Sums of dilates

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$$\{1,2,3\} + \{1,2,3\} + \{1,2,3\} = ?$$

A dilate is...

Definition

A λ-dilate of $A \subset \mathbb{Z}$ is the set

$$\lambda \cdot A = \{\lambda a : a \in A\}.$$

Example

$$2 \cdot \{1, 3, 4\}$$
$$= \{2, 6, 8\}$$

Basic problem

Fix $\lambda_1, \ldots, \lambda_k$. *How large is*

$$\lambda_1 \cdot A + \lambda_2 \cdot A + \cdots + \lambda_k \cdot A?$$

Lower bounds

Theorem

$$|A + A| \geq 2|A| - 1$$ *Folklore*

$$|A + 2 \cdot A| \geq 3|A| - 2$$ *Nathanson'07*

$$|A + 3 \cdot A| \geq 4|A| - O(1)$$ *B., Cilleruelo-Silva-Vinuesa*

Lower bounds

Theorem

$$|A + A| \geq 2|A| - 1 \qquad \textit{One line}$$

$$|A + 2 \cdot A| \geq 3|A| - 2 \qquad \textit{Two lines}$$

$$|A + 3 \cdot A| \geq 4|A| - O(1) \qquad \textit{One page}$$

Difficulty

For $A + 3 \cdot A$ there are two kinds of sets A that achieve the minimum:

- $\{1, 2, 3, 4, 5, 6, \ldots, n\}$, *an arithmetic progression*
- $\{1, 2, 4, 5, 7, 8, \ldots, 3k + 1, 3k + 2\}$, *a union of two residue classes modulo* 3.

For $A + \lambda \cdot A$ there should be up to $2^{\lambda - 2}$ distinct minimizers.

Lower bounds

Theorem

$$|A+A| \geq 2|A|-1 \qquad \textit{One line}$$
$$|A+2\cdot A| \geq 3|A|-2 \qquad \textit{Two lines}$$
$$|A+3\cdot A| \geq 4|A|-O(1) \qquad \textit{One page}$$

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Lower bounds

Theorem (B.)

If $\lambda_1, \ldots, \lambda_k$ are fixed coprime integers, then

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \geq \big(|\lambda_1| + |\lambda_2| + \cdots + |\lambda_k|\big)|A| - o(|A|)$$

The numbers $\lambda_1, \ldots, \lambda_k$ do not have to be pairwise coprime or positive:

$$|6 \cdot A - 10 \cdot A + 15 \cdot A| \geq (6 + 10 + 15)|A| - o(|A|).$$

Plünnecke-type upper bounds

Theorem (Plünnecke'70, Ruzsa'89)

If $|A+A| \leq K|A|$, *then*

$$|\underbrace{A+\cdots+A}_{t \text{ times}}| \leq K^t|A|$$

Corollary

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \leq K^{|\lambda_1|+\cdots+|\lambda_k|}|A|.$$

Plünnecke-type upper bounds

Corollary of Plünnecke's inequality

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \leq K^{|\lambda_1| + \cdots + |\lambda_k|}|A|.$$

Theorem (B.)

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \leq K^{20[\log(1+|\lambda_1|) + \cdots + \log(1+|\lambda_k|)]}|A|.$$

Plünnecke-type upper bounds

Theorem (B.)

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \leq K^{20[\log(1+|\lambda_1|)+\cdots+\log(1+|\lambda_k|)]}|A|.$$

Corollary

If $k \geq 3$ and the set $A \subset \{1, \ldots, N\}$ does not contain a solution to the symmetric linear equation

$$\lambda_1 x_1 + \cdots + \lambda_k x_k = \lambda_1 y_1 + \cdots + \lambda_k y_k,$$

then $|A| = O(N^{1/2 - 1/\log\|\lambda\|_1})$.

Idea #1 for the lower bound

Theorem

For fixed integer $\lambda > 0$

$$|A + \lambda \cdot A| \geq (\lambda + 1)|A| - o(|A|)$$

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For fixed integer $\lambda > 0$

$$|A + \lambda \cdot A| \geq (\lambda + 1 - \varepsilon)|A|$$

for sufficiently large $|A|$

Let $A_i = \{a \in A : a \bmod \lambda = i\}$.

Observation

$$|A + \lambda \cdot A| \geq |A_1 + \lambda \cdot A_1| + \cdots + |A_\lambda + \lambda \cdot A_\lambda|$$

Suffices to establish the theorem for A_i's

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Idea #2 for the lower bound

$A_i = \{a \in A : a \bmod \lambda = i\}$.

Lemma

If $A_1, \ldots, A_\lambda$ are all non-empty, then $|A + \lambda A| \geq (\lambda + 1)|A| - \lambda$.

Proof.

$$\left.\begin{array}{l}\text{All elements of } A_1 + \lambda \cdot A \text{ are congruent to } 1 \text{ modulo } \lambda. \\ \qquad\qquad\vdots \\ \text{All elements of } A_\lambda + \lambda \cdot A \text{ are congruent to } \lambda \text{ modulo } \lambda.\end{array}\right\} \textit{Disjoint}$$

$$\begin{aligned} |A + \lambda \cdot A| &= |A_1 + \lambda \cdot A| + \cdots + |A_\lambda + \lambda \cdot A| \\ &\geq |A_1| + |A| - 1 + \cdots + |A_\lambda| + |A| - 1 \\ &= (|A_1| + \cdots + |A_\lambda|) + \lambda|A| - \lambda. \end{aligned}$$

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$\square$

Proof overview

A is a subset of density $\alpha > 0$ of some interval.

Algorithm to show $|A + \lambda \cdot A| \geq (\lambda + 1 - \epsilon)|A|$

1. If $A_1, \ldots, A_\lambda$ are all non-empty, done.
2. Otherwise average density of non-empty sets among $A_1, \ldots, A_\lambda$ is larger than that of A. Recurse with A replaced by A_i for $i = 1, \ldots, \lambda$.

Each step the average density increases, and the algorithm halts.

An annoying problem

Corollary to Plünnecke

Always $|A + 2 \cdot A| \leq \left(\frac{|A+A|}{|A|}\right)^3$.

Question

Is $|A + 2 \cdot A| \leq \left(\frac{|A+A|}{|A|}\right)^{3-\varepsilon}$ for some $\varepsilon > 0$?