Hierarchical Anderson Model

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References

- K: Spectral Localization in the Hierarchical Anderson Model. Proc. Amer. Math. Soc. 135 (2007), 1431-1440.

- K: Hierarchical Anderson Model. Centre de Recherches Mathmatiques, CRM Proceedings and Lecture Notes Volume 42, 2007.

- Molchanov S.: Hierarchical random matrices and operators. Application to Anderson model. Multidimensional statistical analysis and theory of random matrices (Bowling Green, OH, 1996), 179–194, VSP, Utrecht, 1996.

- Molchanov, S.: Lectures on random media. Lectures on probability theory (Saint-Flour, 1992), 242–411, Lecture Notes in Math., **1581**, Springer, Berlin, 1994.

Plan of the talk

1. Motivation

2. Hierarchical structures and the free hierarchical Laplacian

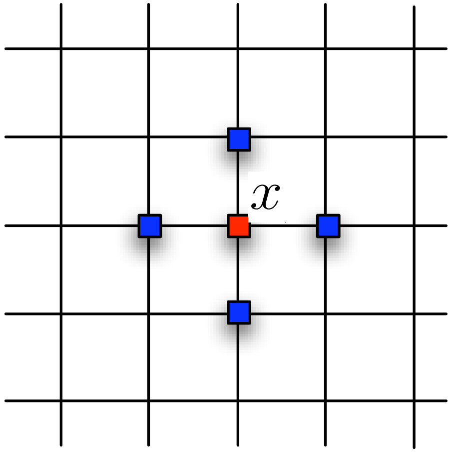
3. Hierarchical Anderson model

4. Spectral localization

5. Fine eigenvalue statistics

1. Motivation

Anderson tight binding model on $\mathbb{Z}^d$



Hilbert space: $l^2(\mathbb{Z}^d)$

Discrete laplacian: $(\Delta\psi)(x) = \sum_{|x-y|=1} \psi(y)$

Anderson model: $(H_\omega\psi)(x) = (\Delta\psi)(x) + \omega(x)\psi(x)$

$\omega(x)$ are i.i.d. random variables $U(-c, c)$.
Anderson conjecture (58) When $d \geq 3$, there exists $c_0 > 0$ such that

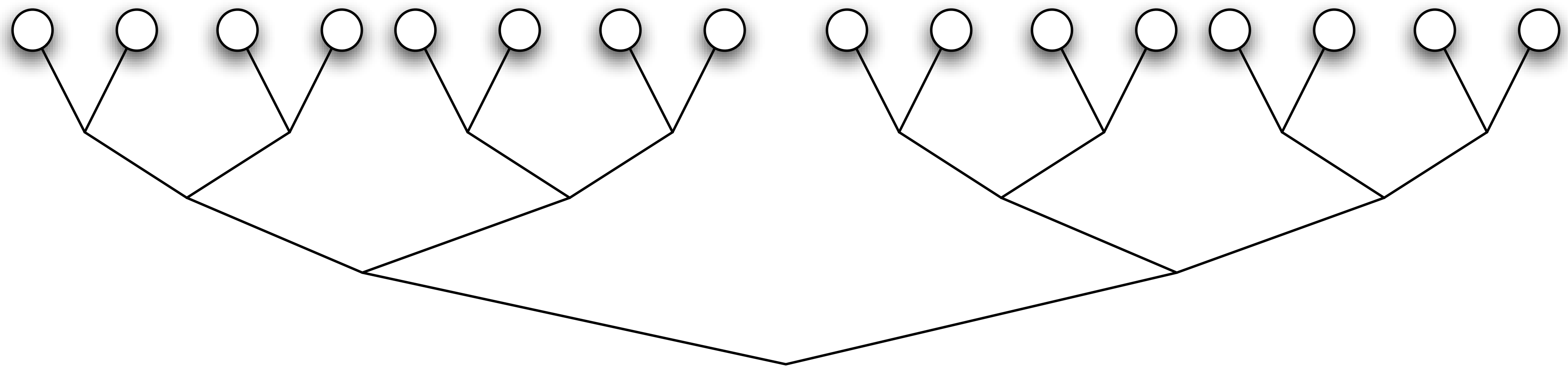
- (localization) $c > c_0 \Rightarrow$ a.s. H_ω has pure point spectrum and the eigenfunctions decay exponentially.
- (extended states) $c < c_0 \Rightarrow$ a.s. H_ω has some absolutely continuous spectrum.

known results $d = 1 \Rightarrow$ localization for all c (Goldsheid-Molchanov- Pastur 77, Kunz-Souillard 80). $d \geq 2 \Rightarrow$ localization for $c > c_0$ (Frohlich-Spencer 83, Aizenman-Molchanov 93). For small c, the presence of ac spectrum on tree graphs (Klein 98, Aizenman-Warzel-Simms 2005, Froese-Hasler-Spitzer 2005)

2. Hierarchical structures and the free hierarchical Laplacian

Example:

Homogeneous hierarchical structure on X of degree $n = 2$.



Hierarchical distance

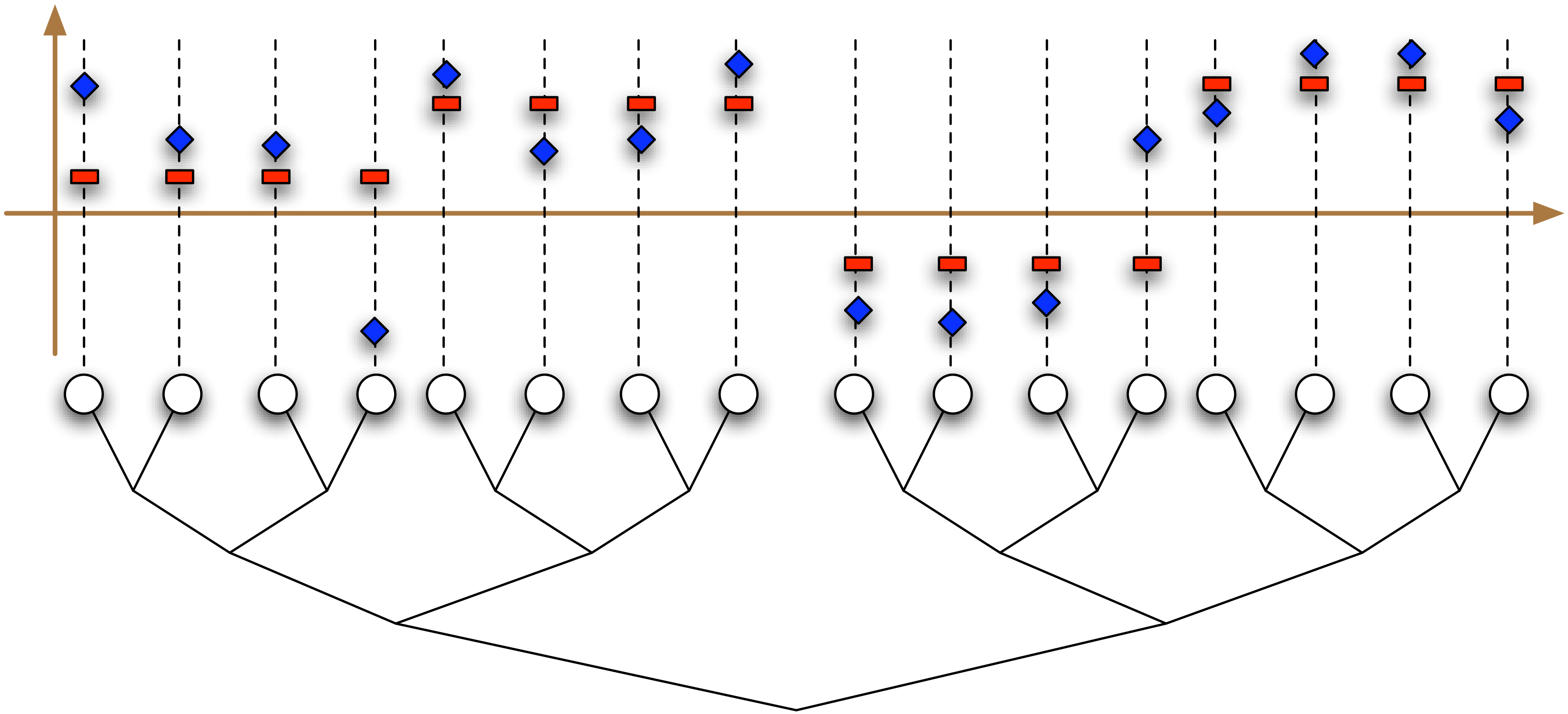
$$d(x, y) = \frac{1}{2} \text{ length of the shortest path } x \to y$$

Hilbert space $l^2(X) := \left\{\psi : X \to \mathbb{C} \text{ such that } \sum_{x \in X} |\psi(x)|^2 < \infty\right\}$.

Averaging operator $E_r : l^2(X) \to l^2(X)$

$$(E_r\psi)(x) := \frac{1}{n^r} \sum_{d(x,y) \leq r} \psi(y), \qquad r = 0, 1, 2, \cdots$$

Example: $\psi \in l^2(X)$ and $E_2\psi$



Hierarchical Laplacian

$$\Delta := \sum_{r=0}^{\infty} p_r E_r,$$

where $p_r \geq 0$ and $\sum_{r=0}^{\infty} p_r = 1$. Assume $p_0 = 0$.

Basic Properties of Δ

- Δ is a self-adjoint operator on $l^2(X)$
- $0 \leq \Delta \leq 1$
- $\sum_{y \in X} \langle \delta_x | \Delta \delta_y \rangle = 1$ and therefore Δ generates a random walk on X.

Explicit diagonalization of Δ

(1) Spectrum of Δ:

$$\lambda_r = \sum_{s=0}^{r} p_s, \qquad r = 0, \ldots, \infty.$$

Each λ_r, $r < \infty$, is an eigenvalue of Δ of infinite multiplicity. The point $\lambda_\infty = 1$ is not an eigenvalue.

(2) $E_r - E_{r+1}$ is the orthogonal projection onto the eigenspace of λ_r and

$$\Delta = \sum_{r=0}^{\infty} \lambda_r (E_r - E_{r+1}).$$

Spectral Measure and Dimension

Notation: $N_r = n^r =$ size of balls of radius r.

For every $x \in X$, the spectral measure μ for δ_x and Δ is given by

$$\mu = \sum_{r=0}^{\infty} \left(\frac{1}{N_r} - \frac{1}{N_{r+1}} \right) \delta(\lambda_r),$$

i.e.

$$\langle \delta_x | f(\Delta) \delta_x \rangle = \int f d\mu = \sum_{r=0}^{\infty} \left(\frac{1}{N_r} - \frac{1}{N_{r+1}} \right) f(\lambda_r).$$

The spectral dimension d is defined by

$$\lim_{t \downarrow 0} \frac{\log \mu([1-t, 1])}{\log t} = \mathrm{d}/2.$$

Motivation for this definition: discrete Laplacian on $l^2(\mathbb{Z}^d)$.

Proposition

Suppose that there exist constants $C_1 > 0, C_2 > 0$ and $\rho > 1$ such that

$$C_1\rho^{-r} \leq p_r \leq C_2\rho^{-r},$$

for r big enough. Then:

(1) The spectral dimension is

$$\mathrm{d}(n, \rho) = 2\frac{\log n}{\log \rho}.$$

Hence $0 < \mathrm{d}(n, \rho) \leq 2$ iff $n \leq \rho$.

(2) The random walk generated by Δ is recurrent if $0 < \mathrm{d}(n, \rho) \leq 2$ and transient if $\mathrm{d}(n, \rho) > 2$.

3. Hierarchical Anderson Model

Random self-adjoint operator

$$(H_\omega \psi)(x) = (\Delta\psi)(x) + \omega(x)\psi(x), \qquad \omega \in \Omega.$$

Probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega := \mathbb{R}^X$, $\mathcal{F}$ is the product σ-algebra in Ω, and $\mathbb{P}$ is a given probability measure on $(\Omega, \mathcal{F})$.

Questions:

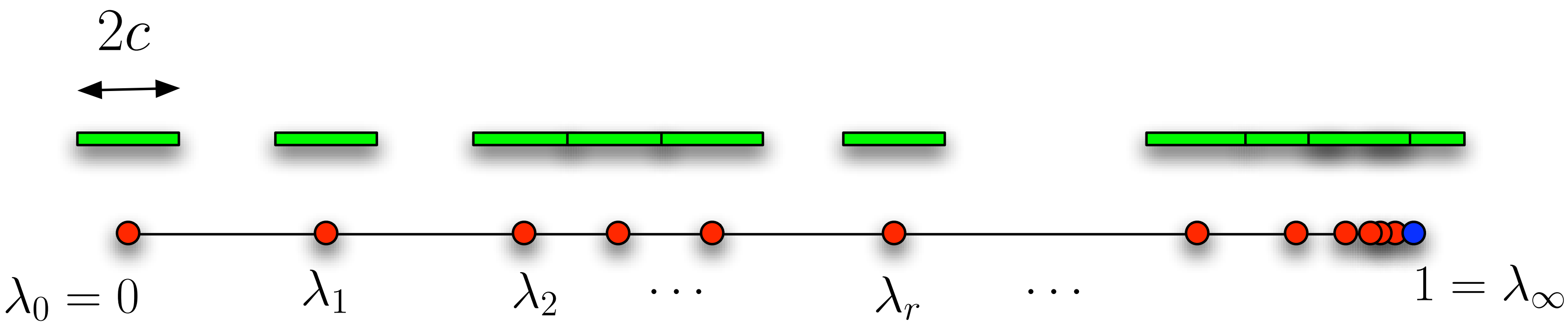
- When is the spectrum of H_ω nonrandom?
- What is the spectral type?
- What are the fine eigenvalue statistics?

Spectrum of H_ω

Assume (1) $\{\omega(x) : x \in X\}$ are i.i.d. with distribution ν, i.e. $\mathbb{P} = \otimes_{x \in X} \nu$ and assume (2) $S := support(\nu)$ is connected. THEN a.s.

$$\mathrm{sp}(\mathrm{H}_\omega) = \mathrm{sp}(\Delta) + \mathrm{S}.$$

Example 1: generic structure of $\mathrm{sp}(\mathrm{H}_\omega)$ when $\omega(x)$ are i.i.d. $U(-c, c)$.



Example 2: If $\omega(x)$ are i.i.d. $N(0,1)$, then $\mathrm{sp}(\mathrm{H}_\omega) = \mathbb{R}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.

4. Spectral Localization

Theorem 1 (Molchanov, 1996): Assume

$$(*) \qquad \sum_{r=1}^{\infty} p_r r^{1+\varepsilon} < \infty, \qquad \text{for some } \varepsilon > 0.$$

Assume $\{\omega(x) : x \in X\}$ are i.i.d. Cauchy random variables. THEN: $\mathrm{sp}_{\mathrm{cont}}(\mathrm{H}_\omega) = \emptyset$ almost surely.

Theorem 2 (K, 2006): Assume

$$(**) \qquad \sum_{r=1}^{\infty} p_r r^{1+\varepsilon} \sqrt{N_r} < \infty, \qquad \text{for some } \varepsilon > 0.$$

THEN:

(1) For all $\omega \in \mathbb{R}^X$, $\mathrm{sp}_{\mathrm{ac}}(\mathrm{H}_\omega) = \emptyset$.

(2) If $\{\omega(x) : x \in X\}$ are i.i.d. with density, then $\mathrm{sp}_{\mathrm{cont}}(\mathrm{H}_\omega) = \emptyset$ almost surely.

4. Fine eigenvalue statistics

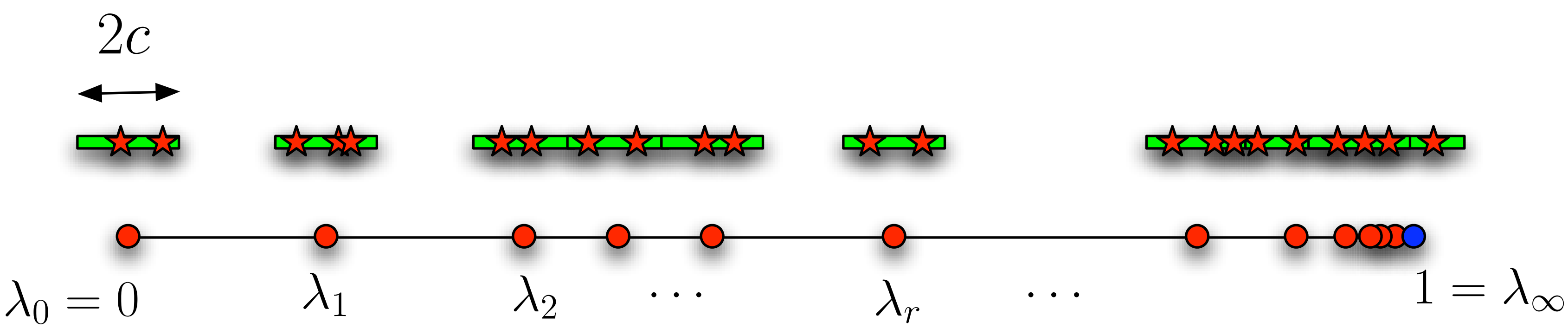
Finite volume

$$B_r := \{y : d(x_0, y) \le r\}, \qquad r = 0, 1, 2, \cdots .$$

Approximations $H_r^\omega : l^2(B_r) \to l^2(B_r)$

$$(H_r^\omega \psi)(x) := \sum_{s=1}^{r} p_s (E_s \psi)(x) + \omega(x)\psi(x).$$

H_r^ω is a random self-adjoint $N_r \times N_r$. Random eigenvalues $e_1, e_2, \cdots, e_{N_r}$
Example 1: $\omega(x)$ are i.i.d. $U(-c, c)$.



If $e \in \mathbb{R}$ is given and $\varepsilon > 0$ is small, then the number of eigenvalues of H_r^ω in $(e-\varepsilon, e+\varepsilon)$ is of size εN_r. To study the local fluctuations of eigenvalues near e we define the random point measure

$$\int f d\xi_r = Trf(N_r(H_r^\omega - e)) = \sum_{j=1}^{N_r} f(N_r(e_j - e)).$$

Theorem Assume that the spectral dimension $\mathrm{d} < 1$ and that $\omega(x)$ are i.i.d. with bounded density. Then, for Lebesgue almost all $e \in \mathbb{R}$, ξ_r converges to a homogeneous Poisson process as $r \to \infty$.