Flat Connections on Riemann Surfaces

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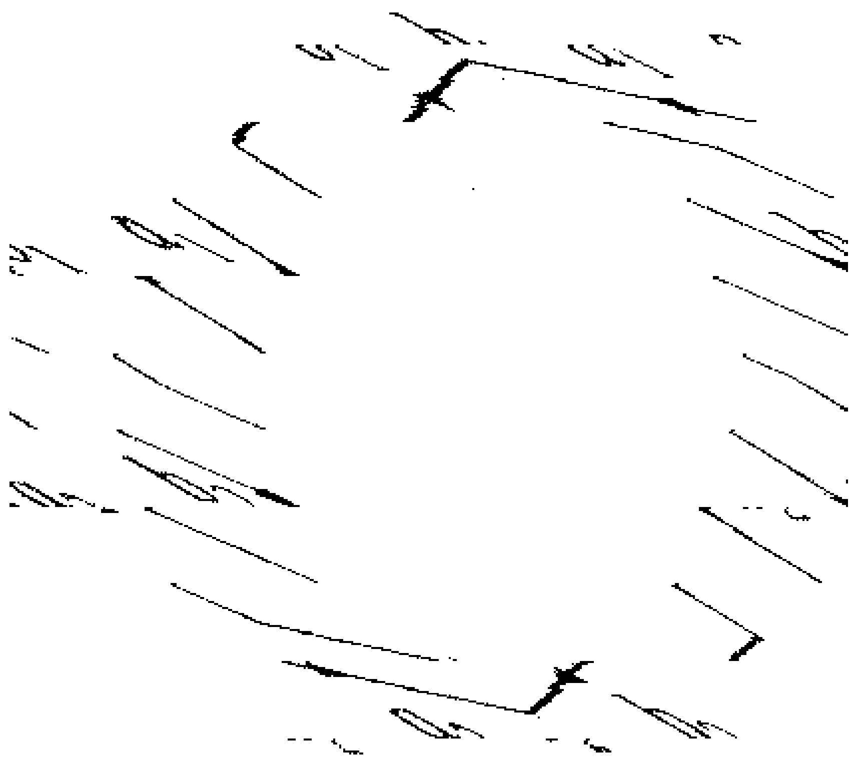
I. Introduction

Let Σ be a compact Riemann surface (2-dim'l orientable manifold).

Σ can be described in different ways depending on how much structure we choose to specify:

Topological description:

As topological spaces, Riemann surfaces (orientable manifolds of dimension 2) are formed by gluing together edges of a polygon with $4g$ sides:



Riemann surfaces

Topological description: Riemann surfaces are classified by their fundamental group (the equivalence classes of loops under deformation). For genus g this is

$$\pi = \pi_1(\Sigma^g) =< a_1, b_1, \ldots, a_g, b_g : \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} = 1 >$$

The a_j, b_j are a basis of $H_1(\Sigma)$, chosen so that their intersection numbers are

$$a_j \cap b_j = 1$$

and all other intersections are zero.

Smooth description: Riemann surfaces of genus g are smooth manifolds, and all smooth structures on a compact Riemann surface of genus g are equivalent up to diffeomorphism.

Holomorphic description: Riemann surfaces may also be described as complex manifolds.

Topological spaces associated to Σ

One may associate topological spaces (*moduli spaces*) which admit different descriptions depending on the amount of structure with which we have equipped our Riemann surface.

A prototype is the *Jacobian*, which can be described in several ways:

Topological description: If we view Σ as a topological space and retain only the structure of its fundamental group, we may define

$$\mathrm{Jac}(\Sigma) = \mathrm{Hom}\,(\pi, U(1)) = U(1)^{2g}$$

Smooth description: If we view Σ as a smooth manifold, the Jacobian has a gauge theory description

$$\mathrm{Jac}(\Sigma) = \text{ flat } U(1) \text{ connections /gauge group}$$

$$\mathcal{A} = \{A = A_1 dy_1 + A_2 dy_2 : A_1, A_2 \in C^\infty(\Sigma)\}$$

$$(\ U(1) \text{ connections})$$

The *gauge group* is $\mathcal{G} = C^\infty(\Sigma, U(1))$; its Lie algebra is $\mathrm{Lie}(\mathcal{G}) = C^\infty(\Sigma)$.

We find that

$$\mathcal{A}_{\text{flat}}/\mathcal{G} = \mathbf{R}^{2g}/\mathbf{Z}^{2g} = U(1)^{2g}.$$

Holomorphic description: If we endow Σ with a complex structure, $\mathrm{Jac}(\Sigma)$ is identified with the moduli space which classifies *holomorphic line bundles* over Σ: this is how the Jacobian arises naturally in algebraic geometry.

What happens when we replace $U(1)$ by a compact *nonabelian* group G (e.g. $G = SU(2)$, or more generally $G = SU(n)$)?

Topological description:

The natural generalization of $\mathrm{Jac}(\Sigma)$ is

$$\mathcal{M}(\Sigma) = \mathrm{Hom}\,(\pi, G)/G$$

where G acts on $\mathrm{Hom}\,(\pi, G)$ by conjugation.

Smooth description: $\mathcal{M}(\Sigma)$ has a natural gauge theory description which generalizes the description of the Jacobian:

$$\mathcal{M}(\Sigma) \cong \text{flat } G \text{ connections on } \Sigma$$

up to gauge equivalence.

Holomorphic description: $\mathcal{M}(\Sigma)$ also has a description in algebraic geometry: it is the moduli space of holomorphic $G^{\mathbf{C}}$ bundles over Σ, with an appropriate GIT stability condition .

Example: $G = U(n)$

$\mathcal{M}(\Sigma)$ is the moduli space of (semistable) holomorphic vector bundles of rank n and degree 0 over Σ.

Different descriptions of moduli spaces

Moduli spaces of flat connections on Riemann surfaces arise in a number of different contexts:

1. **Gauge theory**: The properties of these moduli spaces are a prototype for properties of moduli spaces arising in gauge theory related to manifolds of dimension higher than 2 (e.g. Donaldson or Seiberg-Witten invariants in dimension 4; Floer homology in dimension 3...)

Their properties may be useful in understanding problems involving Riemann surfaces embedded in manifolds of higher dimension.

2. **Topology:** These moduli spaces provide a natural setting for various questions involving topology of manifolds of dimension 2 and 3.

Example: The Casson invariant is an invariant of 3-manifolds; it arises naturally as the intersection number of two Lagrangian submanifolds in a moduli space of flat connections.

3. **Mathematical physics:** These moduli spaces arise from the study of the *Yang-Mills equations* on a manifold of dimension 2.

Many topics of recent interest in quantum field theory are related to them (for example, Chern-Simons gauge theory).

4. **Algebraic geometry:** These moduli spaces have surprising properties in the context of algebraic geometry.

Example: the *Verlinde formula*, a formula for the dimension of the space of holomorphic sections of a line bundle $\mathcal{L}$ over the moduli space. This formula is remarkable since it is usually difficult to explicitly determine the number of holomorphic sections of a bundle over a complex manifold.

5. **Symplectic geometry:**

These moduli spaces are *symplectic manifolds* and may be studied from that point of view.

6. **Relation to symplectic and geometric invariant theory quotients:**

Atiyah and Bott (1982) exhibited these moduli spaces as *symplectic quotients* (an infinite dimensional construction): the space of all connections $\mathcal{A}$ on Σ is acted on by the gauge group $\mathcal{G}$ with moment map the curvature

$$\mu : A \mapsto F_A$$

so the symplectic quotient is the moduli space of flat connections up to equivalence under the action of the gauge group.

These moduli spaces are interesting examples of quotient constructions in symplectic geometry and geometric invariant theory.

Atiyah-Bott; Guillemin-Sternberg; Kirwan; Mumford – The symplectic quotient by a (compact) group G is equivalent to the geometric invariant theory quotient by the complexification $G^{\mathbf{C}}$.

(Mumford, Fogarty, Kirwan, *GIT*)

II. Moduli spaces of flat connections on Riemann surfaces

II.1 General properties

The objects of interest are the flat connections on Σ.

A connection specifies a way to do parallel transport in a principal bundle over Σ with structure group G. If the bundle can be trivialized, it is equivalent to the product bundle $\Sigma \times G$.

There are many different ways to specify the trivialization: the choice of a trivialization is given by an element of the *gauge group*

$$\mathcal{G} = \text{Maps}(\Sigma, G),$$

the (smooth) maps from Σ to G. The Lie algebra of $\mathcal{G}$ is the smooth maps from Σ to Lie(G). The gauge group acts on the space of connections:

$$A \mapsto g^{-1} A g + g^{-1} dg$$

For any closed loop γ in Σ, a connection determines a *holonomy*, which is the group element g such that the image of parallel transport around γ starting at a point $\tilde{\gamma}(0)$ in the fiber above $\gamma(0)$ is obtained by multiplying $\tilde{\gamma}(0)$ by g.

If the connection is *flat* (the curvature F_A is zero), then the parallel transport is not changed by continuous deformations of the loop (as long as these deformations keep the beginning point of the loop fixed). It depends only on the class of the loop as an element of the *fundamental group* π (equivalence classes of loops under deformation).

In fact the action of the gauge group takes the subspace of flat connections to itself.

It is not hard to see that the space of gauge equivalence classes of flat connections is identified with the representations of π into G (mod conjugation).

This space is called the *moduli space* $\mathcal{M}$ of flat connections mod gauge transformations.

Via the holonomy, a flat connection determines a representation of the fundamental group. In fact all representations arise in this way and the correspondence makes the two sets equivalent.

Representations of the fundamental group of Σ

Ex. 1 2-sphere S^2

The fundamental group of the 2-sphere is trivial (any closed loop can be shrunk to a point)

Ex. 2 Torus $S^1 \times S^1$

The fundamental group of the torus is generated by two loops a and b and they commute with each other (the diagram shows that $ab = ba$). Thus the fundamental group is commutative.

Ex. 3: Higher genus

More generally, a Riemann surface of genus g (g-holed torus) is formed by taking a polygon with $4g$ sides and gluing the sides together. The sides of the polygon become the generators of the group.

Now, however, the group is not commutative: from the information that the loop around the outside of the polygon can be shrunk to a point we learn only that the generators satisfy the relation

$$a_1 b_1 (a_1)^{-1} (b_1)^{-1} \dots a_g b_g (a_g)^{-1} (b_g)^{-1} = 1$$

In order to specify a representation ρ of π into a compact Lie group G we must specify the elements A_i, B_i in G to which ρ sends each loop a_i, b_i. In order that it should be a representation we insist that the relation is preserved:

$$A_1 B_1 (A_1)^{-1} (B_1)^{-1} \dots A_g B_g (A_g)^{-1} (B_g)^{-1} = 1 \tag{I}$$

We must also quotient out the action of G by conjugation on the space of representations:

$$g \in G : A_i \mapsto g^{-1} A_i g; B_i \mapsto g^{-1} B_i g.$$

Ex. 4 $G = U(1)$, the circle group. Note that this group is commutative, so the conjugation action is the identity map

$$A_i \mapsto A_i, B_i \mapsto B_i$$

for any $g \in G$. Also, any elements A_i, B_i of G automatically satisfy the relation (I) because $A_i B_i (A_i)^{-1} (B_i)^{-1} = 1$ for any A_i and B_i.

So for this group the space $\mathcal{M}$ is simply $U(1)^{2g}$.

Ex. 5 $\Sigma = S^1 \times S^1$, $G = U(n)$ (not commutative)

In this case if we choose elements A and B in G to represent the two loops a and b in $S^1 \times S^1$, we need to insist that

$$AB = BA$$

(because $ab = ba$ in π). Every element of G is conjugate to a diagonal matrix with unit complex numbers $e^{i\theta}$ along the diagonal (i.e. it can be diagonalized).

If we have diagonalized A, in general the only elements B which commute with it are the diagonal matrices with unit complex number entries. Call the space of such matrices T.

If A and B are both in T, what is left over of the conjugation action (i.e. what elements of G will conjugate T into itself)?

In general the elements of G that will do this act via a finite group isomorphic to the permutation group S_n on n letters, which acts by permuting the diagonal entries.

(Note: this is the *Weyl group* $W = N(T)/T$)

So we find that

$$\mathcal{M} = T \times T/W$$

Ex. 6: The general case

$G = U(n)$, Riemann surface with $g > 1$: In the case when G is a noncommutative group (such as $U(n)$) the moduli space is more complicated, since:

1. The relation between the images of the generators (imposed by the fact that the loop around the boundary of the polyhedron can be shrunk to a point) is no longer automatically satisfied

2. The action of the group by conjugation is now nontrivial

In fact in this case the moduli space $\mathcal{M}$ is not smooth; we replace it by a smooth analogue obtained by cutting out a small disc in Σ and requiring that the representation send the loop around the boundary of the disc not to 1 but to the product of the identity matrix and a root of unity $e^{2\pi i d/n}$ which generates the n-th roots of unity. This moduli space (denoted $M(n, d)$) is in fact smooth, and shares many properties with the more natural space $\mathcal{M}$.

II.2 Connections

The space $\mathcal{A}$ of all connections is simply the vector space of 1-forms tensored with Lie(G).

What about the tangent space to the space of *flat* connections?

The curvature is the following quantity

$$F_A = dA + \frac{1}{2}[A, A]$$

Infinitesimally, if $F_A = 0$ the condition that $F_{A+a} = 0$ translates to

$$da + [A, a] = 0$$

We write this as

$$d_A a = 0$$

d_A is an operator taking Lie(G)-valued differential forms of degree p to Lie(G)-valued differential forms of degree $p + 1$ and satisfying

$$d_A \circ d_A = 0.$$

Tangent space

At the infinitesimal level, the image of 0-forms under d_A is the tangent space to the orbits of the group of gauge transformations. Thus the tangent space to the moduli space $\mathcal{M}$ is the space

$$H^1(\Sigma, d_A) = \frac{\{a \in \Omega^1 \otimes \mathrm{Lie}(G) \ \mid \ d_A a = 0\}}{\{d_A \phi \mid \phi \in \Omega^0 \otimes \mathrm{Lie}(G)\}}.$$

We can see from this that $\mathcal{M}$ has a *symplectic form*, a nondegenerate skew-symmetric pairing on the tangent space.

At the level of the vector space $\mathcal{A}$ of all connections, this just comes from the wedge product on differential forms, combined with an inner product $<, >$ on $\mathrm{Lie}(G)$.

One can see (using Stokes' theorem) that the wedge product descends to a skew-symmetric pairing on $H^1(\Sigma, d_A)$; in fact this pairing is nondegenerate. (If G is abelian, this pairing is just the cup product on de Rham cohomology.)

It follows that $\mathcal{M}$ has a symplectic structure.

II.3 Cohomology of $U(1)$ moduli spaces

Let us revisit the space $\mathcal{M}$ regarded as flat connections modulo gauge transformations in the case when $G = U(1)$.

A connection A is simply a 1-form $\sum_{i=1}^{2} A_i dx^i$ on Σ.

The holonomy of A around a cycle $\gamma(t)$ in Σ is

$$e^{i \int_\gamma A},$$

since the parallel transport satisfies the equation

$$\frac{dg}{dt} = iA(\gamma(t))g(t).$$

The connection A is flat $\iff dA = 0$ in terms of the exterior differential d.

The image of an infinitesimal gauge transformation $\exp i\phi$ (where ϕ is a function on Σ) is the connection $d\phi$.

Thus when we take the quotient of the flat connections by those that arise as the image of infinitesimal gauge transformations we get

$$\frac{\{A|dA = 0\}}{\{d\phi\}}$$

In de Rham cohomology this give the first cohomology $H^1(\Sigma; \mathbf{R})$ of $U(1)^{2g}$, which is the direct sum of $2g$ copies of the first cohomology group of $U(1)$:

$$H^1(\Sigma; \mathbf{R}) \cong \mathbf{R}^{2g}$$

If we want to take the quotient of the space of flat connections not only by those gauge transformations which arise as images of infinitesimal gauge transformations under the exponential map, but rather by the full gauge group, we must divide by an additional $\mathbf{Z}^{2g}$:

$\mathcal{A}_F/\mathcal{G} \cong \mathbf{R}^{2g}/\mathbf{Z}^{2g} \cong (S^1)^{2g}$.

We recover our previous description of the moduli space.

The generators of the cohomology come from the generators of the cohomology for $U(1)$:

$$S^1 = U(1) = \{e^{i\theta}\}$$

so the cohomology is generated by the 1-form $d\theta$.

There is one relation $d\theta \wedge d\theta = 0$ (since there are no nonzero 2-forms on the 1-dimensional manifold $U(1)$).

Thus the cohomology of the moduli space $U(1)^{2g}$ has $2g$ generators $d\theta_i$, $i = 1, \ldots, 2g$ and the only relations are that

$$d\theta_i \wedge d\theta_j = -d\theta_j \wedge d\theta_i$$

(and in particular $d\theta_i \wedge d\theta_i = 0$).

So the cohomology is an exterior algebra on $2g$ generators of degree 1.

II.4 Cohomology: the general case

We would like to find the analogues of these generators and relations for the case of $\mathcal{M}$ when G is a noncommutative group such as $U(n)$ or $SU(n)$.

Here the generators of the cohomology ring are obtained as follows. There is a vector bundle $\mathcal{U}$ (the "universal bundle") over $\mathcal{M} \times \Sigma$, so for each point x in $\mathcal{M}$ the restriction to $x \times \Sigma$ is a vector bundle over Σ. We take a connection A on $\mathcal{U}$ and decompose polynomials in its curvature F_A (for example $\mathrm{Trace}(F_A^n)$) into the product of closed forms on Σ and closed forms on $\mathcal{M}$.

We then integrate these forms over cycles in Σ (a point or 0-cycle, the 1-cycles a_i and b_i, or the 2-cycle given by the entire Riemann surface Σ) to produce closed forms on $\mathcal{M}$, which represent the generators of the cohomology ring of $\mathcal{M}$.

These classes generate the cohomology of $\mathcal{M}$ (if we are allowed to multiply them as well as to add them).

Ex. 1 One important cohomology class is the cohomology class of the symplectic form on $\mathcal{M}$.

Ex. 2 Another important family of classes are those obtained by evaluating the classes on $\mathcal{M} \times \Sigma$ at a point in Σ.

III. Witten's formulas

Witten (1991-2) obtained formulas for intersection numbers in the cohomology of moduli spaces.

In particular, he obtained formulas for the symplectic volume of the moduli spaces.

For $SU(2)$ these formulas are as follows.

Example: $n = 2, d = 1$ **(Donaldson 1992; Thaddeus 1991)**

The structure of the cohomology ring can be reduced to knowing the intersection numbers of all powers of the two even dimensional generators

- $a \in H^4(M(2,1))$
- $f \in H^2(M(2,1))$

These are as follows

(where a is the class arising from the inner product

$$Q(B) = \mathrm{Trace}(B^2)$$

evaluated at a point in Σ, and f is the cohomology class of the symplectic form on $M(2,1)$):

$$\int_{M(2,1)} a^j \exp f$$

$$= \frac{(-1)^j}{2^{g-2}\pi^{2(g-1-j)}} \sum_{n>0} \frac{(-1)^{n+1}}{n^{2g-2-2j}}$$

$$= \frac{(-1)^j}{2^{g-2}\pi^{2(g-1-j)}}(1 - 2^{2g-3-2j})\zeta(2g-2-2j)$$

Here we have used

$$\exp f = \sum_{m \geq 0} \frac{f^m}{m!}$$

and we use the fact that $\int_{M2,1} \alpha = 0$

(where the integral denotes evaluation on the fundamental class of $M(2,1)$) unless the degree of α equals the dimension of $M(2,1)$.

We note that the formulas for intersection numbers can be written in terms of a sum over irreducible representations of G.

Ex.: The symplectic volume of the moduli space $\mathcal{M}$ of flat G connections is given by the "Witten zeta function":

$$\int_{\mathcal{M}} \exp(f) \sim \sum_R \frac{1}{(\dim R)^{2g-2}}$$

where we sum over irreducible representations R of G. In the special case of $SU(2)$ we have

$$\int_{M(2,1)} \exp(f) \sim \sum_n \frac{(-1)^{n+1}}{n^{2g-2}}$$

where we sum over the irreducible representations of $SU(2)$, which are parametrized by their dimension n.

IV. Mathematical proof of Witten's formulas

The moduli space is a symplectic quotient

$$\mu^{-1}(0)/\mathcal{G},$$

where μ is the moment map (collection of Hamiltonian functions whose Hamiltonian flow generates the action of a group $\mathcal{G}$ on a symplectic manifold M):

1. Infinite-dimensional symplectic quotient of the space of all connections $\mathcal{A}$ by the gauge group $\mathcal{G}$: the moment map of a connection A is its curvature F_A, so $\mu^{-1}(0)/\mathcal{G}$ is the moduli space $\mathcal{M}$.

2. Finite-dimensional symplectic quotient of a (finite dimensional) space of flat connections on a punctured Riemann surface, by the action of the finite-dimensional group G

We use formulas (Jeffrey-Kirwan) for intersection numbers in a symplectic quotient, in terms of the restriction to the fixed points of the action of a maximal commutative subgroup of G (for $G = U(n)$ this subgroup is the diagonal matrices $U(1)^n$): the answer is given in terms of

1. The action of G on the tangent space at the fixed points
2. The values of the moment map at the fixed points
3. The restriction of differential forms to the fixed point set of the $U(1)^n$ action

Using these methods we recover Witten's formulas.

Dictionary between physics and mathematics

Mathematics	Physics
• connections $\Omega^1(\Sigma) \otimes \mathbf{g}$	Fields: $\mathcal{A}$
• flat connections $\mathcal{M}$	• Extrema: $\delta L_A = 0$ (Euler-Lagrange)
• generators of the ring $H^*(\mathcal{M})$	• Observables (BRST cohomology classes)
• exterior differential d_A	• BRST differential
• symplectic volume of $\mathcal{M}$	• Partition function
• intersection nos. in $\mathcal{M}$	• Correlation functions

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