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The Vénéreau polynomials relative to $\mathbb{C}^*$-fibrations and stable coordinates

Fields Institute, Toronto, 11 July 2006

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Motivation: Dolgachev-Weisfeiler Conjecture

If $\phi : \mathbb{C}^n \to \mathbb{C}^m$ is a flat morphism, and each fiber is isomorphic to $\mathbb{C}^{n-m}$, then ϕ is a trivial fibration.

Vénéreau Polynomials f_n:

$B = \mathbb{C}[x,y,z,u]$, $f_n \in B$ $(n \geq 1)$, $A_n = \mathbb{C}[x, f_n]$

Let $\phi_n : \mathbb{C}^4 \to \mathbb{C}^2$ be induced by $A_n \hookrightarrow B$.

Kaliman, Vénéreau , Zaidenberg showed:

I. For $n \geq 3$, $\phi_n : \mathbb{C}^4 \to \mathbb{C}^2$ is a trivial fibration.
II. For $n \geq 1$ and $\lambda(x) \in \mathbb{C}[x]$, $f_n - \lambda(x)$ defines a $\mathbb{C}^3$ in $\mathbb{C}^4$, and its generic fiber is also an affine space.
III. Given $n \geq 1$, let $\mathbb{C}^3 \hookrightarrow \mathbb{C}^4$ be defined by f_n, and let $\mathbb{C}^4 \hookrightarrow \mathbb{C}^5$ be the standard embedding. Then the composite embedding $\mathbb{C}^3 \hookrightarrow \mathbb{C}^5$ is rectifiable.

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Open Question: Are ϕ_1 and ϕ_2 trivial?

Theorem (2005): If $\pi : \mathbb{C}^4 \times \mathbb{C}^1 \rightarrow \mathbb{C}^4$ is the standard projection, then $\phi_n \pi : \mathbb{C}^5 \rightarrow \mathbb{C}^2$ is a trivial fibration for each $n \geq 1$.

Algebraically, this means that the Vénéreau polynomials are **stable** x**-variables**, i.e., for each $n \geq 1$, there exist $X, Y, Z \in B[t]$ such that

$$\mathbb{C}[x, y, z, u, t] = \mathbb{C}[x, f_n, X, Y, Z].$$

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Remark: van den Essen, 2002, showed that f_1 is an x-variable of $B[s, t]$....complicated!

Define $\quad p = yu + z^2, \quad v = xz + yp,$

$$w = x^2u - 2xzp - yp^2$$

The **Vénéreau polynomials** are $f_n := y + x^n v$ $(n \geq 1)$.

Rational Generators:

Define a $\mathbb{C}(x)$-derivation θ of $\mathbb{C}(x)[y, z, u]$ by

$$\theta(y) = 0 \ , \ \theta(z) = x^{-1}y \ , \ \theta(u) = -2x^{-1}z \ ,$$

noting that $\theta(p) = 0$. Then

$$y = \exp(p\theta)(y), v = \exp(p\theta)(xz), w = \exp(p\theta)(x^2u) \ .$$

It follows that, for all $n \geq 1$,

$$\mathbb{C}(x)[y, z, u] = \mathbb{C}(x)[y, xz, x^2u] = \mathbb{C}(x)[y, v, w]$$
$$= \mathbb{C}(x)[y + x^n v, v, w] = \mathbb{C}(x)[f_n, v, w].$$

Strategy: Find a locally nilpotent derivation D of B or $B[t]$ with $Dx = Dv = 0$ and $Dy = x^n$. Then

$$x = \exp(vD)(x) \qquad \text{and} \qquad f_n = \exp(vD)(y).$$

Notation: $\partial(a, b, c, d)$ means $\frac{\partial(a,b,c,d)}{\partial(x,y,z,u)}$, and

$\partial(a, b, c, d, e)$ means $\frac{\partial(a,b,c,d,e)}{\partial(x,y,z,u,t)}$.

Proof of I: Define a locally nilpotent derivation of B by $d = \partial(x, \cdot, v, w)$. Then $dx = dv = 0$ and $dy = x^3$. Therefore, if $n \geq 3$, then $\Delta = x^{n-3}d$ is locally nilpotent, and satisfies $\Delta x = \Delta v = 0$, and $\Delta y = x^n$. $\square$

Proof of Theorem: Set $T = xt + p$ in $B[t]$. Then

$$v - yT = xV,\ w + vT = xW',\ W' + VT = xW$$

which yields

$$V = z - yt \qquad \text{and} \qquad W = u + 2zt - yt^2.$$

Define $D = \partial(x, \cdot, v, W, T)$, which is locally nilpotent since

$$\mathbb{C}(x)[y, v, W, T] = \mathbb{C}(x)[y, z, u, t].$$

It follows that $Dx = Dv = 0$, and

$$
\begin{array}{lll}
Dy & & \\
= & \partial(x,y,v,W,T) & \\
= & x^{-1}\partial(x,y,v,W'+VT,T) & : W = x^{-1}(W'+VT) \\
= & x^{-1}\partial(x,y,v,W',T) & : VT = x^{-1}(vT - yT^2) \\
= & x^{-2}\partial(x,y,v,w+vT,T) & : W' = x^{-1}(w+vT) \\
= & x^{-2}\partial(x,y,v,w,T) & \\
= & x^{-2}\partial(x,y,v,w,xt+p) & \\
= & x^{-2}\partial(x,y,v,w,xt) & : p \text{ alg.}/\mathbb{C}[x,y,v,w] \\
= & x^{-1}\partial(x,y,v,w,t) & \\
= & x^{-1}d(y) = x^{-1}x^3 = x^2 \ . &
\end{array}
$$

Therefore, f_2 is an x-variable of $B[t]$.

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What about f_1? The "Strategy" no longer works. However...

...we have:

$$\begin{aligned} x^2 &= \partial(x,y,v,W,T) \\ &= \partial(x,y+xv,v,W,T) \\ &= \partial(x,f_1,v,W,T) \\ &= \partial(x,f_1,v-f_1T,W,T) \\ &= \partial(x,f_1,xV_1,W,T) \\ &= x\partial(x,f_1,V_1,W,T)\ , \end{aligned}$$

which implies

$$\begin{aligned} x &= \partial(x,f_1,V_1,W,T) \\ &= \partial(x,f_1,V_1+f_1T^2,W,T) \\ &= \partial(x,f_1,V_2,W,T) \\ &= \partial(x,f_1,V_2,W,T-(f_1W+V_2^2)) \\ &= \partial(x,f_1,V_2,W,xT_2) \\ &= x\partial(x,f_1,V_2,W,T_2)\ . \end{aligned}$$

Therefore $\partial(x,f_1,V_2,W,T_2)=1$, and since

$$\mathbb{C}(x,f_1,V_2,W,T_2)=\mathbb{C}(x,y,z,u,t),$$

we conclude that $\mathbb{C}[x,f_1,V_2,W,T_2]=\mathbb{C}[x,y,z,u,t]$.
□

We now have $A_n = \mathbb{C}[x, f_n]$ and $B[t] = A_n[X, Y, Z]$. Note that:

$$f_n \text{ is an } x\text{-variable of } B \text{ if and only if}$$
$$t \text{ is an } A_n\text{-variable of } B[t].$$

A *necessary* condition that $A_n[X, Y, Z] = A_n[t]^{[2]}$ is $(t_X, t_Y, t_Z) = (1)$. Using $x = a$ and $f_2 = b$, I get

$t = Z + (aY + abZ + b^2X + bY^2)[aX - 2Y(aZ + bX + Y^2) - a(aY + abZ + b^2X + bY^2)(aZ + bX + Y^2)^2]$.

I checked with a computer algebra system that this ideal equality does, in fact, hold. (The analogous result also holds for f_1.)

Question: Is $t \in \mathbb{C}[a, b, X, Y, Z]$ a variable over $\mathbb{C}[a, b]$?

The Russell-Sathaye Criterion (1979)

Let B be a domain over a UFD A such that

$$B[t] \cong B^{[1]} = A[X, Y, Z] \cong A^{[3]} .$$

If there exists $\alpha \in B$ of the form $\alpha = P + ZQ$, where $P \in A[X]$, $Q \in A[X, Y, Z]$, and $A[P(X)] = A[X]$, then $B \cong A^{[2]}$.

In our current situation, using $A = \mathbb{C}[x, f_1]$ or $A = \mathbb{C}[x, f_2]$, the existence of such an element $\alpha \in \mathbb{C}[x, y, z, u]$ would imply that f_1 or f_2 is an x-variable.

Associated $\mathbb{C}^*$-Fibrations

For $n \geq 1$, define a degree function $\deg_n$ on B by declaring that

$$\deg_n(x) = -2, \quad \deg_n(y) = -n-2,$$
$$\deg_n(z) = n, \quad \deg_n(u) = 3n+2.$$

This is equivalent to the linear $\mathbb{C}^*$-action on $\mathbb{C}^4$ with weights $(-2, -n-2, n, 3n+2)$. In the induced grading of B, f_n is homogeneous of degree $-n-2$, and $A_n = \mathbb{C}[x, f_n]$ is a homogeneous subring of B. *Up to multiples, this is the unique system of weights relative to which the monomials of f_n have the same degree.*

Kaliman and Zaidenberg pointed out that the map $\phi_n : \mathbb{C}^4 \to \mathbb{C}^2$ yields a flat family of affine planes over $\mathbb{C}^2$, and that ϕ_n is an algebraic fiber bundle if and only if f_n is an x-variable. We can further add to their observations that (1) ϕ_n is equivariant relative to the $\mathbb{C}^*$-actions on $\mathbb{C}^4$ and $\mathbb{C}^2$; and (2) the $\mathbb{C}^*$-action on $\mathbb{C}^4$ maps fibers *linearly* to other fibers.

To see this, consider first the planar $\mathbb{C}^*$-action associated to f_n. Given $(a, b) \in \mathbb{C}^2$, the fiber over (a, b) is defined by the B-ideal $(x-a, f_n-b)$. Apply the $\mathbb{C}^*$-action:

$$t \cdot (x - a, f_n - b) = (t^{-2}x - a, t^{-n-2}f_n - b)$$
$$= (x - t^2a, f_n - t^{n+2}b)$$

Thus, the $\mathbb{C}^*$-action maps the fiber over (a, b) to the fiber over $(t^2a, t^{n+2}b)$, and $\mathbb{C}^*$ acts on the base of the fibration with weights $(2, n+2)$. How does $\mathbb{C}^*$ act on the fibers?

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It is easy to see that, over points of the form $(0, b)$ in the base, $\mathbb{C}^*$ acts on the fibers with weights $(n, 3n + 2)$. So consider fibers over (a, b) for $a \neq 0$.

Recall that $\mathbb{C}(x)[y,z,u] = \mathbb{C}(x)[f_n, v, w]$. Similarly, $\mathbb{C}[y,z,u] = \mathbb{C}[f_n(a), v(a), w(a)]$, where $F(a)$ denotes evaluation at $x = a$. (This requires $a \neq 0$.) Therefore, $B = \mathbb{C}[x, f_n(a), v(a), w(a)]$. It follows that, for $t \in \mathbb{C}^*$, the coordinate ring of the fiber over $t \cdot (a,b)$ is $\mathbb{C}[v(t^2a), w(t^2a)]$. Moreover,

$$t \cdot v(a) = t \cdot (az + yp) = at^n z + t^{n-2} yp$$
$$= t^{n-2}((t^2a)z + yp) = t^{n-2} v(t^2a)$$

and

$$t \cdot w(a) = t \cdot (a^2u - 2azp - yp^2)$$
$$= t^{3n-2}((t^2a)^2u - 2(t^2a)zp - yp^2) = t^{3n-2} w(t^2a).$$

Therefore, when $a \neq 0$, we associate to the fiber over (a,b) the linear $\mathbb{C}^*$-action on the plane with weights $(n-2, 3n-2)$. Herein lies a key difference in the cases $n = 1$, $n = 2$, and $n \geq 3$:

- The planar $\mathbb{C}^*$-action associated to a general fiber of Φ_1 is **hyperbolic**, with weights $(1, -1)$.

- The planar $\mathbb{C}^*$-action associated to a general fiber of Φ_2 is **parabolic**, with weights $(0, 4)$.

- The planar $\mathbb{C}^*$-action associated to a general fiber of Φ_n is **elliptic** when $n \geq 3$, with weights $(n-2, 3n-2)$.

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Theorem: If $B = A_n^{[2]}$, there exist $P_n, Q_n \in B$ such that

(a) $B = A_n[P_n, Q_n]$
(b) P_n and Q_n are homogeneous
(c) $\deg_n(P_n) = n$ and $\deg_n(Q_n) = 3n + 2$.

Proof. The assumption $B = A_n^{[2]}$ implies that ϕ_n is a $\mathbb{C}^*$-vector bundle. By the well-known theorem of Masuda, Moser-Jauslin, and Petrie, every algebraic $\mathbb{C}^*$-vector bundle over a $\mathbb{C}^*$-module is trivial. At the level of coordinate rings, this is precisely the statement that there exist homogeneous $P_n, Q_n \in B$ such that $B = A_n[P_n, Q_n]$. To verify the claim about degrees, given $F \in B$, let F^* denote its linear part, i.e., the degree-one summand of F in the standard grading of B. Then $B = \mathbb{C}[x, f_n, P_n, Q_n] = \mathbb{C}[x, f_n^*, P_n^*, Q_n^*] = \mathbb{C}[x, y, P_n^*, Q_n^*]$. We may therefore assume that $P_n^* = ax + by + cz + du$ for some $a, b, d \in \mathbb{C}$ and $c \in \mathbb{C}^*$. Since P_n is homogeneous, it follows that $\deg_n(P_n) = \deg_n(z) = n$. Likewise, we may assume u appears in Q_n^*, and thus $\deg Q_n = \deg_n(u) = 3n + 2$. □

Associated Planar Automorphisms

Set $L = \mathbb{C}(x, f_3)$ and $B_L = L \otimes_{\mathbb{C}} B = L[v, w]$. We know that $B = \mathbb{C}[x, f_3, P_3, Q_3]$, and therefore $L[v, w] = L[P_3, Q_3]$. We may thus view the pair (P_3, Q_3) as an element of $GA_2(L)$, the group of polynomial L-automorphisms of $L[v, w]$, where (v, w) is the identity.

Lemma: P_3 is a triangular variable.

Proof. $P_3 = (-x^{-3} f_3^2) w + (x^{-1} v - x^{-3} f_3 v^2)$. □

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Likewise, set $K = \mathbb{C}(x, f_1)$ and $B_K = K \otimes_{\mathbb{C}} B = K[v, w]$. Suppose $B = \mathbb{C}[x, f_1, P_1, Q_1]$ for some $P_1, Q_1 \in B$. Then $K[v, w] = K[P_1, Q_1]$ and $(P_1, Q_1) \in GA_2(K)$.

Lemma: In this case, neither P_1 nor Q_1 is a triangular variable.