Can spatially extended CA system replicate the logistic map?

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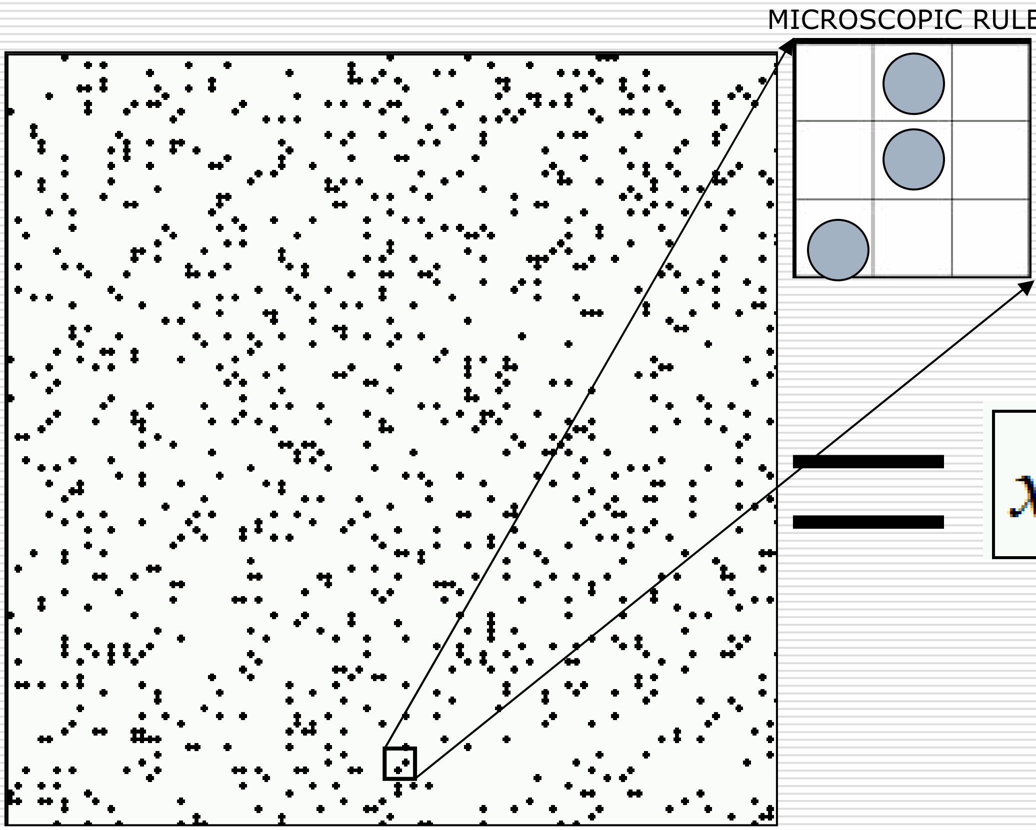
Dzwinel, W., In Search of Cellular Automata Reproducing Chaotic Dynamics Described by Logistic Formula, *Lecture Notes in Computer Science,* ACRI 2006, LNCS 4173, pp. 657 – 666, 2006.

Metaphor



????

$$=\quad x^{n+1} \rightarrow k \cdot x^{n}\left(1-x^{n}\right)$$

populated grid of CA with density x^n

Outline

- Motivations
- CA system with global logistic behavior
  - definition of CA looked for and model assumptions
  - derivation of microscopic CA rules
- Experiments on system dynamics
  - limited motion ability
  - size of neighborhood
  - non-symmetrical neighborhood
- Infinite system size - logistic coupled map lattice.
- Conclusions

Motivation

- understanding the role of spatial structures and their dynamics in population evolution
  - explaining the principal sources of inconsistency between logistic law and experimental findings
    - does the spatio-temporal chaos is accompanied by chaotic changes of the total population density?

Consequences of chaotic evolution

- When the average population density is oscillating or chaotic, the colony becomes vulnerable on external threats, e.g., pestilences
- Additional degrees of freedom defined by space and microscopic interactions between individuals allows for creating "self-preservation instinct" and flexible reaction of the population on the external danger.
- Scrutinizing the ways the adverse effects of chaotic evolution can be eliminated, is fundamental to protect the colony from extinction.
- Conversely, this knowledge can allow for eliminating the structured populations, which are very difficult to destroy (e.g., bacterial biofilms)

Existing models

- COUPLED MAP

Lloyd, A.L., (1995). The coupled Logistic Map: A Simple Model For The Effects of Spatial Heterogeneity on Population Dynamics, *J. Theor. Biol.,* 173, 217-30

- COUPLED MAP LATTICES

Kaneko, K., (1993), Theory and applications of coupled map lattices, John Wiley & Sons New York,

- LATTICE DYNAMICAL SYSTEMS

Chow S-N, Mallet-Paret, J., Van Vleck, E.,S., (1996) Dynamics of lattice differential equations, Int J. Bifurc. and Chaos, 6/ 9, 1605-1621

- MOMENT CLOSURE (IBM MODEL)

David J. Murrell, D.,J., Dieckmann, U., Law,R., (2004) On moment closures for population dynamics in continuous space, *J. Theor. Biol* 229 (2004) 421–432

- PDE (e.g. reaction-diffusion eqs.)

Holmes, E.E., Lewis, M.,A., Banks, J.E., Veit, R.R., (1994). Partial differential equations in ecology: Spatial interactions and population dynamics, *Ecology,* 75(1), 17-29.

Law, R., Murrell, D.J., Dieckmann, U., (2003). Population Growth in Space and Time. Spatial Logistic Equation, *Ecology*, 84 (1), 252-262.

… sources of inconsistency

- heterogeneity of the environment
- population dispersion may not increase linearly with density
- time delay in the operations: of density dependence and in dispersion/competition
- the random collision of individuals may not represent interactions among organisms well.
- creation of spatial patterns
- what more??

… sources of inconsistency

- Here we will focus on the role of more fundamental factors connected with spatio-temporal dynamics:
  - **spatial structures formation,**
  - **motion ability of population members,**
  - **size and type of the neighborhood.**

CA class of models searched

- species living on 2-D, squared lattice $\aleph(l,m)=[\{l,m\},\ l,m=1,\ldots,M]$
- PBC, M>=300 (reflecting, adiabatic, CBPBC checked)
- neighborhood - sum of dispersal and competition $\Omega(l,m)=\Omega_D(l,m,R)\cup\Omega_C(l,m,r)$
  - $\Omega_D(l,m,R_D)\rightarrow$ responsible for reproduction of new individuals in unpopulated sites
  - $\Omega_C(l,m,r_C)\ \rightarrow$ responsible for elimination of populated sites
- initial, random population, with density $x=x_0$

CA class of models searched cd.

- Two repetitive steps
  - random walk with a given time period Δt
  - evaluation round

$$\varepsilon_{lm}^{n\Delta t+1} \rightarrow \mathbf{R}\left[X_{lm}^{nT}, \Omega_D(l,m,R), \Omega_C(l,m,r)\right]$$

$$X_{lm}^{n\Delta t+1} = \begin{cases} 1 \text{ if } & rnd < \varepsilon_{lm}^{n+1} \\ 0 \text{ if } & rnd > \varepsilon_{lm}^{n+1} \end{cases}$$

- We are looking for the set of probabilities $a_{0/1i}$

$$\mathbf{R} = \left\{0 \leq a_{1,i} \leq 1 \wedge 0 \leq a_{0,i} \leq 1; X_{lm}^{n+1} = 1\right\}$$

  - $a_{1,i}$ – probability that populated site X_{lm}=1 will **remain** populated in the next evaluation round
  - $a_{0,i}$ – probability that unpopulated site X_{lm}=0 will **become** populated in the next evaluation round

Assumptions

- for $\Delta t \rightarrow \infty$ random walk $\Leftrightarrow$ random scatter of individuals over the CA lattice after each evaluation round
- $\forall (lm) \in \aleph(l,m)$ the number of $X_{i \in \Omega(l,m)} = 1$, undergoes the Bernoulli distribution
- Let x_i^n the average population density in *n*th evaluation round

… then

$$x^{n+1} \to p x^n \sum_{i=0}^{N_C} a_{1i} \binom{N_C}{i} \left(x^n\right)^i (1-x^n)^{N_C-i} + q(1-x^n) \sum_{i=0}^{N_D} a_{0(N_N-i)} \binom{N_D}{i} \left(x^n\right)^{N_D-i} (1-x^n)^i$$

where $$x^n = \sum_{l,m}^{M} X_{lm}^n / M^2$$

on the other hand

$$x^{n+1} \to k \cdot x^n \left(1-x^n\right)$$

thus after some algebra

$$p \cdot \left(\frac{N_C \cdot a_{1i}}{N_C - i} \right) + q \cdot \left(\frac{N_D \cdot a_{0i+1}}{i+1} \right) = k; \; a_{1N_N} = a_{00} = 0 \qquad (1)$$

for $\quad i = 0, \ldots, N_N - 1$

and then …

- for $N_D=N_C=N_N$ (i.e. $R_D=r_C$) the maximum value of k, $kmax$ $a_{0i}=a_{1i}=0$ and $i=N_N/2$:

$$k_{\max} = 4 \cdot \left(\frac{N_N \left(\frac{p+q}{2} \right) + p}{N_N + 2} \right) \wedge \lim_{N_N \to \infty} \left(k_{\max} \right) = 4 \cdot \left(\frac{p+q}{2} \right) \qquad (2)$$

- the CA model can reproduce the whole logistic map in macro-scale for certain microscopic rules **R** in case of infinite neighborhood $N_N \to \infty$.
- for $N_N=8$ the system reproduces only the part of it, i.e., the fragment for $k \le k_{max} \approx 3.6$

Rules

- The number of rules is infinite.
- A few exemplary rules

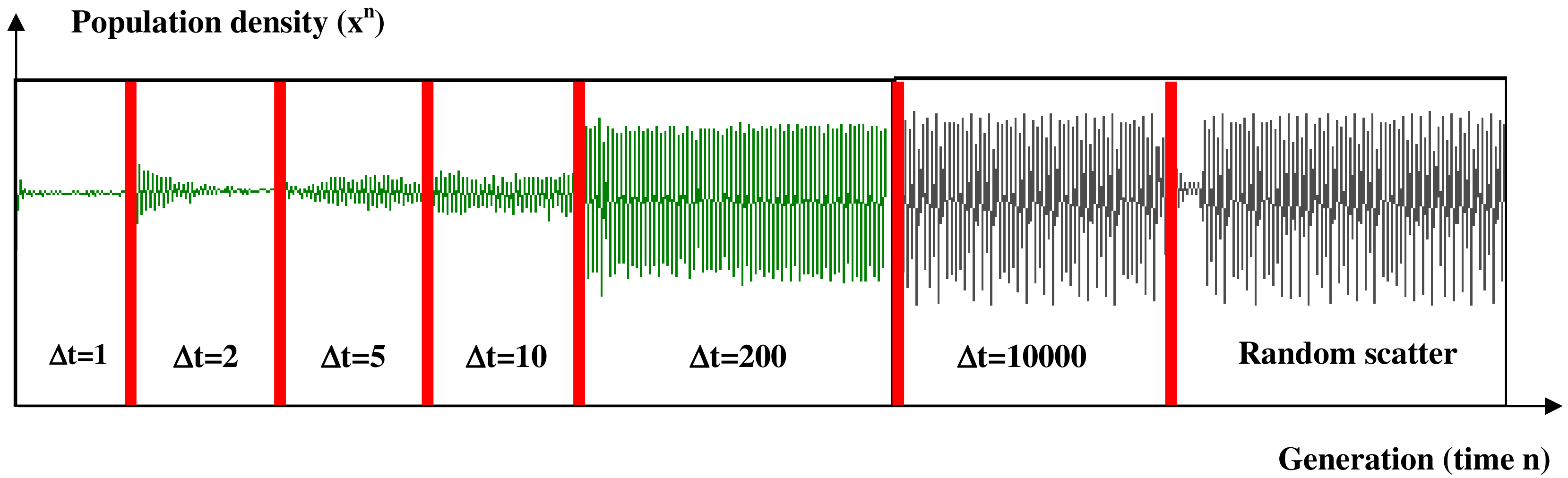
No. neigh. i	1		2		3		4	
	a_{1i}	a_{0i}	a_{1i}	a_{0i}	a_{1i}	a_{0i}	a_{1i}	a_{0i}
0	0	0	0.4	0	1	0	0	0
1	0	0.45	0.7	0.4	1	0.325	0	0.45
2	0.9	0.9	0.9	0.7	1	0.614	0.7	0.386
3	1	0.9	1	0.9	1	0.85	1	1
4	1	1	1	1	1	1	1	1
5	0.9	1	0.9	1	1	1	1	1
6	0.9	0.9	0.7	0.9	0.9	0.7	0.9	0.7
7	0.45	0	0.4	0.7	0.45	0	0.45	0
8	0	0	0	0.4	0	0	0	0

- for totalistic CA, the rule is

$$a_i = k \cdot \frac{(i+1)}{(N_N+1)} \cdot \left(1 - \frac{i}{N_N}\right) = k \cdot Y_i\left(1 - \tilde{Y}_i\right) \wedge k \in (0, k_{\max}] \qquad (3)$$

„logistic” microscopic CA rule builds-up LM

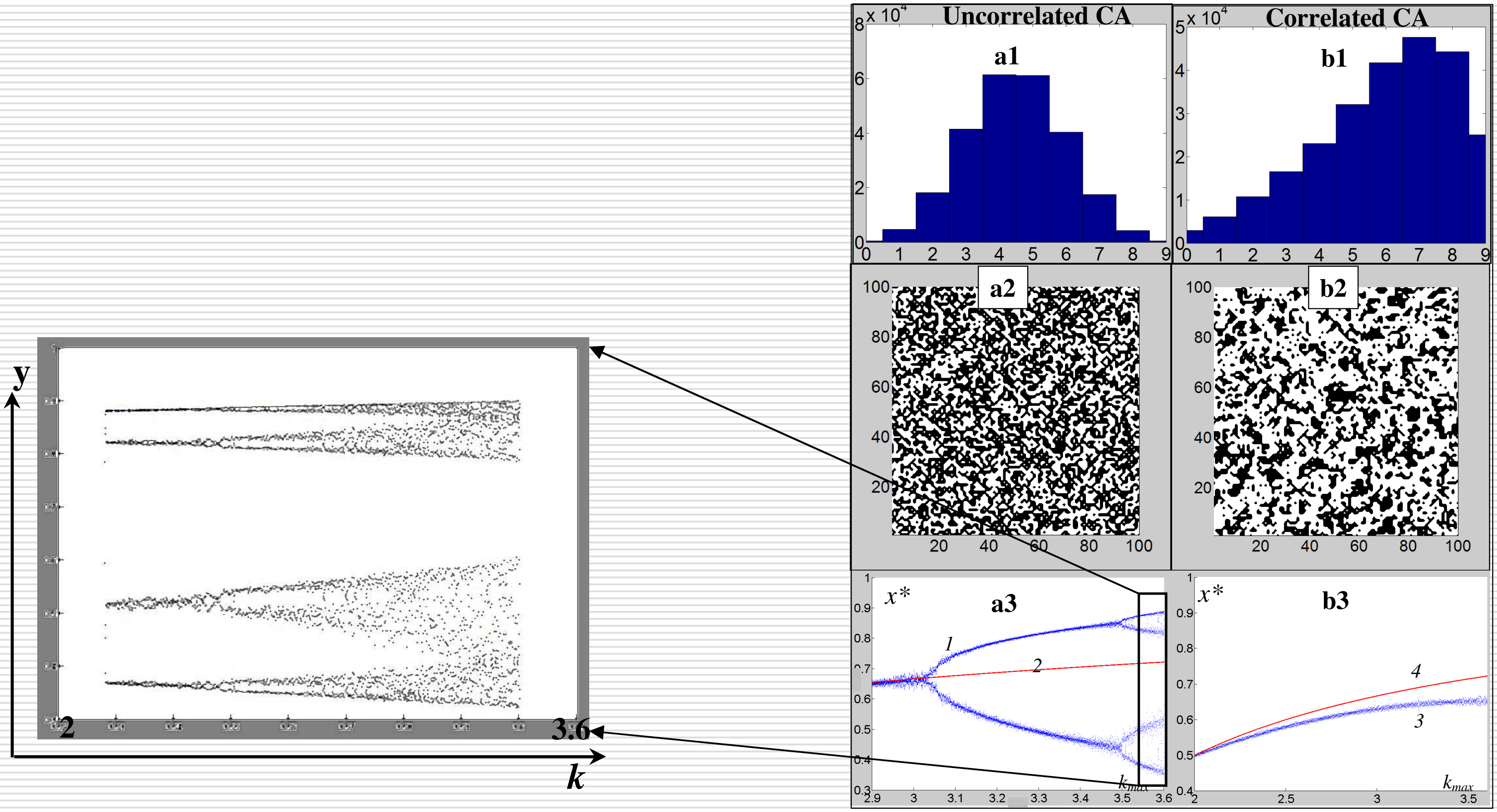
Results – *decrease in mobility*



The fragments of time plots which represent the evolution of the average population density. The value of **Δt** means how many steps of random walk were applied between subsequent evaluation rounds.

Decrease in mobility disables chaotic behavior

Results– *comparison to 0 mobility*



different R_D/r_C ratios

- from

$$p\cdot\left(\frac{N_C\cdot a_{1i}}{N_C-i}\right)+q\cdot\left(\frac{N_D\cdot a_{0i+1}}{i+1}\right)=k;\ a_{1N_N}=a_{00}=0\ \text{for}\qquad i=0,\ldots,N_N-1$$

- we have that

$$1]\ N_C<N_D(r_C<R_D);\quad a_{0N_D}=k\qquad\qquad\Rightarrow k\le k_{\max}=1$$

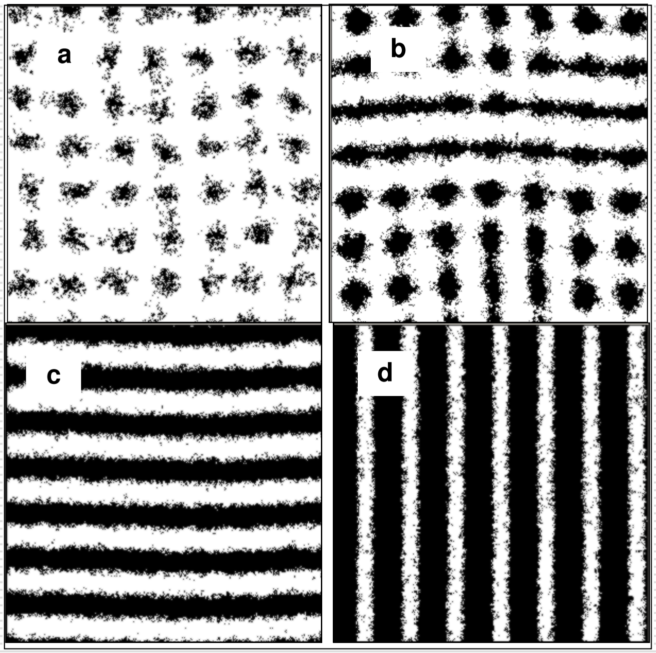
$$2]\ N_C>N_D(r_C>R_D);\quad a_{1N_D}\frac{N_C}{(N_C-N_D)}=k\Rightarrow k\le k_{\max}=\frac{1}{1-\dfrac{N_D}{N_C}}$$

i.e. for 1] – CA replicating LM does not exist

for 2] – it exists only for $1<N_D/N_C<0.75$ (full map)

Non-symmetric neighborhood destroys LM

Results – different R_D/r_C ratios



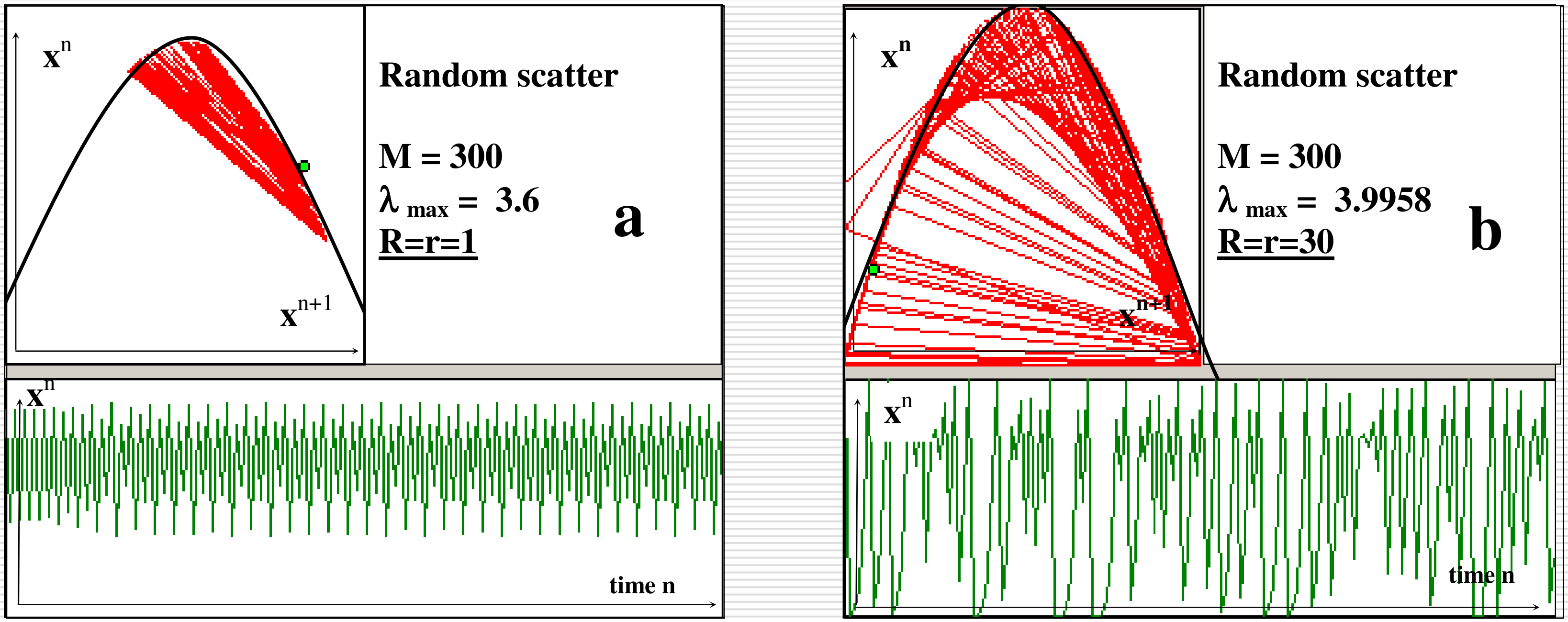
The regular and stable patterns created by the discrete “motionless” CA system for, M=200, R=1 and r=30:

a) k=1.15, t=3000, x^*=*0.13*; b) k=1.4, t=600, x^*=*0.31*;

c) k=2, t=600, x^*=*0.51*; d) k=4, t=600, x^*=*0.73*,

where t is the number of time-steps and x^* is the average population density.

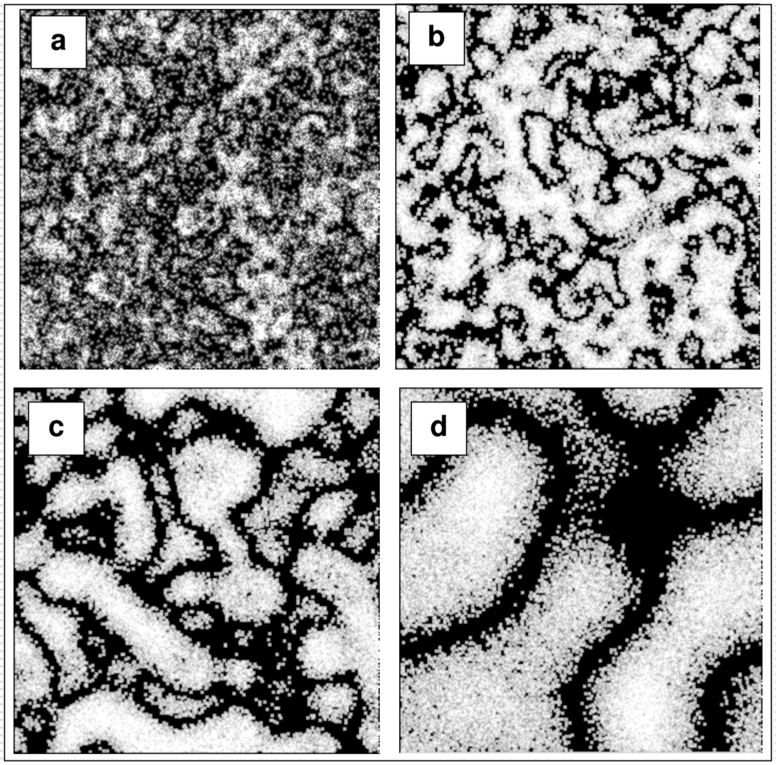
Results- *increasing neighborhood*



The phase diagrams for increasing dispersal and competition radiuses (R_D=r_C).
a) for (R_D=r_C =*1*), the system just becomes to be chaotic.
b) the chaos develops for increasing number of neighbors

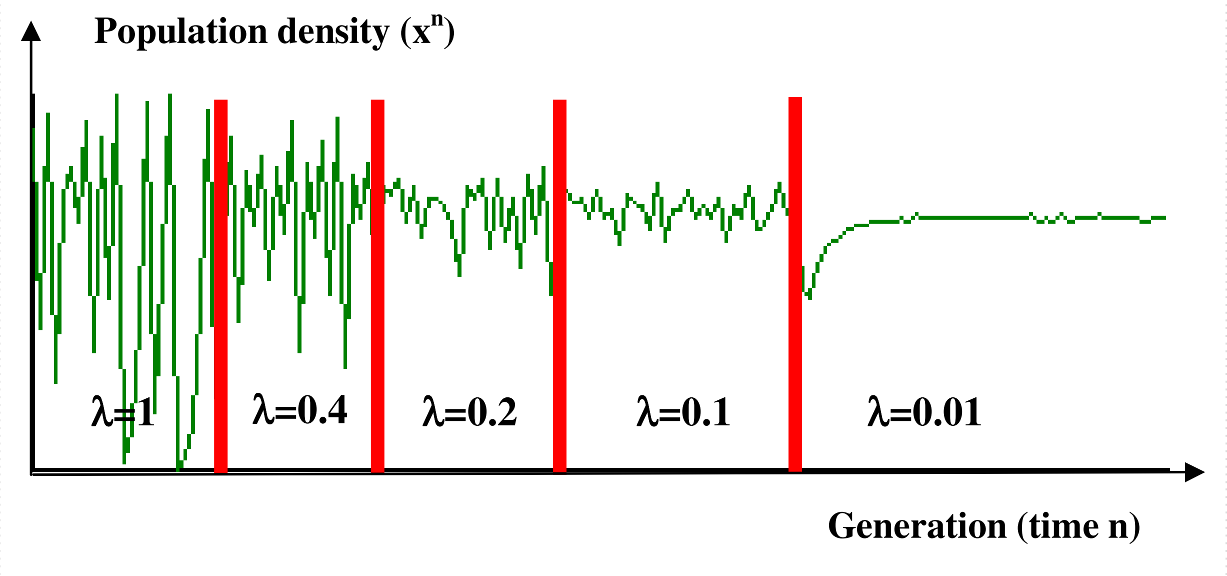
Greater neighborhood builds-up LM

Results- *increasing neighborhood*



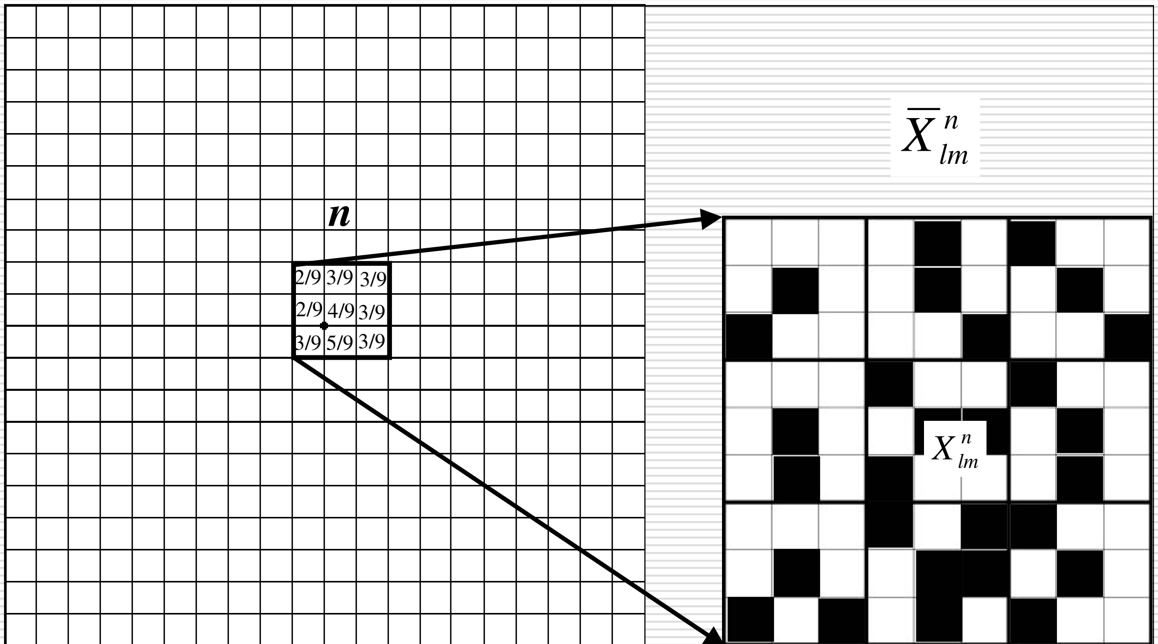
The self-similar patterns produced on the CA 300x300 lattice for the motionless population model assuming *r*=*R*= a) 3 b) 10 c) 20 d) 50. The black dots stand for populated nodes.

Results- *increasing neighborhood*



The fragments of plots showing the changes of population density in time for the motionless “correlated” CA system for various aspect ratios $\lambda=(2{\cdot}R+1)/M$.

Infinite neighborhood limit

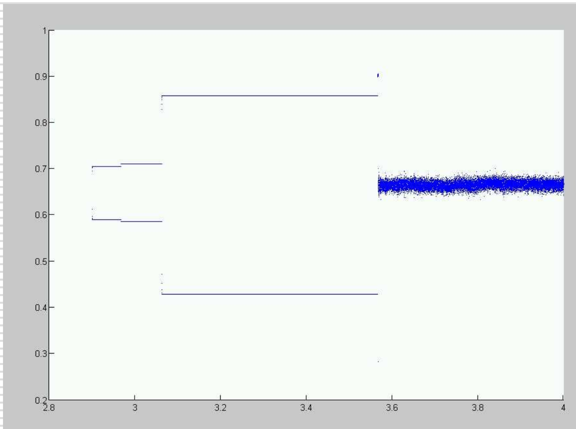


$$\overline{X}_{lm}^{n} = (3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + \frac{4}{9} + \frac{5}{9}) / 9$$

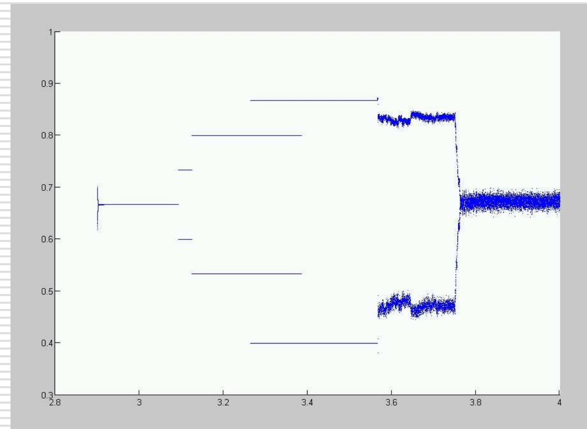
$$X_{lm}^{n+1} = k(=4) \cdot \overline{X}_{lm}^{n}\left(1 - \overline{X}_{lm}^{n}\right) \approx 0.905 \approx \frac{8}{9}$$

Increasing neighborhood

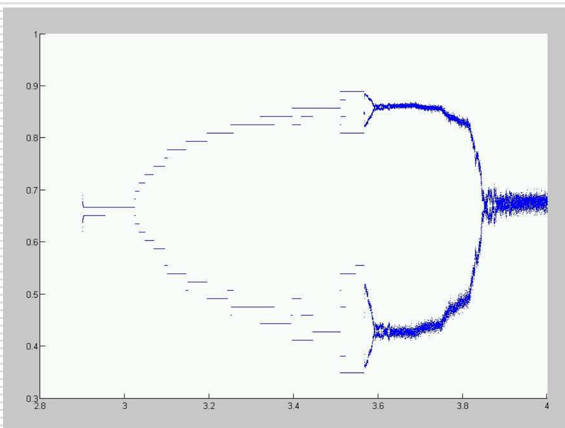
s=9



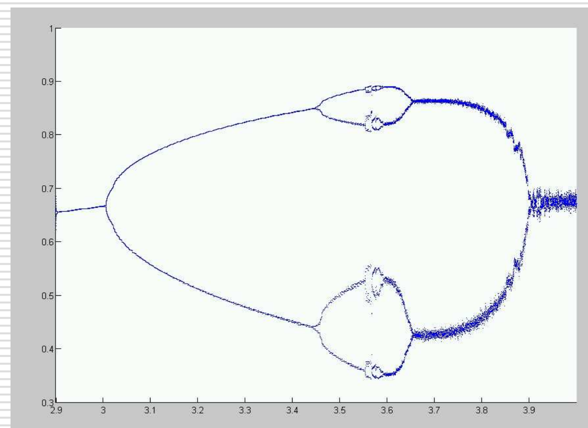
s=16



s=64



s=1024



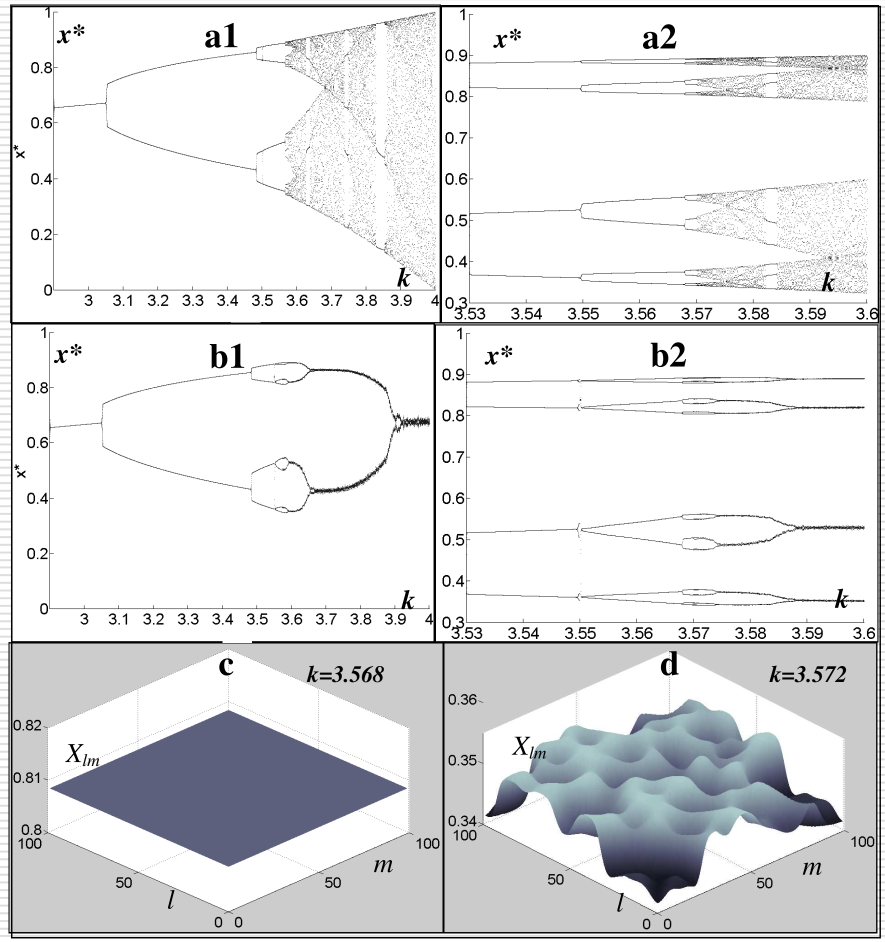
2-D coupled map lattice

$$X_{lm}^{n+1} = k \cdot \overline{X}_{lm}^{\,n} \cdot \left(1 - \overline{X}_{lm}^{\,n}\right)$$

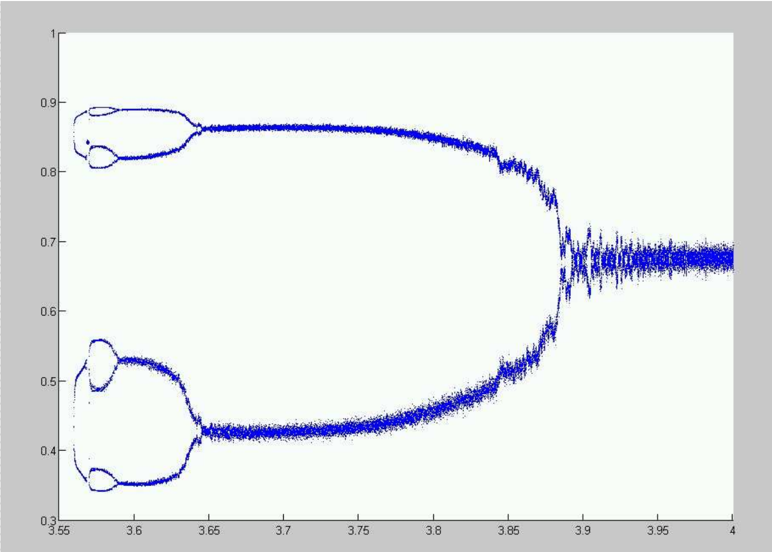
$$\text{where} \quad \overline{X}_{lm} = \sum_{i\in\Omega(l,m)\cup\{l,m\}} X_i \;/\; N$$

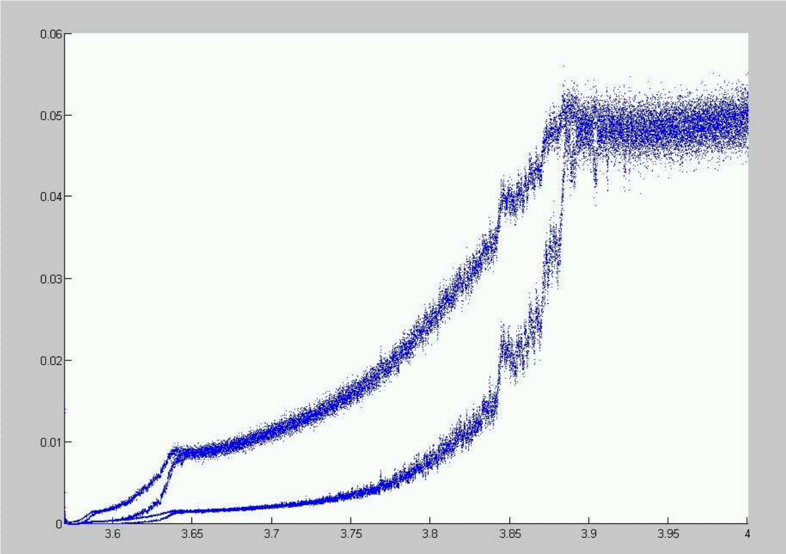
$$x^{n+1} = k \cdot x^{n} \cdot \left(1 - x^{n}\right) - k \cdot \sigma_n^{\;2}$$

$$\sigma_n^{\;2} = E\left[\left(\overline{X}_{lm}^{\,n}\right)^2\right] - E^2\left(\overline{X}_{lm}^{\,n}\right)$$



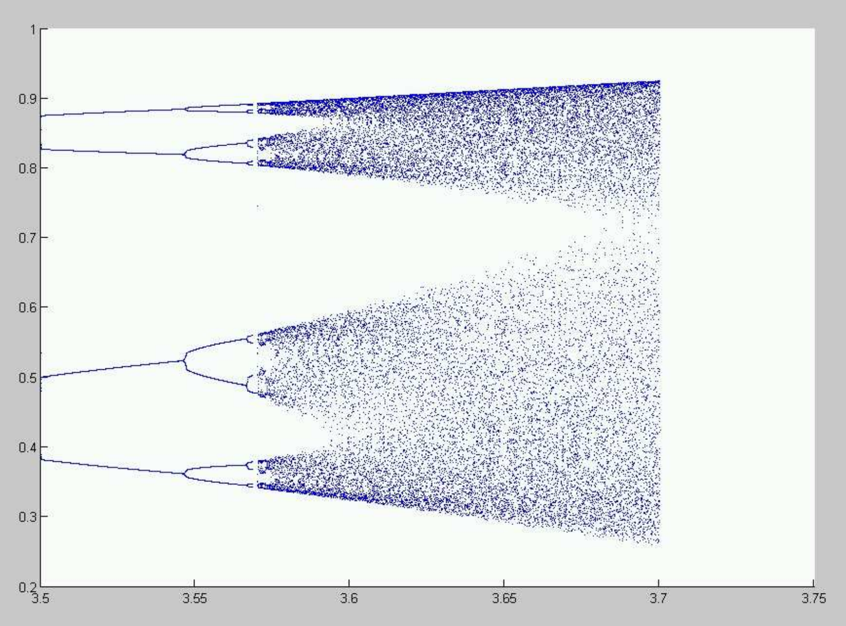
The variance dynamics

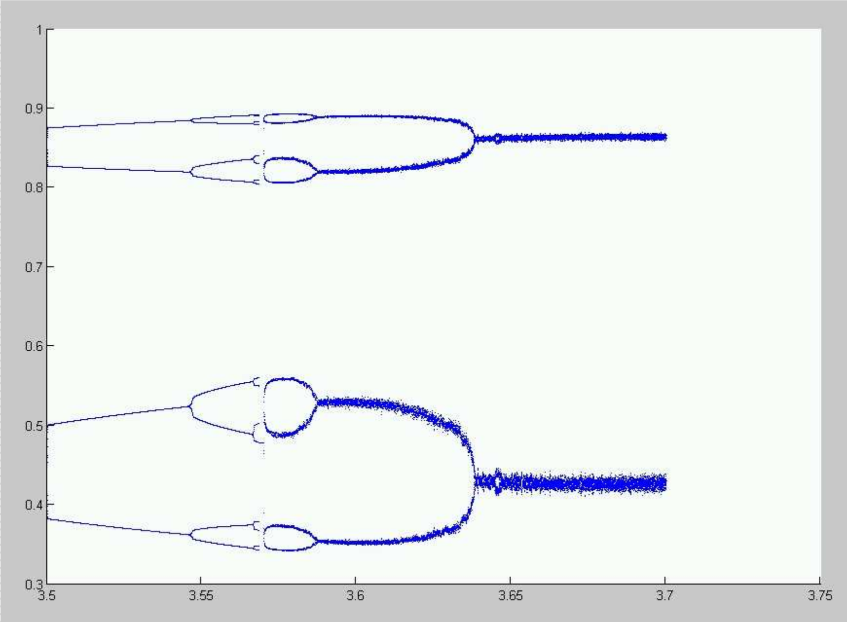




If the variance $\sigma^2=0$ the entire system behaves according to the logistic law. The non-zero variance signals the existence of spatial structures.

Spatio-temporal vs. global chaos





The value of $X_{10,10}$ for increasing k

The value of average population density *x* for increasing k

Conclusions

- There exist a class of stochastic CA which global behavior replicates LM, assuming:
  - certain class of microscopic rules (specifically, but not exceptionally, micro-logistic law)
  - untied motion of individuals destroying patterns
  - sufficiently large neighborhood
- The populations with different dispersion and competition mostly cannot mimic LM.
- Motionless populations creates spatio-temporal structures, which stops global chaotic behaviour (in terms of x)
- For large neighborhood, motionless populations can replicate exact LM up to accumulation point k_∞ due to strong synchronization producing uniform distribution of population.
- For $k>k_\infty$ instead the chaotic behavior of x, the spatio-temporal chaos is observed.