Chaos and synchronization in discrete systems

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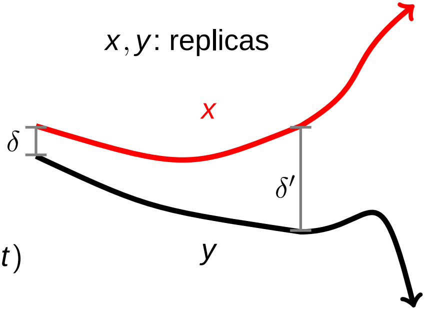
Outline

- Chaos is related to the exponential amplification of a small difference between two initially close trajectories, and therefore to unpredictability.
- Mathematically, it is related to the positivity of the largest Lyapunov exponent (LLE).
- The synchronization threshold of two replicas (Pecora-Carroll) is related to the LLE.
- High-dimensional systems are different. In the evolution of “chaotic” CA for instance, a *finite* difference may *propagate* to the whole system (unpredictability). This happens also in continous systems (coupled map lattices).
- It is possible to define *discrete* (Boolean) derivatives and Jacobian, and therefore *finite-distance* Lyapunov exponents.
- These exponents are related to the replica synchronization threshold (while normal LLE are not).

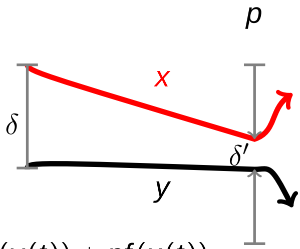
Low-dimensional chaos



$$
\begin{aligned}
x(t+1) &= f(x(t)), \\
y(t+1) &= f(y(t)), \\
\delta = |x - y| &\rightarrow 0 \\
\delta(t+1) &\simeq |f'(x(t))|\delta(t) \\
\delta(t) &\simeq \delta_0 \exp(\lambda t)
\end{aligned}
$$

- Sensitivity to infinitesimal perturbations
- Exponential amplification of distances
- Lyapunov exponent $\lambda > 0$
- Unpredictability

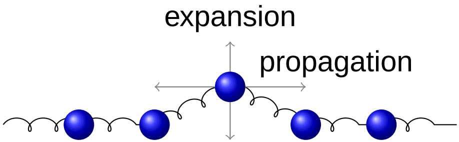
Replica synchronization



$$
\begin{aligned}
x(t+1) &= f(x(t)), \\
y(t+1) &= (1-p)f(y(t)) + pf(x(t)), \\
\delta = |x-y| &\to 0 \\
\delta(t+1) &\simeq (1-p)f'(x(t))\delta \\
\delta(t\to\infty) &\to 0 \quad \text{for} \quad p_c = 1-\exp(-\lambda)
\end{aligned}
$$

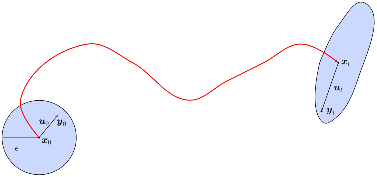
- The “force” p needed to achieve synchronization is related to Lyapunov exponent for *vanishing distances*
- But one may “push” from large distances..

High-dimensional systems



- Coupled “oscillators” (coupled maps, cellular automata)
- $x_i(t+1) = f(g(x_{i-1}(t), x_i(t), x_{i+1}(t)))$
- g is the coupling: linear (diffusive) or nonlinear
- f is the map: chaotic or stable (periodic, fixed point)
- A perturbation may amplify exponentially in time by the action of f, but only linearly through the coupling

Lyapunov spectrum



- $\boldsymbol{x}(t+1) = \boldsymbol{F}(\boldsymbol{x}(t))$
- $\delta(t+1) = \boldsymbol{J}(\boldsymbol{x}(t))\delta(t)$
- $J_{ij} = \dfrac{\partial F_i(\boldsymbol{x}(t))}{\partial x_j}$ is the Jacobian
- For one-dimensional systems with nearest neighbors coupling, the Jacobian is three-diagonal
- The eigenvalues of the time product of Jacobians $\boldsymbol{J}(\boldsymbol{x}(t))$ constitue the Lyapunov spectrum $\lambda_0 \geq \lambda_1 \dots$
- Diffusive coupling generally reduces the Largest Lyapunov Exponent (LLE)

Replica synchronization for extended systems

There are many ways of “pushing” together two extended replicas. For instance

- Uniform:

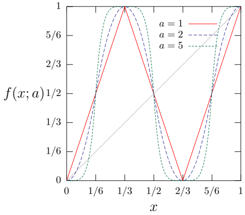
$$y_i(t+1) = (1-p)F_i(\boldsymbol{y}(t)) + pF_i(\boldsymbol{x}(t))$$

- Pinching:

$$y_i(t+1) = \begin{cases} F_i(\boldsymbol{y}(t)) & \text{with probability } 1-p \\ F_i(\boldsymbol{x}(t)) & \text{with probability } p \end{cases}$$

Uniform synchronization of chaotic maps gives results similar to low-dimensional systems: $p_c = 1 - \exp(-\lambda_0)$
Pinching synchronization depends on coupling: uncoupled chaotic maps synchronizes for $p_c = 0$. In general p_c is larger for larger couplings [Bagnoli Cecconi, PLA **260**, 9-17 (2001)]

Cellular automata as discrete dynamical systems



Let us consider Boolean deterministic CA

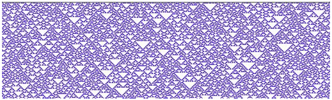
- $x(t+1) = \boldsymbol{F}(\boldsymbol{x}(t))$
- $x_i(t+1) = f(x_{i-1}(t), x_i(t), x_{i+1}(t))$ (elementary CA)
- The coupling is nonlinear
- The map f, if considered as a continuous system, is *superstable*
- All Lyapunov exponents are $-\infty$

CA classes (Wolfram like)

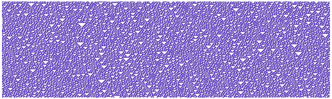
1. Few attractors, short cycles, short transients. Example: rule 0. Insensitivity to perturbations
2. Many attractors, short cycles, short transients. Example: majority rule 232. Indifference to perturbations
3. Few attractors, long cycles, short transients (chaotic patterns). Example: rule 150 (XOR)
4. Long transients (universal computation?), ending in class 1 or 2. Complex, long–lived structures. Examples: rule 110, Life in 2D.

Chaotic CA (class 3)

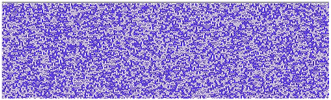
R 22



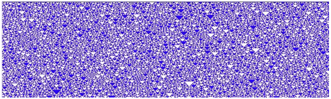
R 30



R 105



R 150



Boolean derivatives

$$
\begin{aligned}
J_{i,j} &= \frac{\partial F_i(\boldsymbol{x})}{\partial x_j} \\
&= F_i(x_0, \ldots, x_j \oplus 1, \ldots, x_{N-1}) \oplus F_i(x_0, \ldots, x_j, \ldots, x_{N-1}) \\
&= \begin{cases} 1 & F_i \text{ changes when } x_j \text{ changes} \\ 0 & F_i \text{ does not change when } x_j \text{ changes.} \end{cases}
\end{aligned}
$$

The concept may be extended to discrete systems, with a suitable metric and modular operations.

Boolean calculus

Many results from calculus can be extended to Boolean derivatives, for instance the McLaurin expansion

$$f(x,y) = \left(\frac{\partial f}{\partial x}\right)_{\substack{x=0\\y=0}} x \oplus \left(\frac{\partial f}{\partial y}\right)_{\substack{x=0\\y=0}} y \oplus \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=0\\y=0}} xy$$

[Bagnoli, IJMPC **3**, 307(1992)]

Boolean derivatives of R150

x_0	x_1	x_2	f	f'_{x_0}	f'_{x_1}	f'_{x_2}
0	0	0	0	1	1	1
0	0	1	1	1	1	1
0	1	0	0	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	1	1
1	1	1	0	1	1	1

Boolean derivatives of R22

x_0	x_1	x_2	f	f'_{x_0}	f'_{x_1}	f'_{x_2}
0	0	0	0	1	1	1
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	0	0	1	1
1	0	0	1	1	1	1
1	0	1	0	1	0	1
1	1	0	0	1	1	0
1	1	1	0	0	0	0

Elementary cellular automata

$$J_{i,j} = \begin{cases} \dfrac{\partial f(x_{i-1}, x_i, x_{i+1})}{\partial x_j} & |i-j| \leq 1 \\ 0 & |i-j| > r\,. \end{cases}$$

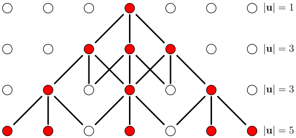
$$\boldsymbol{J} = \begin{pmatrix} J_{1,1} & J_{1,2} & 0 & 0 & \cdots & 0 & J_{1,N} \\ J_{2,1} & J_{2,2} & J_{2,3} & 0 & \cdots & 0 & 0 \\ 0 & J_{3,2} & J_{3,3} & J_{3,4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ J_{N,1} & 0 & 0 & 0 & \cdots & J_{N-1,N} & J_{N,N} \end{pmatrix}$$

where $J_{i,j} = 0, 1$.

Damage spreading and tangent space

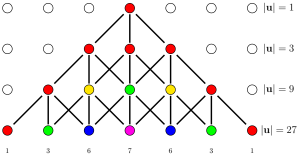
damage spreading

$$\boldsymbol{u}^{(t)} = \boldsymbol{J}(\boldsymbol{u}^{(t-1)}) \odot \boldsymbol{u}^{(t-1)}$$



paths in tangent space

$$\boldsymbol{u}^{(t)} = \boldsymbol{J}(\boldsymbol{u}^{(t-1)}) \cdot \boldsymbol{u}^{(t-1)}$$



In damage spreading the scalar product is computed modulo two ($\odot$)

Largest Lyapunov exponent of CA

- The definition of tangent space is similar to that of continuous systems

$$\boldsymbol{u}(t+1) = \boldsymbol{J}(\boldsymbol{x}(t))\boldsymbol{u}(t)$$

- $u_i(t)$ is the number of possible different paths in tangent space ending at site i at time t
- The tangent space is formed by all possible ways of propagating a perturbation of vanishing size (1 site).
- A path joins “ones” in the product of Jacobians
- $|\boldsymbol{u}(t)| = |\boldsymbol{u}_0| \exp(\lambda t)$
- λ is the largest Lyapunov exponent
- $\lambda \geq 0$ if there are “percolating” paths (time-directed percolation in tangent space)

Random matrix approximation

μ: average number of ones in the product of Jacobians

$$\boldsymbol{M}(t) = \begin{cases} 1 & \text{with probability } \mu \text{ if } |i-j| \le r \\ 0 & \text{with probability } 1-\mu \text{ if } |i-j| \le r \\ 0 & \text{otherwise} \end{cases}$$

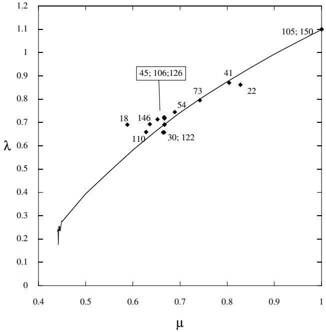
$$\boldsymbol{u}(t+1) = \boldsymbol{M}(t)\boldsymbol{u}(t); \quad |\boldsymbol{u}(t)| \sim \exp(\lambda t) \quad \lambda > 0 \text{ if } \mu > \mu_c$$

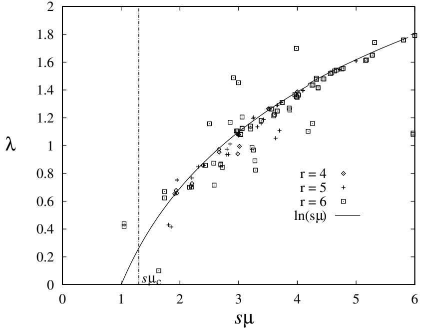
Mean field approximation:

$$M_{i,j} = \begin{cases} \mu & \text{if } |i-j| \le r \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_{MF} = \log(2r+1) > \lambda$$

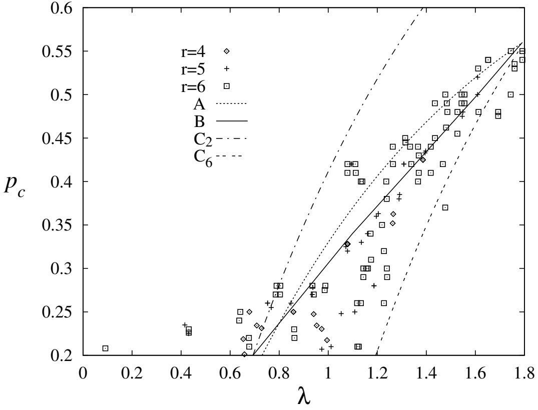
LLE of elementary and totalistic CA





[Bagnoli Rechtman Ruffo, PLA **172**, 34 (1992)]

Pinching synchronization of CA



[Bagnoli Rechtman, PRE **59**, R1307 (1999)]

Conclusions

- Boolean (finite-distance) Lyapunov exponents characterize unstable trajectories in CA
- Under perturbation in the function or in the connectivity (like in Kauffman networks) λ characterizes chaotic CA
- Pinching synchronization and Boolean Lyapunov exponents are related in CA
- Pinching synchronization is not related to standard Lyapunov exponents [Bagnoli Rechtman, PRE **73** 026202 (2006)]
- Finite-distance Lyapunov exponents: indicators of "physical" unpredictability?