Order Independence in Asynchronous Cellular Automata

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Sequential Dynamical Systems – Definitions

- An SDS is a triple consisting of:
  - A graph Y with vertex set $\{1, 2, \ldots, n\}$.
  - For each vertex i a state $y_i \in K$ (e.g. $\mathbb{F}_2 = \{0, 1\}$) and a local function $F_i \colon K^n \longrightarrow K^n$

$$F_i(\mathbf{y} = (y_1, y_2, \ldots, y_n)) = (y_1, \ldots, y_{i-1}, \underbrace{f_i(\mathbf{y}[i])}_{\text{vertex function}}, y_{i+1}, \ldots, y_n) \ .$$

  - A word w of length m over $\mathrm{v}[Y] = \{1, 2, \ldots, n\}$.
- The SDS map generated by the triple $(Y, (F_i)_1^n, w)$ is

$$\boxed{[\mathfrak{F}_Y, w] = F_{w(m)} \circ F_{w(m-1)} \circ \cdots \circ F_{w(1)} \ .}$$

Example: Asynchronous Cellular Automata

Let $Y = \text{Circ}_n$, the circular graph on n vertices.

If $k = a_7a_6a_5a_4a_3a_2a_1a_0$ in binary, then *Wolfram rule* k is defined by $\text{wolf}^{(k)}\colon (y_{i-1}, y_i, y_{i+1}) \mapsto z_i$ by the following table.

$y_{i-1}y_iy_{i+1}$	111	110	101	100	011	010	001	000
z_i	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0

Let $\text{Wolf}^{(k)}\colon \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$ be the corresponding local function, and $\mathfrak{Wolf}_n^{(k)} = (\text{Wolf}^{(k)})$ the sequence of local functions of Circ_n.

The SDS $[\mathfrak{Wolf}_n^{(k)}, \pi]$, where $\pi \in S_Y$, is an *asynchronous cellular automata (*ACA*)*.

Tags of Wolfram rules

We can arrange the binary digits of k in the following table.

	110	100	
111	101	010	000
	011	001	

	a_6	a_4	
a_7	a_5	a_2	a_0
	a_3	a_1	

	p_3	
p_4		p_1
	p_2	

There are 4 possibilities for each pair:

$$\text{`1'} = \boxed{1 \quad 1} \qquad \text{`0'} = \boxed{0 \quad 0} \qquad \text{`-'} = \boxed{1 \quad 0} \qquad \text{`x'} = \boxed{0 \quad 1}$$

The *tag* of Rule k is $p_4 p_3 p_2 p_1$, where each $p_i \in \{0, 1, -, \mathtt{x}\}$.

The substring $p_4 p_1$ represents the symmetric part of Rule k, and $p_3 p_2$ represents the antisymmetric part.

w-independence

- A sequence $\mathfrak{F}_Y$ is π-*independent* (w-independent) if $\mathrm{Per}[\mathfrak{F}_Y, w] = \mathrm{Per}[\mathfrak{F}_Y, w']$ for all w and w' in S_Y (W_Y).

Proposition

$\mathfrak{F}_Y$ is π-independent iff it is w-independent.

Main theorem

Theorem (Hansson, Mortveit, Reidys, 2005)

Of the 16 symmetric Wolfram rules, exactly 11 are w-independent for all $n > 3$.

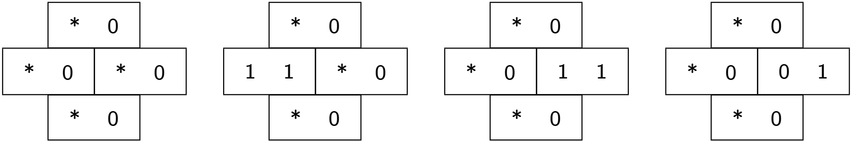
Theorem (Macauley, McCammond, Mortveit, 2007)

Of the 256 Wolfram rules, exactly 104 *are w-independent. More precisely,* $\mathfrak{Wolf}_n^{(k)}$ *is w-independent for all* $n > 3$ *iff* $k \in \{$*0, 1, 4, 5, 8, 9, 12, 13, 28, 29, 32, 40, 51, 54, 57, 60, 64, 65, 68, 69, 70, 71, 72, 73, 76, 77, 78, 79, 92, 93, 94, 95, 96, 99, 102, 105, 108, 109, 110, 111, 124, 125, 126, 127, 128, 129, 132, 133, 136, 137, 140, 141, 147, 150, 152, 153, 156, 157, 160, 164, 168, 172, 184, 188, 192, 193, 194, 195, 196, 197, 198, 199, 200, 201, 202, 204, 205, 206, 207, 216, 218, 220, 221, 222, 223, 224, 226, 228, 230, 232, 234, 235, 236, 237, 238, 239, 248, 249, 250, 251, 252, 253, 254, 255*$\}$.

These 104 rules constitute 41 distinct classes up to equivalence (inversion and reflection).

Major classes of *w*-independent Wolfram rules

- The 104 Wolfram rules fall into one of three categories:
  - Invertible rules
  - Rules of the following form



  - One of 6 exceptional cases: 32, 40, 152, 184, 28, 29.

The main technique used in the second case was the use of *potential functions*.

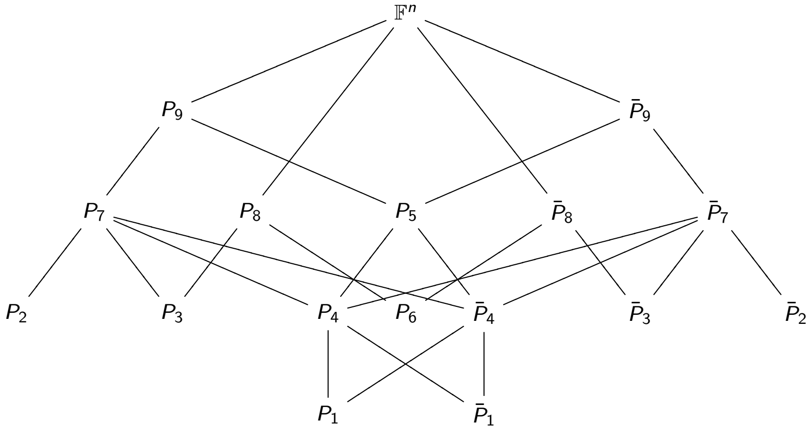
Periodic point sets

- Disregarding the constant states $\mathbf{0}$ and $\mathbf{1}$, the following are the only sets of periodic points that arise up to inversion:

$P_{n,1}$: {No '11', '000'},
$P_{n,2}$: {No '11', '010'},
$P_{n,3}$: {No '11', '101'},
$P_{n,4}$: {No '000', '111', '1100'},
$P_{n,5}$: {No '000', '111'},
$P_{n,6}$: {No '101', '010'},
$P_{n,7}$: {No '11'},
$P_{n,8}$: {No '101'},
$P_{n,9}$: {No '111'}
$\mathbb{F}_2^n$

Periodic point poset

- The sets of periodic points form the following poset:



Definitions

Proposition

If $\mathfrak{F}_Y$ is w-independent, then each F_i is bijective on $P := \mathrm{Per}(\mathfrak{F}_Y)$.

Let $[\mathfrak{F}_Y, \omega]^*$ denote the restriction of $[\mathfrak{F}_Y, \omega]$ to the set of periodic points.

If $W' \subseteq W_Y$ then the group

$$H(\mathfrak{F}_Y, W') = \langle [\mathfrak{F}_Y, \omega]^* \colon \omega \in W' \rangle$$

is called the *dynamics group* of $\mathfrak{F}_Y$ and W'.

- Full dynamics group: $G(\mathfrak{F}_Y) := H(\mathfrak{F}_Y, W_Y) = \langle F_i^* \colon F_i \in \mathfrak{F}_Y \rangle$,
- Permutation dynamics group: $H(\mathfrak{F}_Y) := H(\mathfrak{F}_Y, S_Y) = \langle [\mathfrak{F}_Y, \pi]^* \colon \pi \in S_Y \rangle$.

Computation

- The dynamics group is the homomorphic image of a Coxeter group: $|F_i| \leq 2$ and $|F_iF_j| = m_{ij}$ for $m_{ij} \in \{1, 2, 3, 4, 6, 12\}$.
- Of the 41 non-equivalent rules, only 15 of them have a non-trivial dynamics group.
  - $S_{L(n)}$ or $A_{L(n)}$: Rules 1, 9, 110, 126.
  - $\mathbb{Z}_2^n$: Rules 28, 29, 51.
  - A_n or A_{n-1}: Rules 54, 57
  - $GL(2, n)$: Rule 60.
  - Not sure: Rules 73, 105, 108, 150, 156

Flips

▶ For each of the 8 neighborhood state configurations (y_{i-1}, y_i, y_{i+1}), Wolfram rule k can be though of as either preserving, or "flipping" the value y_i.

# flips	0	1	2	3	4	5	6	7	8
# w-independent rules	1	8	26	34	26	4	4	0	1
# of rules	1	8	28	56	70	56	28	8	1
Percentage	100%	100%	93%	61%	37%	7%	14%	0%	100%

All 5 w-independent rules with more than 5 flips are invertible.

This can be extended to SDSs. Call $0 \mapsto 1$ and *up-flip* and $1 \mapsto 0$ a *down-flip*. Define the *signature* of $\mathfrak{F}_Y$ to be the number of up-flips minus the number of down-flips.

The signature is an indication of stability, and a good starting point for the study of update-order stochastic SDSs.

Table of the 104 rules

	p_3 / p_2	– / –	– / 0	0 / –	0 / 0	– / 1	1 / –	1 / 1	– / x	x / –	x / x
$p_4 p_1$		72	64	8	0	74	88	90	66	24	18
––	132	204	196	140	132	206	220	222	198	156	150
0–	4	76	68	12	4	78	92	94	70	28	
–0	128	200	192	136	128	202	216	218	194	152	
1–	164	236	228	172	164	238	252	254	230	188	
–1	133	205	197	141	133	207	221	223	199	157	
10	160	232	224	168	160	234	248	250	226	184	
01	5	77	69	13	5	79	93	95	71	29	
00	0	72	64	8	0						
x0	32		96	40	32						
0x	1	73	65	9	1						
–x	129	201	193	137	129				195	153	147
x–	36	108				110	124	126	102	60	54
x1	37	109				111	125	127			
1x	161					235	249	251			
11	165	237				239	253	255			
xx	33	105							99	57	51

Table: The 104 *w*-independent rules arranged by symmetric and asymmetric parts of their tags.

Table of the 104 rules, arranged by m_{ij}

	p_3 / p_2	x / x	- / x	x / -	- / -	- / 0	0 / -	0 / 0	- / 1	1 / -	1 / 1
$p_4 p_1$		18	66	24	72	64	8	0	74	88	90
0x	1				6	6	6	6			
x1	37				6				6	6	6
-x	129	12	4	4	6	6	6	6			
x-	36	12	4	4	6				6	6	6
xx	33	2	12	12	3						
--	132	3	6	6	1	1	1	1	1	1	1
-0	128		1	1	1	1	1	1	1	1	1
1-	164		1	1	1	1	1	1	1	1	1
10	160		1	1	1	1	1	1	1	1	1
-1	133		2	2	1	1	1	1	1	1	1
0-	4		2	2	1	1	1	1	1	1	1
01	5		2	2	1	1	1	1	1	1	1
00	0				1	1	1	1			
11	165				1				1	1	1
1x	161								1	1	1
x0	32					1	1	1			

Table: The 104 *w*-independent rules arranged by Coxeter number m_{ij}

Future research

- Finish analyzing the dynamics groups.
- Analyze the other 152 rules.
- Extend these ideas and techniques to general SDSs.
- Use these ideas and techniques to study stochastic systems.
- Compare to the dyamics of classical (synchronous) CAs.

SDS – Collaborators, Papers, Info

Joint work with: Jon McCammond, Henning Mortveit

SDS **course web page with link to papers:**

Web: http://www.math.vt.edu/people/hmortvei/class_home/4984_15748.html

NDSSL:

Web: http://ndssl.vbi.vt.edu