Classifying
Fuzzy Cellular Automata

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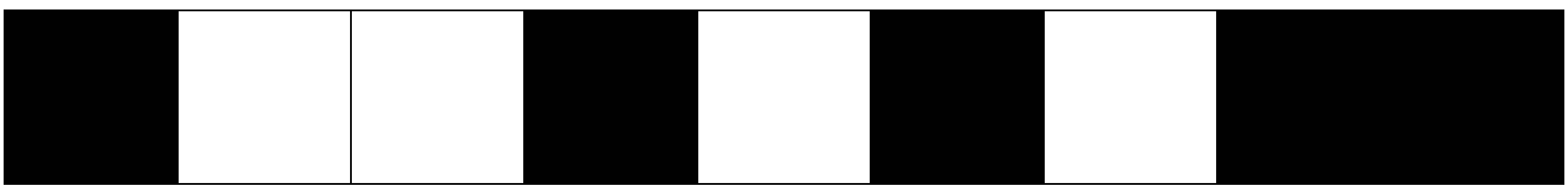
A (k,r)-CA is a regular uniform lattice, finite or not, each cell containing a discrete value. The state of the automaton at cell i for time t is given by x_i^t where x_i^t takes one of k possible values and

$$x_i^{t+1} = g(x_{i-r}^t, \ldots, x_i^t, \ldots, x_{i+r}^t),$$

defines the evolution of the CA. g is sometimes called the local rule of the CA. Here we consider (2,1)-CA, or ECA/FCA….



1	0	0	1	0	1	0	1	1

A local rule is then defined by the 8 possible local configurations a cell can detect in its immediate two-sided neighborhood:

(000, 001, 010, 011, 100, 101, 110, 111) à (r_0,….r_7)

0 1 0 1 1 0 1 0

The name of the rule = 90= $r_0 2^0 + r_1 2^1 + \cdots + r_7 2^7$

A local rule has a disjunctive normal form

$$g(x_1, x_2, x_3) = \bigvee_{i|r_i=1} \bigwedge_{j=1:3} x_j^{d_{ij}}$$

where d_{ij} is the j-th digit, from left to right, of the binary expression of i, and x^0 (resp. x^1) stands for $\neg x$ (resp. x).

Example

$90 = 01011010_2$

$r_i = 1, \quad i = 1, 3, 4, 6$ only

$$6 = \mathbf{0} \cdot 2^0 + \mathbf{1} \cdot 2^1 + \mathbf{1} \cdot 2^2 + \mathbf{0} \ldots$$

$$d_{61} = 0, d_{62} = 1, d_{63} = 1$$

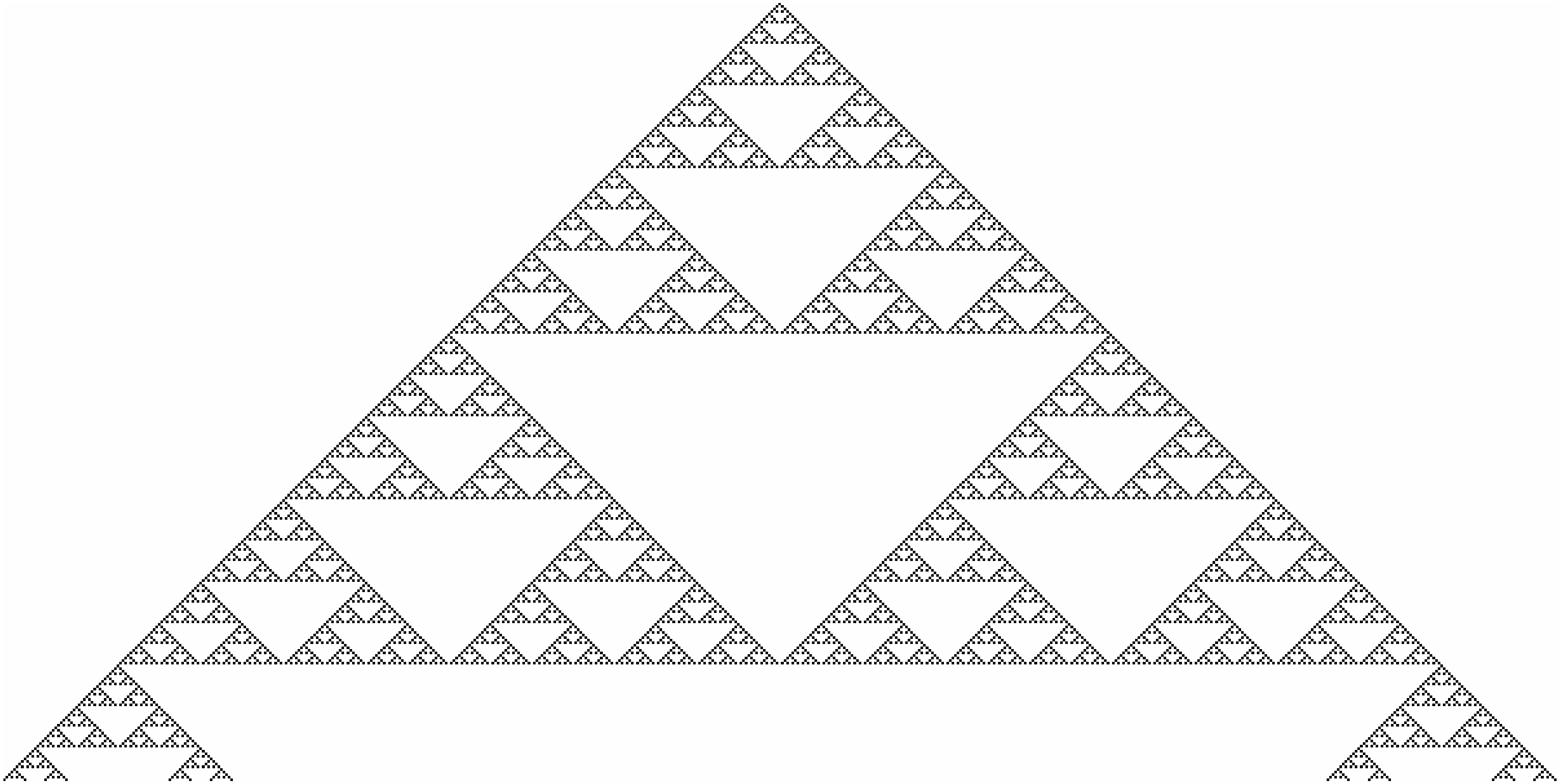
Boolean Rule 90

$$g_{90}(x_1, x_2, x_3) = (x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3)$$

Fuzzy Rule 90

$$\wedge \equiv \cdot, \vee \equiv +, \neg x \equiv 1 - x$$

$$g_{90}(x_1, x_2, x_3) = x_1 + x_3 - 2x_1 x_3$$



Rule 90– Boolean/Fuzzy

Fuzzy Rule 90

cf. Flocchini-Geurts-Mingarelli-Santoro, Physica D: 2000

Evolution from a single seed "a"

$$x_i^t = \begin{cases} \frac{1}{2}(1-(1-2a)^{f(t,i)}) & \textit{if } t+i \textit{ is even and } i \in \{-t,\cdots,t\} \\ 0 & \textit{otherwise} \end{cases}$$

$$f(t,i) = \binom{t}{\frac{t+i}{2}};$$

… Evolution to ½ for arbitrary number of seeds

Fuzzy Rule 110

cf. Mingarelli, WSEAS Trans. Computers, **2** (4) (2003), 1102-1107.

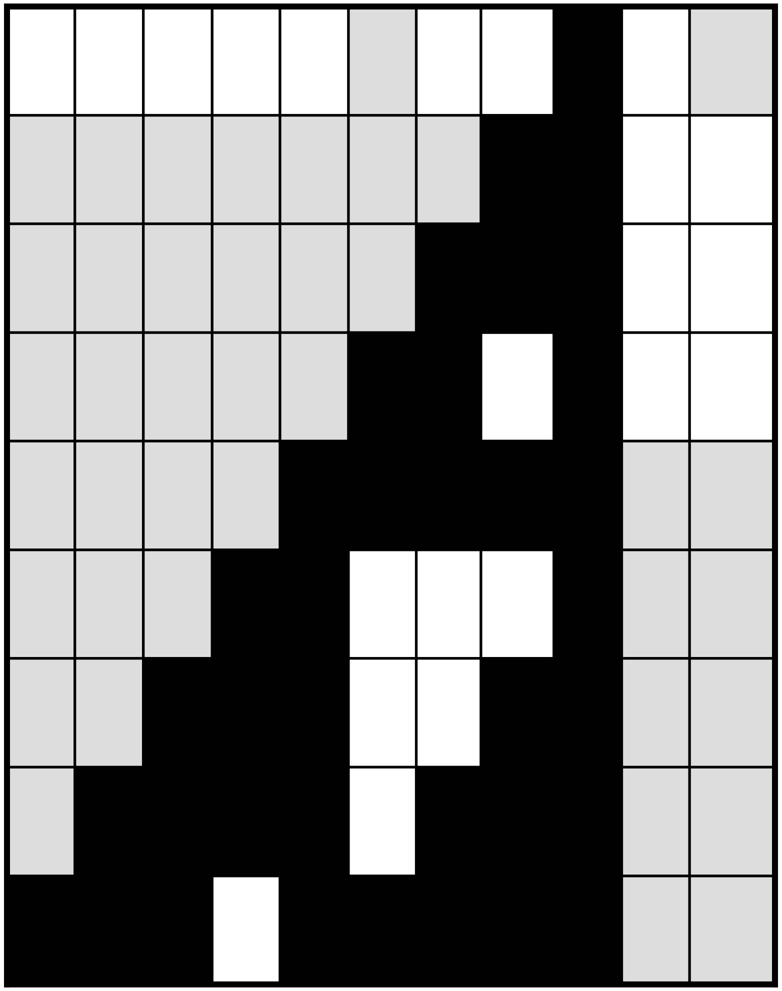
$$g_{110}(x, y, z) = y + z - yz - xyz$$

Fixed Points? 1.618, 0, 0.618 (attracting)

$$\phi = (\sqrt{5}-1)/2$$

Expect: Convergence to the golden number

Fuzzy Rule 110



Boolean

								.5		
							.5	.5		
						.5	.75	.5		
					.5	.87	.67	.5		
				.5	.43	.66	.54	.5		
			.5	.97	.67	.51	. 59	.5		
		.5	.98	.66	.50	.60	.65	.5		
	.5	.99	.66	.51	.60	.66	.63	.5		
.	..	..	..	..	..	..	..	..		

Fuzzy

Fuzzy Rule 110

- Let L_0^- (resp. L_k^-) denote the LIMIT of the first (resp. k-th) left-most diagonal (non-zero sequence).
- Result: For a single seed "a" in a zero background,

$$L_0^- = a, L_1^- = 1.$$

$$L_2^- = \frac{1}{1+a},\ L_3^- = \frac{1}{2},\ L_4^- = \frac{a+1}{a+2},\ L_5^- = \frac{2}{3}.$$

$$L_{2n}^- = \frac{a\,F_{n-1} + F_n}{a\,F_n + F_{n+1}},\ L_{2n-1}^- = \frac{F_n}{F_{n+1}}$$

where F_n is the n-th Fibonacci number: F_0=1, F_1=1, F_2=2, …

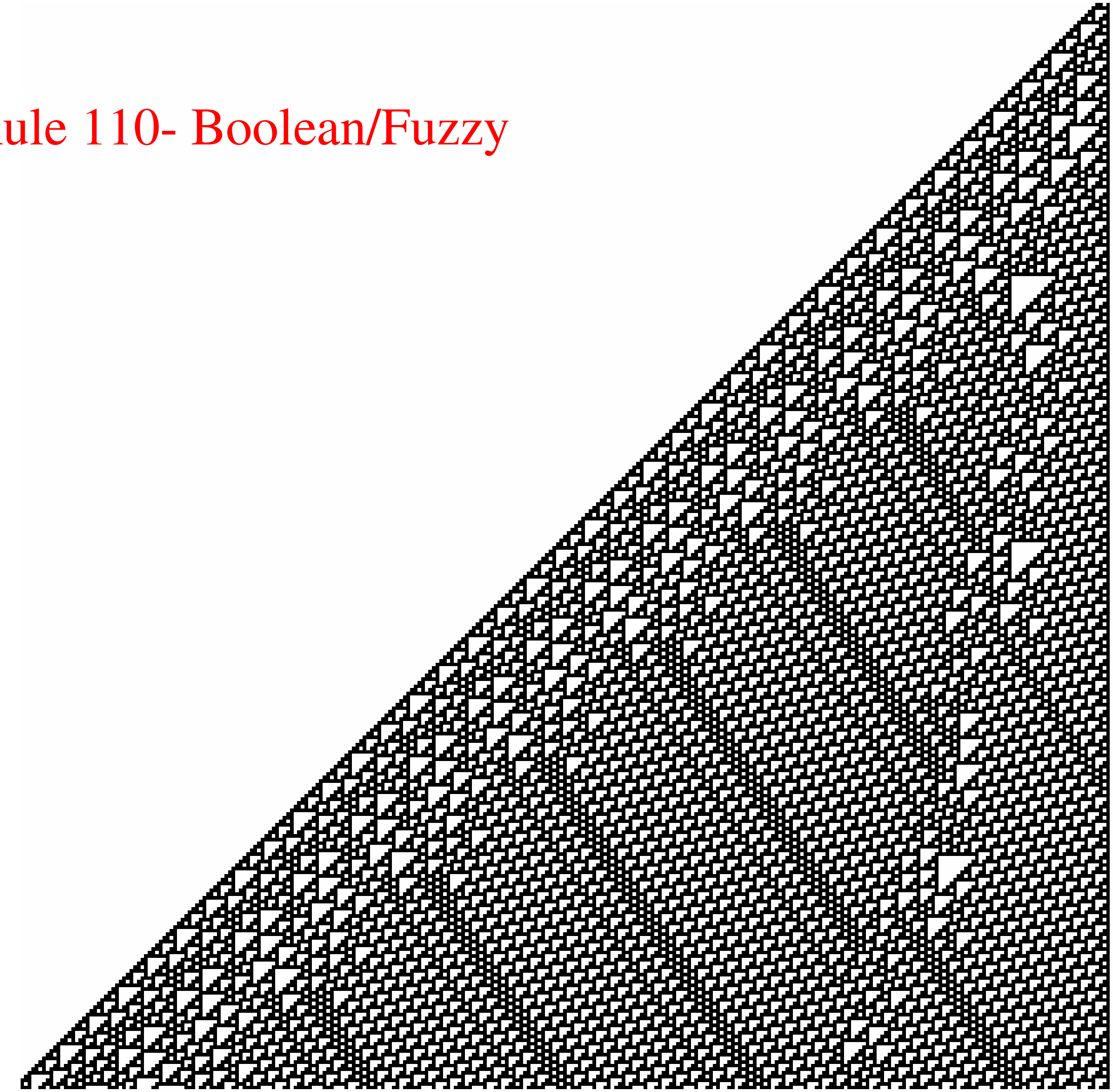
Fuzzy Rule 110

BONUS!

The left-diagonal limits themselves converge to a limit that is independent of the seed value, a. In fact,

$$\lim_{k \to \infty} L_k^- = \frac{\sqrt{5}-1}{2},$$

Rule 110- Boolean/Fuzzy



Notes:

1. Sum of all Rules =128

2. All rules are linear in each variable separately

3. All rules have a **discrete set of fixed points** except: 170, 172, 184, 202, 204 216, 226, 228, 240 (called exceptional rules)

ABM: **J. Cellular Automata,** 1 (2) (2006), 141-164

1. Bearing of such dynamics on Boolean CA's??
2. New CA classification scheme?

First ECA Classification Scheme

(S. Wolfram, 1980's)

- Classification based on "visual" attributes of space-time diagrams of 255 boolean CA's
- 4 main classes
- This complete classification impossible to find

The Four Wolfram Classes : Starting from a randomly chosen initial configuration an ECA rule has an ….

1. "evolution that leads to a homogeneous (stable) configuration"

   (homogeneous)

2. "evolution that leads to periodically repeating patterns"

   (regular)

3. "evolution that leads to "chaotic" patterns"

   (chaotic)

4. "evolution that leads to complex patterns, generated by mobile interacting structures which are relatively long lived."

   (complex)

Classification??

- David Hilbert: a satisfactory solution to any classification problem should provide an algorithm for determining whether a given object in the set has a specific property or not.
- Has a bearing on Wolfram's classification of ECA …

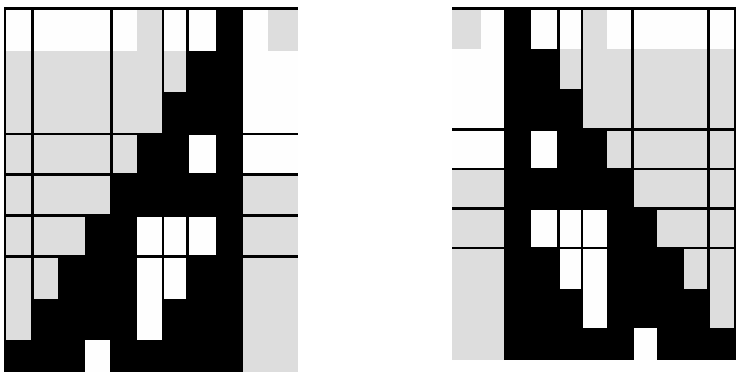
Wish-List for Classifying ECA

- "Equivalent" ECA should belong to the same Class
- Corresponding "equivalent" FCA should have the same fixed points and these f.p. …
- … should be of the same type (attracting/repelling)

F-equivalence

Two FCA are F-equivalent if their space-time diagrams are "mirror reflections" of each other…

Example: FCA 110 and FCA 124



EXAMPLES

30 135 86 149

F-equivalent rules

179 50 179 50

Properties of F-equivalent FCA

1) Every FCA is F-equivalent to only ONE other FCA (maybe itself)
2) Corresp. local rules have same f.p.
3) … same type of f.p.(attr./rep.)

Classifying FCA's:

Given any FCA, local rule g(x,y,z), **EVEN** rule number: Then, **for any x** in (0,1], the function G:[0,1]->[0,3] defined by

$$\mathbf{G(x) = g(x,0,0)+g(0,x,0)+g(0,0,x)}$$

has the property that

$$\frac{G(x)}{x} = \begin{cases} 0 & \text{if} \quad 0 < x \leq 1, \\ 1 & \text{if} \quad 0 < x \leq 1, \\ 2 & \text{if} \quad 0 < x \leq 1, \\ 3 & \text{if} \quad 0 < x \leq 1. \end{cases}$$

Definition

These invariants then define 4 natural classes--
Assuming the rule number is even, an FCA is

CLASS 1:	**iff**	**$G(x)/x = 0$**
CLASS 2:	**iff**	**$G(x)/x = 1$**
CLASS 3:	**iff**	**$G(x)/x = 2$**
CLASS 4:	**iff**	**$G(x)/x = 3$.**

Consequences

F-equivalent rules must belong to the same Class…

All the exceptional rules are Class 2

(FCA 170,172,184,202,204,216,226,228,240)

There are three times as many additive rules in Classes 2 and 3 than there are in 1 and 4

(1,3,3,1)

Class 1	Class 2	Class 3	Class 4
0	2, 4, 10	6, 14, 18	22
8	12, 16, 24	20, 26, 28	30
32	34, 36, 42	38, 46, 50	54
40	44, 48, 56	52, 58, 60	62
64	66, 68, 74	70, 78, 82	86
72	76, 80, 88	84, 90, 92	94
96	98, 100, 106	102, 110, 114	118
104	108, 112, 120	116, 122, 124	126
128	130, 132, 138	134, 142, 146	150
136	140, 144, 152	148, 154, 156	158
160	162, 164, 170	166, 174, 178	182
168	172, 176, 184	180, 186, 188	190
192	194, 196, 202	198, 206, 210	214
200	204, 208, 216	212, 218, 220	222
224	226, 228, 234	230, 238, 242	246
232	236, 240, 248	244, 250, 252	254

Table 2. Classification of even numbered rules according to the values of the G-function

Comparisons of those rules for which an explicit classification was given by Wolfram [NKS, p.232].

Examples of Wolfram classes (not complete):

Class 1: 0, 32, 72, 104, 128, 160, 200 and 232

Class 2: 4, 36, 76, 108, 132, 164, 204 and 236

Class 3: 18, 50, 90, 122, 146, 178, 218, and 250

Class 4: 22, 54, 94, 126, 150, 182, 222, and 254

Contrast: Wolfram places ECA 110 in Class 4 (Class 3 according to above classification).

- Idea is to find a classification that encompasses as many of the common characteristics of all rules as is possible. Deviations are to be expected.

Future work:

• Relationship of this to other classifications (e.g., Li-Packard, Cattaneo et al, Culik, Gutowicz, etc?)

•Relationship of this classification with asymptotics of space-time diagrams (in preparation) …