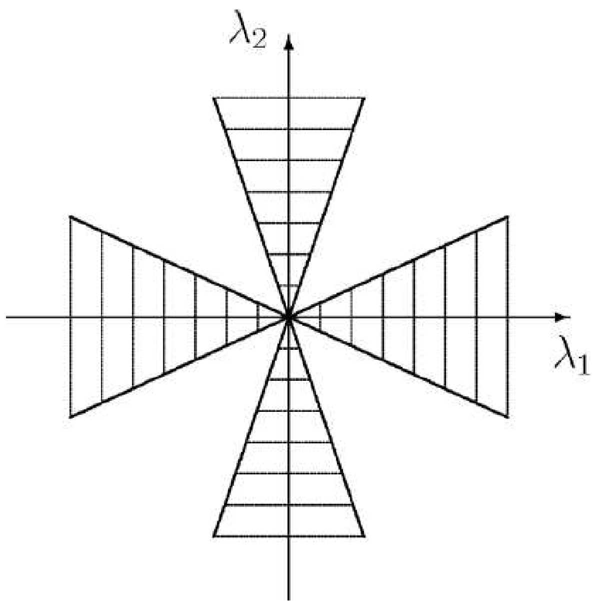
The Effects of Forced Euclidean Symmetry-Breaking

2 simultaneous TSB perturbations – RW

Theorem 2 (Boily:2005)
Let $k \in \{1,2\}$. *Set* $a_k = \mathrm{Re}\left[D_1 H_k(-iv, i\overline{v}, 0)\right]$. *If* $a_k \neq 0$, *there is a wedge-shaped region of the form*

$$\mathcal{W}_k = \{\lambda \in \mathbb{R}^2 \mid |\lambda_j| < W_{k,j}|\lambda_k|,\ j \neq k\},$$

where $W_{k,j} > 0$ *and* λ_k *is small, such that for all* $0 \neq \lambda \in \mathcal{W}_k$, (4) *has a unique family of perturbed rotating waves* $\mathcal{S}_\lambda^k$, *where* $[\mathcal{S}_\lambda^k]$ *is generically aways from* ξ_k, *and whose stability is exactly determined by the sign of* $a_k\lambda_k$.



The Effects of Forced Euclidean Symmetry-Breaking

2 simultaneous TSB perturbations – Proof (I)

1. WLOG, assume $k = 1$.
2. Shift the point of view to $z = p - \xi_1 + ie^{it}v$.
3. Set $\lambda_2 = \lambda_1 \varepsilon_2$, $\lambda_1 \neq 0$.
4. (4) rewrites as $\dot{z} = \lambda_1 e^{it} K(ze^{-it}, \overline{z}e^{it}, t, \lambda_1, \varepsilon_2)$.
5. Set $a_1 = \mathrm{Re}\left[D_1 H_1(-iv, i\overline{v}, 0)\right]$.
6. Near $(z, \lambda_1, \varepsilon_2) = (0, 0, 0)$ the time 2π−map of the above system is

$$P(z, \overline{z}, \lambda_1, \varepsilon_2) = z + 2\pi\lambda_1\left[a_1 z + O\left(|z|^2\right) + O\left(\lambda_1, \varepsilon_2\right) + \mathrm{h.o.t.}\right]$$

The Effects of Forced Euclidean Symmetry-Breaking

2 simultaneous TSB perturbations – Proof (II)

7. By the IFT, there is a unique family $z(\lambda_1, \varepsilon_2) \not\equiv 0$ of fixed points of P with $z(0,0) = 0$.

8. Eigenvalues $\eta_{1,2}$ of $DP(z(\lambda_1, \varepsilon_2), \lambda_1, \varepsilon_2)$ satisfy

$$|\eta_{1,2}(\lambda_1, \varepsilon_2)| = 1 + 4\pi a_1 \lambda_1 + O\left(\lambda_1^2, \varepsilon_2\right).$$

9. If $a_1 \neq 0$, the fixed points are hyperbolic, with stability $a_1 \lambda_1$.

10. Each fixed point $z(\lambda_1, \varepsilon_2)$ of P corresponds to a perturbed rotating wave $\mathcal{S}_{\lambda_1,\lambda_2}$ of (4).

11. $[\mathcal{S}_{\lambda_1,\lambda_2}] = \xi_1 + O(1)$ as $\lambda_1 \to 0$ and $\lambda_2 \neq 0$.

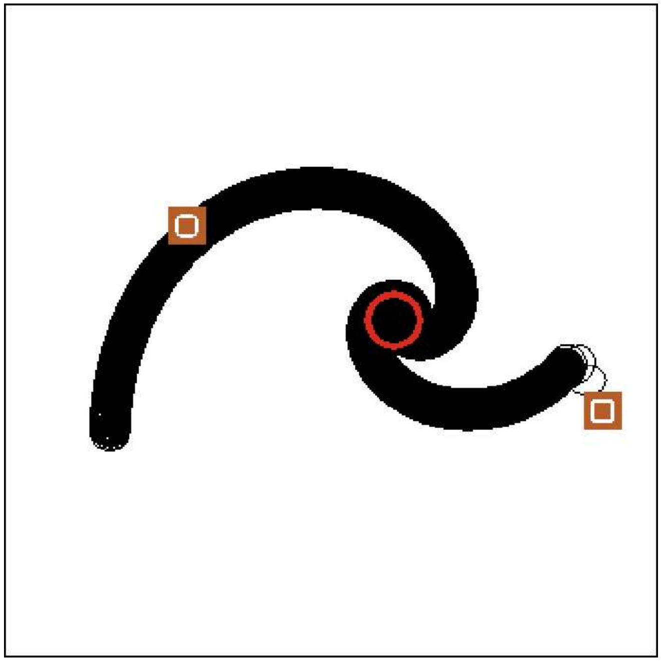
Numerical Simulations

Perturbed FHN with 2 TSB terms

Consider the perturbed FHN system of equations:

$$u_t = \frac{10}{3}\left(u - \frac{u^3}{3} - v\right) + \phi_1 + \Delta u$$
$$v_t = 0.3(u + 0.6 - 0.5v - \phi_2),$$

with $\phi_j(x) = \sqrt{2}\cos(0.05\pi)0.12f(x - c_j)$, for $j = 1, 2$, and $f(x) = \exp(-0.00086\|x\|^2)$, $c_1 = (9, 0)$, $c_2 = (-10, 5\sqrt{3})$.



Numerical Simulations

Perturbed FHN with 4 TSB terms

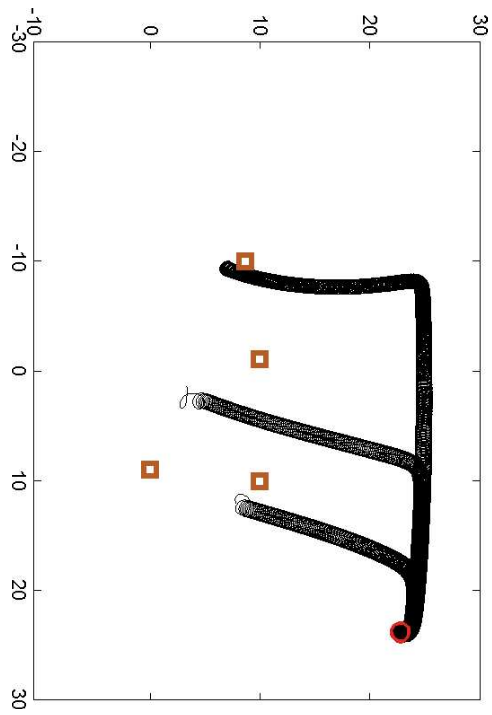
Consider the perturbed FHN system of equations:

$$u_t = \frac{10}{3}\left(u - \frac{u^3}{3} - v\right) + \phi_1 + \Delta u$$
$$v_t = 0.3(u + 0.6 - 0.5v - \phi_2),$$

with

$$\phi_1(x) = 0.12 f_1(x - c_1) - 0.10 f_2(x - c_2)$$
$$\phi_2(x) = -0.12 f_1(x - c_3) + 0.08 f_3(x - c_4),$$

$f_j(x) = \exp(a_j \|x\|^2)$, $j = 1, 2, 3$, $a_1 = -0.00086$, $a_{2,3} = -0.0008$, $c_1 = (9, 0)$, $c_2 = (-1, 10)$, $c_3 = (-10, 5\sqrt{3})$, $c_4 = (10, 10)$.



Numerical Simulations

Homotopy/Hysteresis in a modified Oregonator (I)

Consider the modified Oregonator:

$$\begin{aligned} u_t &= 20\left(u - u^2 - (1.4v + \phi)\tfrac{u-0.002}{u+0.002}\right) + \Delta u \\ v_t &= u - v + 0.6\Delta v, \end{aligned} \tag{5}$$

with

$$\phi(x) = \sum_{j=1}^{2} \alpha_j \exp\left(-\|x - c_j\|^2\right),$$

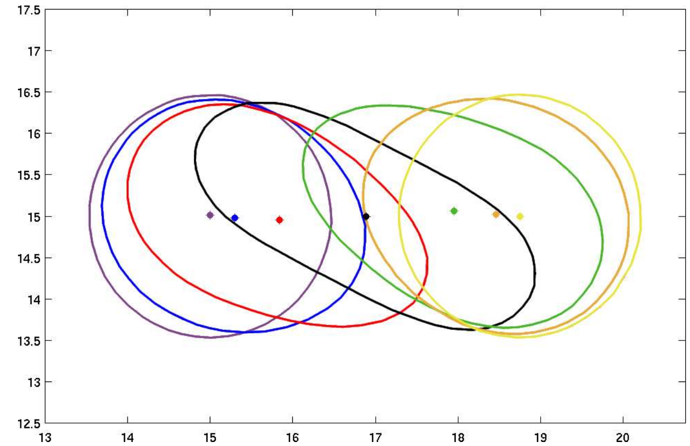
where $\alpha_j, \in \mathbb{R}$, $c_1 = (15, 15)$ and $c_2 = (18.75, 15)$.

Numerical Simulations

Homotopy/Hysteresis in a modified Oregonator (II)

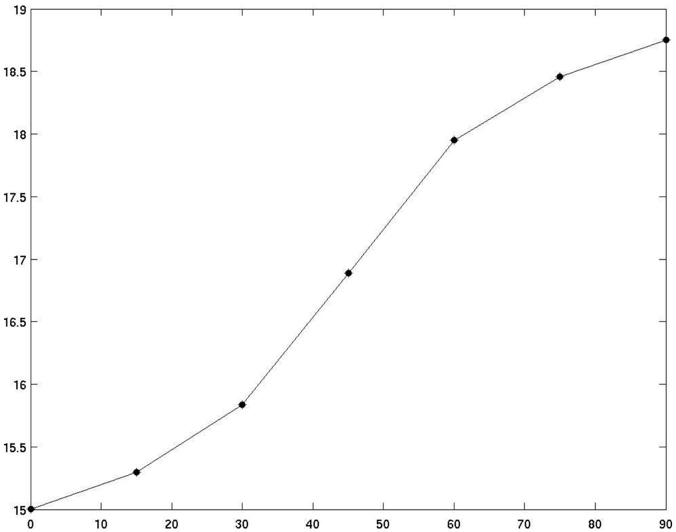
Along $\alpha(\tau) = \gamma_1(t) = 0.01(\cos(\tau), \sin(\tau))$ in parameter space, (5) undergoes homotopy of perturbed rotating waves.

Along the path $\alpha(\tau) = \gamma_2(t) = \frac{1}{10}\gamma_1(\tau)$ in parameter space, (5) undergoes hysteresis of perturbed rotating waves.

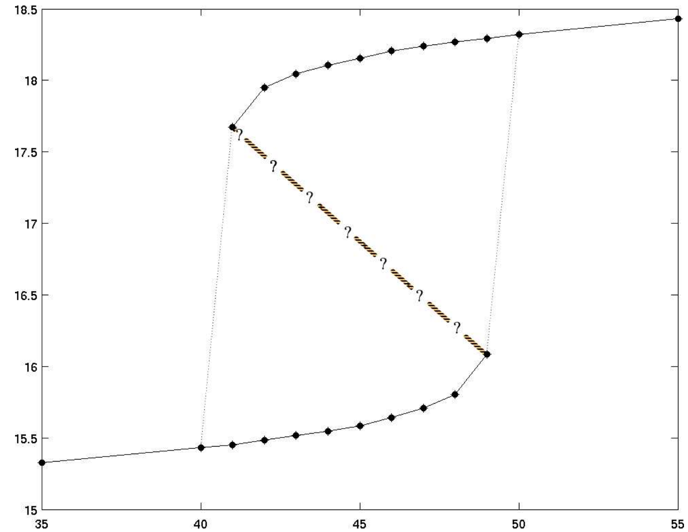


Numerical Simulations

Homotopy/Hysteresis in a modified Oregonator (III)



Along $\gamma_1(\tau)$



Along $\gamma_2(\tau)$

Characterization of Spiral Anchoring

The set-up

Let $n = 2$, $0 \neq \xi \in \mathbb{R}^2$, $\eta \in \{0, \xi\}$, $\Lambda_0 = \{(\lambda_1, 0)\}, \Lambda_\xi = \{(0, \lambda_2)\}$ and $P : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ be a real analytic map with:

(P1) $P(x, 0) \equiv x$ and $DP(x, 0) \equiv I$;

(P2) $\exists \omega_* > 0$ such that $P(\eta, \Lambda_\eta) \equiv \eta$ for all $\|\Lambda_\eta\| < \omega_*$;

(P3) η has the same stability for all $0 < \|\Lambda_\eta\| < \omega_*$, and ...

Characterization of Spiral Anchoring

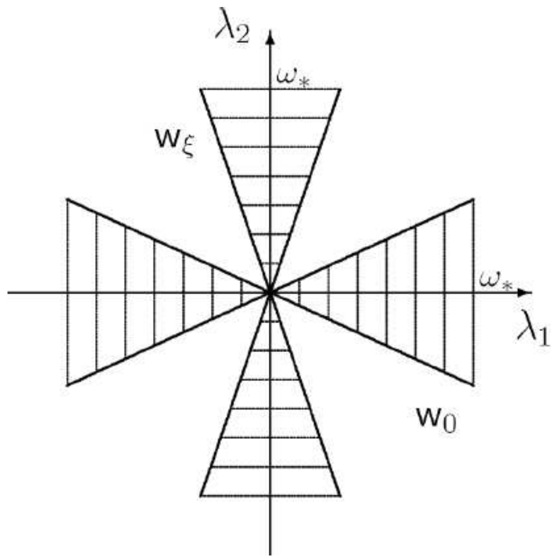
The set-up (II)

(P4) ... around each axis Λ_η, there is a parameter wedge region w_η in which P has a (locally) unique fixed point-manifold $x_\eta(\lambda)$ such that, for all $0 \neq \lambda \in \mathrm{w}_\eta$,

(i) $P(x_\eta(\lambda), \lambda) \equiv x_\eta(\lambda)$;

(ii) $x_\eta(\lambda) \to \eta$ as $\lambda \to \Lambda_\eta - \{0\}$, and

(iii) $x_\eta(\lambda)$ shares its stability with η, in (P3).



Characterization of Spiral Anchoring

The general mapping

The time $2\pi-$map appearing in the proof of theorem 2 has the form

$$\mathcal{P}(x,\lambda) = x + 2\pi\Big[\lambda_1\mathcal{F}_0(x,\lambda_1) + \lambda_1\lambda_2\mathcal{J}(x,\lambda) + \lambda_2\mathcal{G}_\xi(x,\lambda_2)\Big],$$

where $0 \neq \xi \in \mathbb{R}^2$,

$$D_x\mathcal{F}_0(x,\lambda_1) = \begin{pmatrix} a(\lambda_1) & -b(\lambda_1) \\ b(\lambda_1) & a(\lambda_1) \end{pmatrix} \quad \text{and} \quad D_x\mathcal{G}_\xi(x,\lambda_2) = \begin{pmatrix} c(\lambda_2) & -d(\lambda_2) \\ c(\lambda_2) & d(\lambda_2). \end{pmatrix}$$

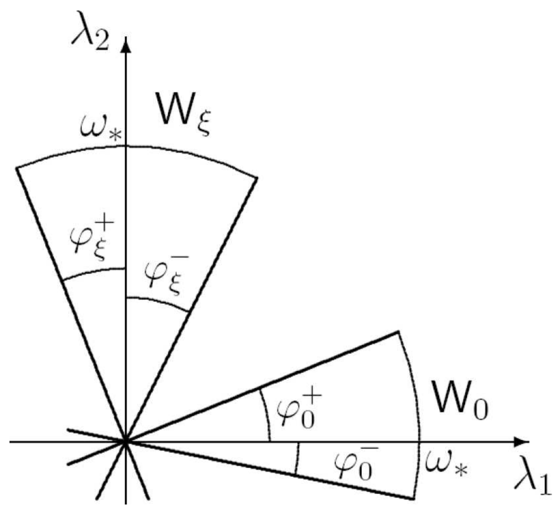
Proposition 3 *If $\mathcal{F}_0, \mathcal{J}, \mathcal{G}_\xi$ are real-analytic, and if $a(0), c(0) \neq 0$, then $\mathcal{P}$ satisfies $(P1) - (P4)$.*

Characterization of Spiral Anchoring

The specific mapping

Proposition 3 also holds for the truncated map

$$P(x,\lambda) = x + 2\pi\Big[\lambda_1 \mathcal{F}_0(x,0) + \lambda_2 \mathcal{G}_\xi(x,0)\Big]$$
$$= x + 2\pi\Big[\lambda_1 F_0(x) + \lambda_2 G_\xi(x)\Big].$$



Let $\rho > 0$ and define the map $P_\rho : \mathbb{R}^2 \times [0, 2\pi] \to \mathbb{R}^2$ by

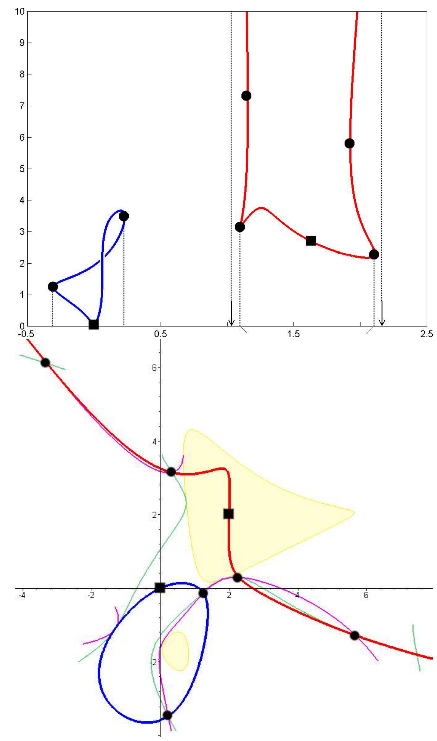
$$P_\rho(x,s) = x + 2\pi\rho\Big[\cos(s)F_0(x) + \sin(s)G_\xi(x)\Big].$$

Characterization of Spiral Anchoring

Analysis of the specific mapping

The fixed point branches in the bifurcation diagram of P_ρ are in one-to-one correspondence with the curves in the zero set

$$\mathcal{Z}_{\mathbb{R}^2}\left(\det[F_0(x) \quad G_\xi(x)]\right) = \mathcal{C}_{\mathsf{B}} \sqcup \mathcal{C}_\infty$$



C_0: fixed point branch through the origin

C_ξ: fixed point branch through ξ

Two types of catastrophes: fold and ∞

Characterization of Spiral Anchoring

Bifurcation diagrams

Bifurcation diagrams of P_ρ are $2\pi-$periodic in s.

Elements of $\mathcal{C}_{\text{B}}$ are loops and elements of $\mathcal{C}_\infty$ give rise to two $\infty-$catastrophes.

The number of fold catastrophes on any $C \in \mathcal{C}_{\text{B}}$ is even.

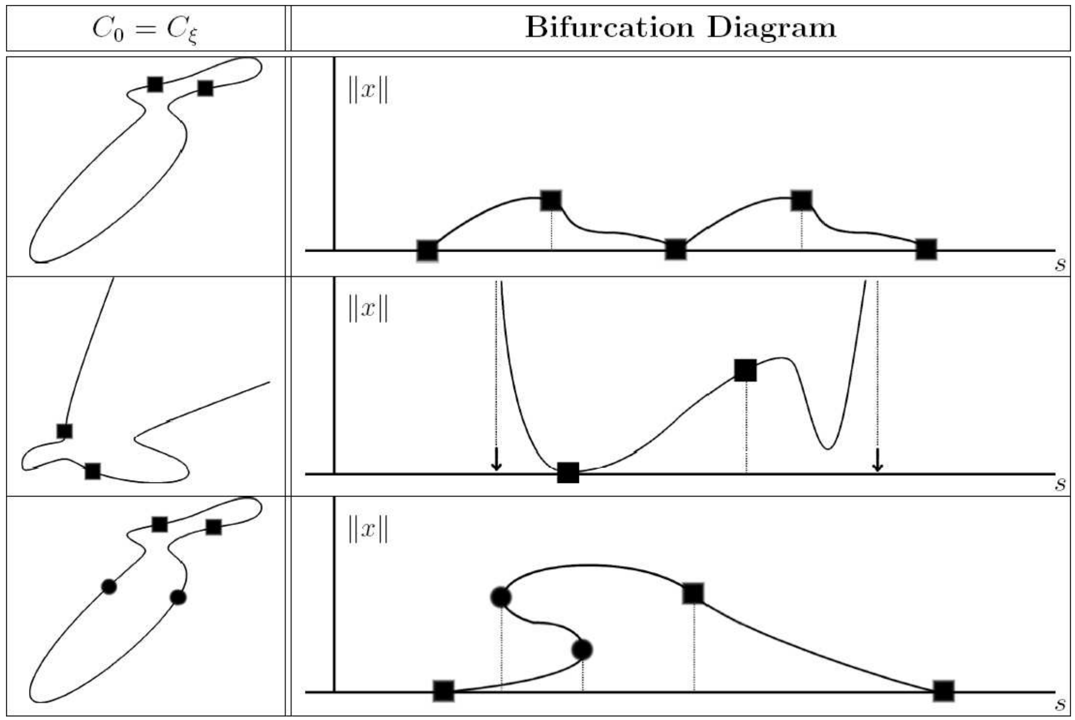
Catastrophes cannot occur at $0, \xi$.

Catastrophes persist under small perturbations.

The bifurcation diagrams of the general mapping and the specific mapping are (locally) topologically equivalent.

Characterization of Spiral Anchoring

Some bifurcation diagrams – $C_0 = C_\xi$



Conjectures and Related Work

Publications

Spiral anchoring under n TSB perturbations, with LeBlanc and Matsui, submitted to *J. Nonlin. Sci.* (2006).

Spiral anchoring under combined TSB and RSB perturbations, submitted to *Nonlinearity* (2006).

Epicyclic drifting, submitted to *SIADS* (2006).

Higher codimension phenomena, waiting for numerical confirmation.

Modified bidomain experiments: with Ethier, not yet submitted.