VANDERBILT UNIVERSITY

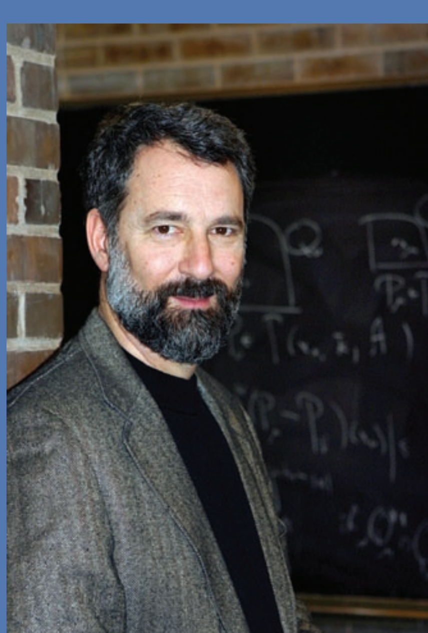
Department of Mathematics • Nashville, Tennessee, USA

The 24th Southeastern Analysis Meeting

in conjunction with The 23rd Annual

SHANKS LECTURE

honoring Baylis and Olivia Shanks



featuring

Charles Fefferman

Princeton University

Wednesday, March 5 to Sunday, March 9, 2008

Supported by the Shanks Endowment, the National Science Foundation, and Vanderbilt University

PLENARY SPEAKERS:

Ron Douglas
Texas A&M University

Charles Fefferman
Princeton University

Guihua Gong
University of Puerto Rico

Michael Lacey
Georgia Institute of Technology

John McCarthy
Washington University, St. Louis

Vladimir Peller
Michigan State University

Stefan Richter
University of Tennessee, Knoxville

Richard Rochberg
Washington University, St. Louis

Kristian Seip
Norwegian University of Science and Technology

Sergei Treil
Brown University

Alexander Volberg
Michigan State University

Jingbo Xia
SUNY at Buffalo

ORGANIZING COMMITTEE:

Brett Wick
University of South Carolina

Daoxing Xia
Vanderbilt University

Dechao Zheng
Vanderbilt University

ADDITIONAL INFORMATION:

Email: seam2008@vanderbilt.edu

Web: www.math.vanderbilt.edu/~seam2008

Postal mail:
SEAM 2008
Department of Mathematics
Stevenson Center 1326
Vanderbilt University
Nashville, TN 37240, USA

Phone: (615) 322-6672

Fax: (615) 343-0215

VANDERBILT UNIVERSITY

Beurling type theorem on the Bergman space via the Hardy space of the bidisk

Dechao Zheng

Vanderbilt University

This is a joint work with Shunhua Sun.

Beurling Theorem

The famous Beurling theorem classifies the invariant subspaces of the multiplication operator by the coordinate function z on the Hardy space of the unit disk, the unilateral shift S :

$$Se_n = e_{n+1}$$

for $e_n = z^n$.

Beurling Theorem

The famous Beurling theorem classifies the invariant subspaces of the multiplication operator by the coordinate function z on the Hardy space of the unit disk, the unilateral shift S :

$$Se_n = e_{n+1}$$

for $e_n = z^n$.

Theorem (Beurling Theorem)

Let $\mathcal{M}$ be in Lat S.

$$\mathcal{M} = \theta H^2,$$

for some inner function θ.

Beurling Theorem

The famous Beurling theorem classifies the invariant subspaces of the multiplication operator by the coordinate function z on the Hardy space of the unit disk, the unilateral shift S :

$$Se_n = e_{n+1}$$

for $e_n = z^n$.

Theorem (Beurling Theorem)

Let $\mathcal{M}$ be in Lat S.

$$\mathcal{M} = \theta H^2,$$

for some inner function θ. This is equivalent to

(1)

$$\mathcal{M} \ominus S(\mathcal{M}) = \mathbb{C}\theta;$$

(2)

$$\mathcal{M} = [\mathcal{M} \ominus S(\mathcal{M})].$$

Here $[A] = span_{n\geq 0} z^n A$ for a subset A of H^2.

Bergman shift

- Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.

Bergman shift

- Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.
- Let dA denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$.

Bergman shift

- Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.
- Let dA denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$.
- The Bergman space

$$L_a^2 = \{f \in L^2(\mathbb{D}, dA) : f \text{ is analytic in } \mathbb{D}\}$$

Bergman shift

- Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.
- Let dA denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$.
- The Bergman space

$$L_a^2 = \{f \in L^2(\mathbb{D}, dA) : f \quad \text{is analytic in } \mathbb{D}\}$$

- Let $e_n = \sqrt{n+1}z^n$. Then $\{e_n\}_{n=0}^{\infty}$ form an orthonormal basis of L_a^2.

Bergman shift

- Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$.
- Let dA denote normalized Lebesgue area measure on the unit disk $\mathbb{D}$.
- The Bergman space

$$L_a^2 = \{f \in L^2(\mathbb{D}, dA) : f \ \text{ is analytic in } \mathbb{D}\}$$

- Let $e_n = \sqrt{n+1}z^n$. Then $\{e_n\}_{n=0}^{\infty}$ form an orthonormal basis of L_a^2.
- M_z is the Bergman shift

$$M_z e_n = \sqrt{\frac{n+1}{n+2}}e_{n+1}.$$

Theorem (Apostol-Bercovici-Foias-Pearcy Theorem)

Let $\mathcal{M}$ be in $LatM_z$.

$$dim\{\mathcal{M} \ominus M_z(\mathcal{M})\} \leq \infty.$$

Dimension of the wandering subspace

Theorem (Apostol-Bercovici-Foias-Pearcy Theorem)

Let $\mathcal{M}$ be in $LatM_z$.

$$dim\{\mathcal{M} \ominus M_z(\mathcal{M})\} \leq \infty.$$

- Hedenmalm, Hedenmalm-Richter-Seip
- Borichev, Borichev-Hedenmalm-Volberg

Beurling type Theorem

Theorem (Aleman-Richter-Sundberg Theorem)

Let $\mathcal{M}$ be in $LatM_z$.

$$\mathcal{M} = [\mathcal{M} \ominus M_z(\mathcal{M})],$$

where $[\mathcal{M} \ominus M_z(\mathcal{M})] = span_{k \geq 0} M_z^k(\mathcal{M} \ominus M_z(\mathcal{M}))$.

Beurling type Theorem

Theorem (Aleman-Richter-Sundberg Theorem)

Let $\mathcal{M}$ be in $LatM_z$.

$$\mathcal{M} = [\mathcal{M} \ominus M_z(\mathcal{M})],$$

where $[\mathcal{M} \ominus M_z(\mathcal{M})] = span_{k\geq 0} M_z^k(\mathcal{M} \ominus M_z(\mathcal{M}))$.

This result is a breakthrough in understanding of the invariant subspaces of the Bergman space and becomes a fundamental theorem in the function theory on the Bergman space.

Beurling type Theorem

Theorem (Aleman-Richter-Sundberg Theorem)

Let $\mathcal{M}$ be in $LatM_z$.

$$\mathcal{M} = [\mathcal{M} \ominus M_z(\mathcal{M})],$$

where $[\mathcal{M} \ominus M_z(\mathcal{M})] = span_{k \geq 0} M_z^k(\mathcal{M} \ominus M_z(\mathcal{M}))$.

This result is a breakthrough in understanding of the invariant subspaces of the Bergman space and becomes a fundamental theorem in the function theory on the Bergman space.

Different proofs of the Beurling type theorem were given by

- Shimorin (2001),
- McCullough-Richter (2002) and
- Olofsson (2005).

Outline of the proof of the Aleman, Richter and Sundberg Theorem via the Hardy space of the bidisk

Outline of the proof of the Aleman, Richter and Sundberg Theorem via the Hardy space of the bidisk

- Lift the Bergman shift M_z as the compression $\mathcal{B}$ of the isometry T_z on a nice subspace $\mathcal{H}$ of the Hardy space of the bidisk.

Outline of the proof of the Aleman, Richter and Sundberg Theorem via the Hardy space of the bidisk

- Lift the Bergman shift M_z as the compression $\mathcal{B}$ of the isometry T_z on a nice subspace $\mathcal{H}$ of the Hardy space of the bidisk.
- Lift an invariant subspace $\mathcal{M}$ of $\mathcal{B}$ to be an invariant subspace $\widetilde{\mathcal{M}}$ of the isometry T_z.

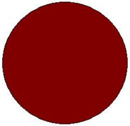
Outline of the proof of the Aleman, Richter and Sundberg Theorem via the Hardy space of the bidisk

- Lift the Bergman shift M_z as the compression $\mathcal{B}$ of the isometry T_z on a nice subspace $\mathcal{H}$ of the Hardy space of the bidisk.
- Lift an invariant subspace $\mathcal{M}$ of $\mathcal{B}$ to be an invariant subspace $\widetilde{\mathcal{M}}$ of the isometry T_z.
- Do Wold decomposition and identify wandering subspaces.

Outline of the proof of the Aleman, Richter and Sundberg Theorem via the Hardy space of the bidisk

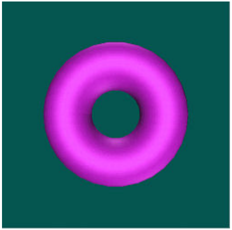
- Lift the Bergman shift M_z as the compression $\mathcal{B}$ of the isometry T_z on a nice subspace $\mathcal{H}$ of the Hardy space of the bidisk.
- Lift an invariant subspace $\mathcal{M}$ of $\mathcal{B}$ to be an invariant subspace $\widetilde{\mathcal{M}}$ of the isometry T_z.
- Do Wold decomposition and identify wandering subspaces.
- Establish an identity.

Bergman Space $L_a^2(\mathbb{D})$



$$L_a^2(\mathbb{D})$$
$$f(z)$$
$$M_z$$
$$M$$

Hardy Space $H^2(\mathbb{T}^2)$



$$\mathcal{H}$$
$$f(z), \quad f(w)$$
$$\mathcal{B}$$
$$\mathcal{M}$$

$H^2(\mathbb{T}^2)$ AND TOEPLITZ OPERATORS

- $\mathbb{T}$ denotes the unit circle.

$H^2(\mathbb{T}^2)$ and Toeplitz operators

- $\mathbb{T}$ denotes the unit circle.
- The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$.

$H^2(\mathbb{T}^2)$ and Toeplitz operators

- $\mathbb{T}$ denotes the unit circle.
- The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$.
- The Hardy space $H^2(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2, d\sigma)$, each function in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in $\mathbb{D}^2$ with the square summable Fourier coefficients.

$H^2(\mathbb{T}^2)$ and Toeplitz operators

- $\mathbb{T}$ denotes the unit circle.
- The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$.
- The Hardy space $H^2(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2,\ d\sigma)$, each function in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in $\mathbb{D}^2$ with the square summable Fourier coefficients.
- Let P be the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $H^2(\mathbb{T}^2)$.

$H^2(\mathbb{T}^2)$ and Toeplitz operators

- $\mathbb{T}$ denotes the unit circle.
- The torus $\mathbb{T}^2$ is the Cartesian product $\mathbb{T} \times \mathbb{T}$.
- The Hardy space $H^2(\mathbb{T}^2)$ is the subspace of $L^2(\mathbb{T}^2,\ d\sigma)$, each function in $H^2(\mathbb{T}^2)$ can be identified with the boundary value of the function holomorphic in $\mathbb{D}^2$ with the square summable Fourier coefficients.
- Let P be the orthogonal projection from $L^2(\mathbb{T}^2, d\sigma)$ onto $H^2(\mathbb{T}^2)$.
- The Toeplitz operator on $H^2(\mathbb{T}^2)$ with symbol f in $L^\infty(\mathbb{T}^2, d\sigma)$ is defined by

$$T_f(h) = P(fh),$$

for $h \in H^2(\mathbb{T}^2)$.
Clearly, T_z and T_w are a pair of doubly commuting pure isometries on $H^2(\mathbb{T}^2)$.

Lift the Bergman shift to the compression of the isometry T_z on a nice subspace of the Hardy space of the bidisk.

Lift the Bergman shift to the compression of the isometry T_z on a nice subspace of the Hardy space of the bidisk.

- For each integer $n \geq 0$, let

$$p_n(z,w) = \sum_{i=0}^{n} z^i w^{n-i} = \frac{z^{n+1} - w^{n+1}}{z - w}.$$

  Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^{\infty}$.
  Then

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus cl\{(z-w)H^2(\mathbb{T}^2)\}.$$

Lift the Bergman shift to the compression of the isometry T_z on a nice subspace of the Hardy space of the bidisk.

- For each integer $n \geq 0$, let
$$p_n(z,w) = \sum_{i=0}^{n} z^i w^{n-i} = \frac{z^{n+1} - w^{n+1}}{z - w}.$$
Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^{\infty}$.
Then
$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus cl\{(z-w)H^2(\mathbb{T}^2)\}.$$
- Let $P_{\mathcal{H}}$ be the orthogonal projection from $L^2(\mathbb{T}^2,\ d\sigma)$ onto $\mathcal{H}$.

$$\mathcal{B} \stackrel{def}{=} P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}.$$

Lift the Bergman shift to the compression of the isometry T_z on a nice subspace of the Hardy space of the bidisk.

- For each integer $n \geq 0$, let

$$p_n(z,w) = \sum_{i=0}^{n} z^i w^{n-i} = \frac{z^{n+1} - w^{n+1}}{z-w}.$$

  Let $\mathcal{H}$ be the subspace of $H^2(\mathbb{T}^2)$ spanned by functions $\{p_n\}_{n=0}^{\infty}$.
  Then

$$H^2(\mathbb{T}^2) = \mathcal{H} \oplus cl\{(z-w)H^2(\mathbb{T}^2)\}.$$

- Let $P_{\mathcal{H}}$ be the orthogonal projection from $L^2(\mathbb{T}^2,\ d\sigma)$ onto $\mathcal{H}$.

$$\mathcal{B} \overset{def}{=} P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}.$$

- The Bergman shift $M_z \cong \mathcal{B}$ via the following unitary operator $U : L^2_a(D) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z,w)}{n+1}.$$

- $$Uz^n = \frac{p_n(z,w)}{n+1}.$$
  - Douglas, Paulsen and Yang
  - Richter
  - Sun and Yu.

- $$Uz^n = \frac{p_n(z,w)}{n+1}.$$
  - Douglas, Paulsen and Yang
  - Richter
  - Sun and Yu.
- $$Uf(z,w) = \int_{\mathbb{D}} \frac{f(\lambda)}{(1-\bar{\lambda}z)(1-\bar{\lambda}w)} dA(\lambda)$$
  - Ahern-Youssfi

- $$Uz^n = \frac{p_n(z,w)}{n+1}.$$
  - Douglas, Paulsen and Yang
  - Richter
  - Sun and Yu.
- $$Uf(z,w) = \int_{\mathbb{D}} \frac{f(\lambda)}{(1-\bar{\lambda}z)(1-\bar{\lambda}w)} dA(\lambda)$$
  - Ahern-Youssfi
- The extension of U^* to $H^2(\mathbb{T}^2)$ is the restriction of functions of the Hardy spaces to the diagonal and maps the Hardy spaces to the Bergman spaces. This phenomenon was discovered by Rudin in 60s and further was explored by Duren-Shields and Horowitz-Oberlin in 70s.
  $$U^*F(z) = F(z,z)$$
  for $F(z,w) \in H^2(\mathbb{T}^2)$.

Indeed, $\mathcal{B}$ was called to be super-isometrically dilatable, i.e.,

$$\mathcal{B}^{n+m} = P_{\mathcal{H}} T_z^n T_w^m|_{\mathcal{H}},$$

and

$$\mathcal{B}^* = T_z^*|_{\mathcal{H}} = T_w^*|_{\mathcal{H}},$$

for the pair of doubly commuting pure isometries T_z and T_w on the Hardy space $H^2(\mathbb{T}^2)$.

Advantage:

• Functions in $H^2(\mathbb{T}^2)$ behave better than those in the Bergman space.

Indeed, $\mathcal{B}$ was called to be super-isometrically dilatable, i.e.,

$$\mathcal{B}^{n+m} = P_{\mathcal{H}} T_z^n T_w^m|_{\mathcal{H}},$$

and

$$\mathcal{B}^* = T_z^*|_{\mathcal{H}} = T_w^*|_{\mathcal{H}},$$

for the pair of doubly commuting pure isometries T_z and T_w on the Hardy space $H^2(\mathbb{T}^2)$.
Advantage:

• Functions in $H^2(\mathbb{T}^2)$ behave better than those in the Bergman space.

Disadvantage:

• $P_{\mathcal{H}}$

The Dirichlet space $\mathcal{D}$ consists of analytic functions on the unit disk whose derivative is in the Bergman space L^2_a.

Theorem

For each $f(z,w)$ in $H^2(\mathbb{T}^2)$, f is in $\mathcal{H}$ if and only if there is a function $\tilde{f}(z)$ in $\mathcal{D}$ such that

$$f(z,w) = \frac{\tilde{f}(z) - \tilde{f}(w)}{z - w},$$

for two distinct points z, w in the unit disk.

The Dirichlet space $\mathcal{D}$ consists of analytic functions on the unit disk whose derivative is in the Bergman space L^2_a.

Theorem

For each $f(z,w)$ in $H^2(\mathbb{T}^2)$, f is in $\mathcal{H}$ if and only if there is a function $\tilde{f}(z)$ in $\mathcal{D}$ such that

$$f(z,w) = \frac{\tilde{f}(z) - \tilde{f}(w)}{z - w},$$

for two distinct points z, w in the unit disk.

Suppose that $e(z,w)$ is in $\mathcal{H}$.

- If $e(z,z) = 0$ for each z in the unit disk, then $e(z,w) = 0$ for (z,w) on the torus.
- $e(z,w) = e(w,z)$.
- Let $E(z) = e(z,0)$. Then

$$e(\lambda,\lambda) = \lambda E'(\lambda) + E(\lambda),$$

for each $\lambda \in \mathbb{D}$.

LIFT EACH INVARIANT SUBSPACE OF $\mathcal{B}$ AS AN INVARIANT SUBSPACE OF T_z

For an invariant subspace $\mathcal{M}$ of $\mathcal{B}$, define the lifting $\widetilde{\mathcal{M}}$ to be the direct sum

$$\mathcal{M} \oplus [z - w]$$

where $[z - w] = cl\{(z - w)H^2(\mathbb{T}^2)\}$.

LIFT EACH INVARIANT SUBSPACE OF $\mathcal{B}$ AS AN INVARIANT SUBSPACE OF T_z

For an invariant subspace $\mathcal{M}$ of $\mathcal{B}$, define the lifting $\widetilde{\mathcal{M}}$ to be the direct sum

$$\mathcal{M} \oplus [z - w]$$

where $[z - w] = cl\{(z - w)H^2(\mathbb{T}^2)\}$.

THEOREM (Richter)

The mapping

$$\eta : \mathcal{M} \to \widetilde{\mathcal{M}}$$

is a one-to-one correspondence between invariant subspaces of $\mathcal{B}$ and invariant subspaces of T_z containing $[z - w]$.

To see that $\widetilde{\mathcal{M}}$ is an invariant subspace of T_z, for each f in $\widetilde{\mathcal{M}}$, we write

$$f = f_1 + f_2$$

for f_1 in $\mathcal{M}$ and f_2 in $[z - w]$.

To see that $\widetilde{\mathcal{M}}$ is an invariant subspace of T_z, for each f in $\widetilde{\mathcal{M}}$, we write

$$f = f_1 + f_2$$

for f_1 in $\mathcal{M}$ and f_2 in $[z-w]$. Since $[z-w]$ is an invariant subspace of T_z, $T_z f_2 = z f_2$ is in $[z-w]$. An easy computation gives

$$\begin{aligned} T_z f_1 &= \mathcal{B} f_1 + P_{[z-w]}(z f_1) \\ &\in \mathcal{M} \oplus [z-w] = \widetilde{\mathcal{M}}. \end{aligned}$$

Thus we have

$$\begin{aligned} T_z f &= \mathcal{B} f_1 + T_z f_2 \\ &\in \mathcal{M} \oplus [z-w] = \widetilde{\mathcal{M}}, \end{aligned}$$

to get that $\widetilde{\mathcal{M}}$ is an invariant subspace of the isometry T_z.

For an invariant subspace $\mathcal{M}$ of $\mathcal{B}$, let the operator $\mathcal{B}^*_{\mathcal{M}}$ on $\mathcal{M}$ denote the compression of T^*_z on $\mathcal{M}$, i.e.,

$$\mathcal{B}^*_{\mathcal{M}} q = P_{\mathcal{M}} T^*_z q = P_{\mathcal{M}} \mathcal{B}^* q$$

for q in $\mathcal{M}$.

For an invariant subspace $\mathcal{M}$ of $\mathcal{B}$, let the operator $\mathcal{B}^*_{\mathcal{M}}$ on $\mathcal{M}$ denote the compression of T_z^* on $\mathcal{M}$, i.e.,

$$\mathcal{B}^*_{\mathcal{M}} q = P_{\mathcal{M}} T_z^* q = P_{\mathcal{M}} \mathcal{B}^* q$$

for q in $\mathcal{M}$.

Since the Bergman shift is bounded below, we have the following lemma.

LEMMA

Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}$. Then $\mathcal{B}^*_{\mathcal{M}} \mathcal{B}$ is invertible on $\mathcal{M}$.

Let $\mathcal{M}_0$ be the wandering space of $\mathcal{B}$ on $\mathcal{M}$. Let

$$\mathcal{M}_{00} = \{-h_g + zP_{\mathcal{H}}g - wg(w) : \quad (h_g, g) \in \mathcal{BM} \times H^2(\mathbb{T}), \\ h_g = \mathcal{B}[\mathcal{B}^*_{\mathcal{M}}\mathcal{B}]^{-1}P_{\mathcal{M}}g\}.$$

Theorem

Let $\mathcal{M}$ be an invariant subspace of $\mathcal{B}$. Let $\widetilde{\mathcal{M}}$ be the lifting of $\mathcal{M}$. Then $\widetilde{\mathcal{M}}$ is an invariant subspace of the isometry T_z and has the following decomposition:

$$\widetilde{\mathcal{M}} = \oplus_{n=0}^{\infty} z^n \mathcal{L}_{\widetilde{\mathcal{M}}}$$

where $\mathcal{L}_{\widetilde{\mathcal{M}}}$ is the wandering space of T_z on $\widetilde{\mathcal{M}}$ given by

$$\mathcal{L}_{\widetilde{\mathcal{M}}} = \mathcal{M}_0 \oplus \mathcal{M}_{00}.$$

Theorem (Aleman-Richter-Sundberg Theorem)

Let $\mathcal{M}$ be in Lat $\mathcal{B}$.

$$\mathcal{M} = [\mathcal{M} \ominus \mathcal{B}(\mathcal{M})],$$

where $[\mathcal{M} \ominus \mathcal{B}(\mathcal{M})] = \mathit{span}_{k \geq 0} \mathcal{B}^k(\mathcal{M} \ominus \mathcal{B}(\mathcal{M}))$.

Let $\mathcal{M}_0$ denote the wandering subspace $\mathcal{M} \ominus \mathcal{B}\mathcal{M}$ of $\mathcal{B}$ on $\mathcal{M}$.

Let $\mathcal{N}$ be the orthogonal complement of $[\mathcal{M}_0]$ in $\mathcal{M}$.

We will show that $\mathcal{N} = \{0\}$.

Step 1.

First we show

$$\mathcal{N} \subset \{\sum_{n=0}^{\infty} z^n u_n : u_n = -h_n + zP_{\mathcal{H}}g_n - wg_n(w) \in \mathcal{M}_{00}\}.$$

Recall

$$\begin{aligned} \mathcal{M}_{00} &= \{-h_g + zP_{\mathcal{H}}g - wg(w) : \quad (h_g, g) \in \mathcal{BM} \times H^2(\mathbb{T}), \\ &\qquad h_g = \mathcal{B}[\mathcal{B}^*_{\mathcal{M}}\mathcal{B}]^{-1}P_{\mathcal{M}}g\}. \end{aligned}$$

Thus each function q in $\mathcal{N}$ has the following form

$$q = \sum_{n=0}^{\infty} z^n u_n. \tag{1}$$

Step 2.

For a function q in $\mathcal{N}$ with the above representation, let

$$q_1 = \sum_{n=1}^{\infty} z^{n-1} u_n. \tag{2}$$

Next we show that for each q in $\mathcal{N}$,

$$q_1 = \mathcal{B}^*_{\mathcal{M}} q$$

and q_1 is still in $\mathcal{N}$.

$q = \sum_{n=0}^{\infty} z^n u_n, \; q_1 = \sum_{n=1}^{\infty} z^{n-1} u_n$

Step 3. We show

$$q = \mathcal{B}(\mathcal{B}_{\mathcal{M}}^{*}\mathcal{B})^{-1} q_1.$$

Use

$$\begin{aligned} q &= zq_1 + [-h_0 + zP_{\mathcal{H}} g_0 - w g_0(w)] \\ &= \mathcal{B}q_1 - h_0 + [(z - \mathcal{B})q_1 + zP_{\mathcal{H}} g_0 - w g_0(w)], \end{aligned}$$

to get

$$q = \mathcal{B}q_1 - h_0$$

and

$$h_0 = -\mathcal{B}(\mathcal{B}_{\mathcal{M}}^{*}\mathcal{B})^{-1}(1 - \mathcal{B}_{\mathcal{M}}^{*}\mathcal{B})q_1.$$

Hence

$$\begin{aligned} q &= \mathcal{B}q_1 - h_0 \\ &= \mathcal{B}q_1 + \mathcal{B}(\mathcal{B}_{\mathcal{M}}^{*}\mathcal{B})^{-1}(1 - \mathcal{B}_{\mathcal{M}}^{*}\mathcal{B})q_1 \\ &= \mathcal{B}(\mathcal{B}_{\mathcal{M}}^{*}\mathcal{B})^{-1} q_1. \end{aligned}$$

Step 4.

We show that $\mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q$ is in $\mathcal{N}$ for each q in $\mathcal{N}$. By **Step 3**, hence $\mathcal{B}(\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}|_{\mathcal{N}}$ is the inverse of $\mathcal{B}^*_{\mathcal{M}}|_{\mathcal{N}}$.

Step 5.

For each $q = \sum_{n=0}^{\infty} z^n u_n$ in $\mathcal{N}$ as in **Step 1**, let $q_k = (\mathcal{B}_{\mathcal{M}}^*)^k q$. Then

$$\|u_{k-1}\|^2 - \|u_k\|^2 = \|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2.$$

Step 5.

For each $q = \sum_{n=0}^{\infty} z^n u_n$ in $\mathcal{N}$ as in **Step 1**, let $q_k = (\mathcal{B}_{\mathcal{M}}^*)^k q$. Then

$$\|u_{k-1}\|^2 - \|u_k\|^2 = \|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2.$$

To prove the above equality, by **Step 2** we have that $q_k = \sum_{n=k}^{\infty} z^{n-k} u_n$, and hence

$$\|q_k\|^2 = \sum_{n=k}^{\infty} \|u_n\|^2.$$

Step 6.

For each q in $\mathcal{N}$, let $q_k = (\mathcal{B}_{\mathcal{M}}^*)^k q$. Then

$$\begin{aligned} &\|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2 \\ &= \langle (\mathcal{B}_{\mathcal{M}}^* \mathcal{B})^{-1} q_k, q_k \rangle + \langle (\mathcal{B}\mathcal{B}_{\mathcal{M}}^*) q_k, q_k \rangle - 2\langle q_k, q_k \rangle. \end{aligned} \tag{3}$$

Step 6.

For each q in $\mathcal{N}$, let $q_k = (\mathcal{B}^*_{\mathcal{M}})^k q$. Then

$$\begin{aligned} &\|q_{k-1}\|^2 + \|q_{k+1}\|^2 - 2\|q_k\|^2 \\ &= \langle (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1} q_k, q_k\rangle + \langle (\mathcal{B}\mathcal{B}^*_{\mathcal{M}}) q_k, q_k\rangle - 2\langle q_k, q_k\rangle. \end{aligned} \tag{3}$$

This follows from **Step 4.**

Step 7.

The Dirichlet space $\mathcal{D}$ consists of analytic functions on the unit disk whose derivatives are in the Bergman space L^2_a. We will get a representation of functions in $\mathcal{M}$.
For each f in $\mathcal{M}$, there is a function $g(z)$ in $H^2(\mathbb{T}) \cap \mathcal{D}$ such that

$$f(z,w) = -P_{\mathcal{H}}g - \frac{zg(z) - wg(w)}{z - w}. \tag{4}$$

Step 8. Key identity

For a function q in the invariant subspace $\mathcal{M}$ of $\mathcal{B}$, let $f = (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q$. By (4) in **Step 7,** there is a function g in $H^2(\mathbb{T}) \cap \mathcal{D}$ such that

$$f(z,w) = -P_{\mathcal{H}}g - \frac{zg(z) - wg(w)}{z - w}.$$

Let $\mathcal{M}^{\perp}$ denote the orthogonal complement of $\mathcal{M}$ in $\mathcal{H}$. Then

$$\begin{aligned}
& \langle (\mathcal{B}^*_{\mathcal{M}}\mathcal{B})^{-1}q, q\rangle + \langle (\mathcal{B}\mathcal{B}^*_{\mathcal{M}})q, q\rangle - 2\langle q, q\rangle \\
= \; & \frac{1}{2}\|P_{\mathcal{M}^{\perp}}\mathcal{B}P_{\mathcal{M}^{\perp}}[\frac{g(z) - g(w)}{z - w}]\|^2 - \|P_{\mathcal{M}^{\perp}}[\frac{g(z) - g(w)}{z - w}]\|^2.
\end{aligned}$$

Step 9.

Finally we show that $\mathcal{N} = \{0\}$.

For each q in $\mathcal{N}$, by **Step 1**, we write $q = \sum_{n=0} z^n u_n$ with $\|q\|^2 = \sum_{n=0}^{\infty} \|u_n\|^2$, for u_n in $\mathcal{M}_{00}$. Let $q_k = (\mathcal{B}_{\mathcal{M}}^*)^k q$. **Step 8** gives that

$$\langle (\mathcal{B}_{\mathcal{M}}^* \mathcal{B})^{-1} q_k, q_k \rangle + \langle (\mathcal{B}\mathcal{B}_{\mathcal{M}}^*) q_k, q_k \rangle - 2\langle q_k, q_k \rangle \leq 0.$$

By **Steps 5 and 6**, we have

$$\|u_{k-1}\|^2 - \|u_k\|^2 \leq 0,$$

to get that the sequence $\{\|u_k\|^2\}$ of nonnegative numbers increases, but is summable. Hence $\|u_k\|^2 = 0$ for $k \geq 0$. This implies that $q = 0$, to complete the proof.