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Span of translates in $L^p(\mathbb{R})$, and zeros of Fourier transform

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Joint work with A. Olevskii

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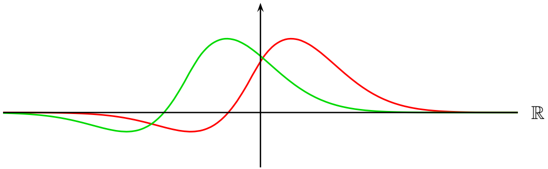

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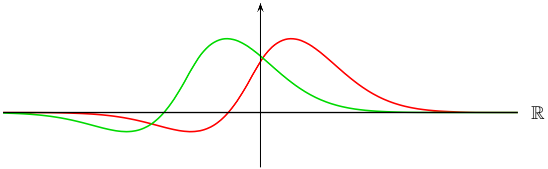

- **Problem: given such V, when is it dense in $L^p(\mathbb{R})$?**

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- Partial results due to Beurling, Edwards, Herz, Kinukawa, Newman, Segal.
- Beurling (1951): Let $V \subset (L^1 \cap L^p)(\mathbb{R})$.

$$\dim Z(V) < 2 - \frac{2}{p} \quad \Rightarrow \quad V \text{ is dense in } L^p(\mathbb{R}).$$

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For $1 < p < \frac{4}{3}$ it is impossible to characterize the translation invariant subspaces V which are dense in $L^p(\mathbb{R})$ in terms of $Z(V)$.

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Main Result

For $1 < p < \frac{4}{3}$ it is impossible to characterize the translation invariant subspaces V which are dense in $L^p(\mathbb{R})$ in terms of $Z(V)$.

- Moreover: characterization of generators φ in $L^p(\mathbb{R})$ is impossible in terms of the zeros of $\widehat{\varphi}$.

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Theorem

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- The same for $\ell_p(\mathbb{Z})$.
- Probably for more general non-compact groups.
- $\frac{4}{3}$ = conjugate of 4, uses a special property of the ℓ_4 norm:

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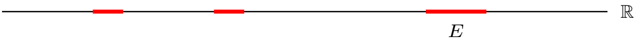

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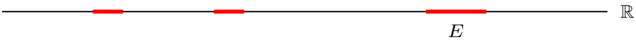

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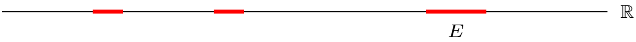

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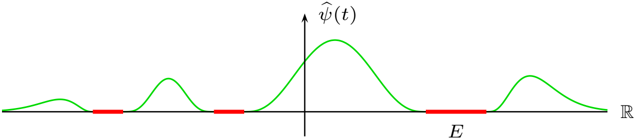

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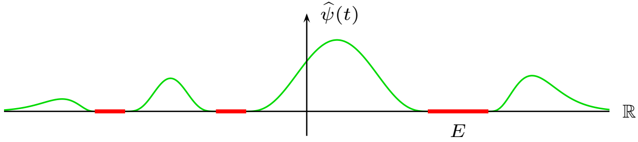

$$\int_{\mathbb{R}} \psi(x-\lambda)\, \widehat{S}(x)\, dx = \langle \widehat{\psi}(t)\, e^{-i\lambda t}, S \rangle = 0. \qquad \square$$

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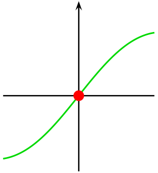
The support of δ' is $\{0\}$.

$\langle f, \delta' \rangle = -f'(0)$



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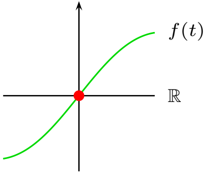

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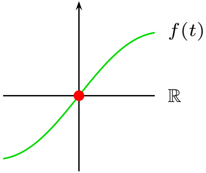

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- Related to theory of *spectral synthesis*.

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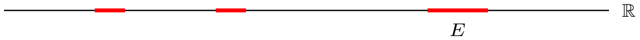

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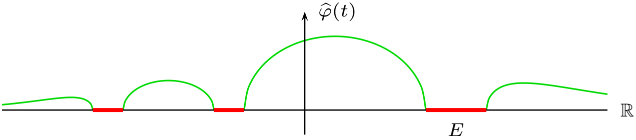

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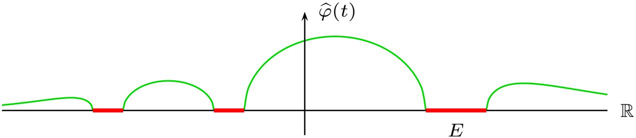

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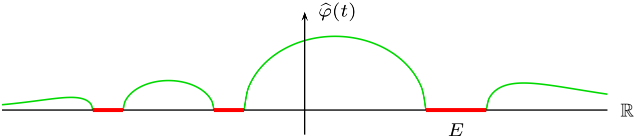

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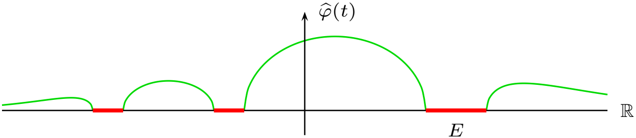

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- The arithmetic structure of E plays a crucial role.

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