Nilsequences in ergodic theory

Bernard Host
Université de Marne la Vallée

Joint work with

Bryna Kra
Northwestern University

What is a nilsequence?

Notation: for $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $e(t) = \exp(2i\pi t)$.

A trigonometric polynomial is a sequence $\underline{a} = (a_n \colon n \in \mathbb{Z})$ of the form:

$$a_n = \sum_{j=1}^{J} \lambda_j \, e(nt_j) \ .$$

An almost periodic sequence is a uniform limit of trigonometric polynomials.

Equivalent definition: Let G be a compact abelian group, $\alpha \in G$ and $f \in \mathcal{C}(G)$. Then the sequence $\underline{a} = (f(\alpha^n) \colon n \in \mathbb{Z})$ is an almost periodic sequence.

Let G be a k step nilpotent Lie group and Γ a discrete cocompact subgroup.
Then $X = G/\Gamma$ is called a k step nilmanifold.

Choose $\tau \in G$ and define $T\colon X \to X$ by $Tx = \tau \cdot x$.
Then (X, T) is a k step nilsystem.

Choose $x_0 \in X$ and a continuous function f on X.
Then the sequence $\underline{a} = (f(\tau^n \cdot x_0)\colon n \in \mathbb{Z})$ is a basic k step nilsequence.

$\underline{a}$ is called a smooth nilsequence if f is smooth.

A k step nilsequence is a uniform limit of basic nilsequences.

k step nilsequences form a shift invariant subalgebra of ℓ^∞.

Examples of nilsequences

- 1 step nilsequences are almost periodic sequences.

- A smooth 1 step nilsequence has the form:

$$a_n = \sum_{j=1}^{\infty} \lambda_j \, e(t_j n) \text{ where } \sum_{j=1}^{\infty} |\lambda_j| < +\infty \, .$$

- If p is a polynomial of degree k with real coefficients then

$$\Big(e(p(n)\colon n \in \mathbb{Z}\Big)$$

is a k step smooth nilsequence.
Example: $(e(n^2\alpha + n\beta)\colon n \in \mathbb{Z})$ is a 2 step nilsequence.

- The Heisenberg group is

$$H = \left\{ (x,y,z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x,y,z \in \mathbb{R} \right\}$$

It is a 2 step nilpotent Lie group.

$$\Gamma = \Big(\{ (j,k,\ell) \colon j,k,\ell \in \mathbb{Z} \Big\}$$

is a discrete cocompact subgroup.

$X = H/\Gamma$ is the Heisenberg nilmanifold.

Recall that:

$$\theta(u,z) = \sum_{k\in\mathbb{Z}} \exp(-\pi i k^2 z + 2i\pi k u) \quad \text{for } z,u \in \mathbb{C},\ \text{Im}(z) > 0\ .$$

The sequence $\underline{b} = (b_n \colon n \in \mathbb{Z})$ given by

$$b_n = \exp(-\pi n^2\beta^2)\,\theta(n\alpha + in\beta, i)$$

is a 2 step smooth nilsequence arising from the Heisenberg nilmanifold.

• The sequences of this type and the quadratic exponential sequences $(e(n^2\alpha + n\beta))$ "span" the algebra of 2 step nilsequences.

Nilsequences in ergodic theory

Sequences of this type are common in the literature, but it seems that the family of nilsequences and their name were introduced in [Bergelson, H, Kra (2005)]:

Theorem. *Let (X, μ, T) be an ergodic system, $k \geq 1$ an integer and $f \in L^\infty(\mu)$. Then the sequence (a_n) given by*

$$a_n = \int f.T^n f.T^{2n} f. \cdots .T^{kn} f \, d\mu$$

is the sum of a k step nilsequence and a sequence tending to zero in density.

Here and below, $T^n f = f \circ T^n$.

Recent results [H, Kra]

Wiener-Wintner Theorem. *Let (X, μ, T) be an ergodic system and $\phi \in L^\infty(\mu)$. Then there exists $X_0 \subset X$ with $\mu(X_0) = 1$ such that*

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(T^n x_0)\, e(nt) \text{ converges as } N \to +\infty$$

for every $x_0 \in X_0$ and every $t \in \mathbb{T}$.

The important point is that X_0 does not depend on t.

Theorem (Generalization). *Under the same hypothesis,*

$$\frac{1}{N}\sum_{n=0}^{N-1} \phi(T^n x_0)\, a_n) \text{ converges as } N \to +\infty$$

for every $x_0 \in X_0$ and every nilsequence $\underline{a} = (a_n \colon n \in \mathbb{Z})$.

Theorem. *Let* $\underline{b} = (b_n)$ *be a bounded sequence and assume that*

$$\frac{1}{N}\sum_{n=0}^{N-1} a_n b_n$$

converges for every k *step nilsequence* $\underline{a} = (a_n)$. *Then, for every system* (Y, ν, S) *and all functions* $f_1, \ldots, f_k \in L^\infty(\nu)$, *the averages*

$$\frac{1}{N}\sum_{n=0}^{N-1} b_n \cdot S^n f_1 \cdot S^{2n} f_2. \cdots .S^{kn} f_k$$

converge in $L^2(\nu)$.

[Q. Chu]: Generalization with integer polynomials $p_1(n), \ldots, p_k(n)$ substituted for $n, \ldots, kn$

There are many sequences $\underline{b}$ satisfying the hypotheses of this theorem:

$$\frac{1}{N}\sum_{n=0}^{N-1} a_n b_n \textit{ converges for every step nilsequence } \underline{a}\ .$$

For example, $\Big(\{p(n)\} \colon n \in \mathbb{Z}\Big)$ or $\Big(e(p(n)) \colon n \in \mathbb{Z}\Big)$ where $p(n)$ is a generalized polynomial.

But also the Thue-Morse sequence and several similar sequences.

Corollary. *Let* (X, μ, T) *be an ergodic system and* $\phi \in L^\infty(\mu)$*. Then there exists* $X_0 \in X$ *with* $\mu(X_0) = 1$ *such that:*

For every $x_0 \in X_0$*, for every system* (Y, ν, S)*, for every* $k \geq 1$*, for all functions* $f_1, \ldots, f_k \in L^\infty(\nu)$*, the averages*

$$\frac{1}{N} \sum_{n=0}^{N-1} \phi(T^n x_0) \cdot S^n f_1 \cdot S^{2n} f_2. \cdots .S^{kn} f_k$$

converge in $L^2(\nu)$*.*

Integer polynomials $p_1(n), \ldots, p_k(n)$ can be substituted for $n, \ldots, kn$ [Q. Chu]

Nilsequences do not appear in this statement: They are used as tools in the proofs, playing a role similar to the role played classically by complex exponentials.

The key: Relation with some norms or seminorms

• The Gowers norms are defined inductively for functions defined on $\mathbb{Z}/N\mathbb{Z}$:

$$\|f\|_{U(1)} = \mathbb{E}_n f(n) \ ;$$
$$\|f\|_{U(k+1)} = \left(\mathbb{E}_k \|\bar{f}\cdot\sigma^k f\|_{U(k)}^{2^k}\right)^{1/2^{k+1}}$$

where $\sigma^k f(n) = f(n+k)$.

Green and Tao conjecture (and prove for $k \leq 3$) a relation between Gowers norms and nilsequences.

• HK seminorms can be defined inductively for bounded functions defined on an ergodic system (X, μ, T):

$$|||f|||_1 = \left| \int f \, d\mu \right| \; ;$$

$$|||f|||_{k+1} = \lim_{N \to +\infty} \left(\sum_{n=0}^{N-1} |||\bar{f} \cdot T^n f|||_k^{2^k} \right)^{1/2^{k+1}}$$

These seminorms are linked to nilsystems and thus to nilsequences:

Theorem (H, Kra (2005)). *Let (X, μ, T) be an ergodic system, $f \in L^\infty(\mu)$ and $\epsilon > 0$.*
There exist a $(k-1)$ step nilsystem (Y, ν, S), a factor map $\pi\colon X \to Y$ and a smooth function h on Y with $|||f - h \circ \pi|||_k < \epsilon$.

• We define similar "seminorms" for some bounded sequences.
Let σ be the shift on $\ell^\infty(\mathbb{Z})$.
For $\underline{b} = (b_n) \in \ell^\infty(\mathbb{Z})$:

$$\|\underline{b}\|_{\bullet 1} = \left(\lim_{H\to+\infty} \frac{1}{H} \sum_{h=0}^{H-1} \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} \overline{b}_n \cdot b_{n+h}\right)^{1/2}$$

assuming the existence of the limit in N for every h.

$$\|\underline{b}\|_{\bullet k+1} = \left(\lim_{H\to+\infty} \frac{1}{H} \sum_{h=0}^{H-1} \|\overline{\underline{b}} \cdot \sigma^h \underline{b}\|_{\bullet k}^{2^k}\right)^{1/2^{k+1}}$$

assuming the existence of $\|\overline{\underline{b}} \cdot \sigma^h \underline{b}\|_{\bullet k}$ for every h.

The 'seminorms" $\|\cdot\|_{\bullet k}$ can be interpreted (at least partially) in terms of nilsequences.

We define (geometrically) a norm $\|\!|\underline{a}|\!\|_k^*$ on the space of smooth $k-1$ nilsequences and show:

Theorem. *For every smooth $k-1$ step nilsequence $\underline{a}$ and for every bounded sequence $\underline{b}$ such that $\|\underline{b}\|_{\bullet k}$ exists,*

$$\limsup_{N\to+\infty}\left|\frac{1}{N}\sum_{n=0}^{N-1} a_n b_n\right| \le \|\!|\underline{a}|\!\|_k^* \cdot \|\underline{b}\|_{\bullet k}$$

For a complete duality we need to modify the definition of the "seminorms".

Questions

Until now, each time nilsequences occur in ergodic theory, they are linked to the limit of some multiple averages.

We believe that they can be used in some larger class of questions, namely each time that the question involves some kind of "parallelograms" or of "parallelepipeds" (without averages).

This conjecture belongs more to Fourier analysis than to ergodic theory.

Classical Fourier analysis on ℓ^1

Let $\underline{a} = (a_n) \in \ell^1(\mathbb{Z})$. The correlation of this sequence is:

$$c_k = \sum_{n \in \mathbb{Z}} a_n \, \overline{a_{n+h}} \ .$$

We remark that

$$\sum_k |c_k|^2 = \|\underline{a}\|_{U(2)}^4$$

where

$$\|\underline{a}\|_{U(2)} := \Big(\sum_{h,k,\ell \in \mathbb{Z}} a_n \overline{a_{n+k}} \, \overline{a_{n+h}} a_{n+h+k} \Big)^{1/4} \ .$$

It is classical that the sequence $\underline{c}$ is the Fourier transform of some function $f \geq 0$ on $\mathbb{T}$:

$$c_k = \int_{\mathbb{T}} f(t) e(-kt) \, dt \ .$$

A quadratic Fourier analysis?

Consider now the double correlation sequence $\underline{d}$ given by:

$$d_{k,\ell} = \sum_{n\in\mathbb{Z}} a_n \, \overline{a_{n+k}} \, \overline{a_{n+\ell}} \, a_{n+k+\ell} \; .$$

We have

$$\sum_{k,\ell\in\mathbb{Z}} |d_{k,\ell}|^2 = \|\underline{a}\|_{U(3)}^8$$

where

$$\|\underline{a}\|_{U(3)}$$

$$= \Big(\sum_{n,k,\ell,m\in\mathbb{Z}} a_n \, \overline{a_{n+k}} \, \overline{a_{n+\ell}} \, a_{n+k+\ell} \, \overline{a_{n+m}} a_{n+k+m} \, a_{n+\ell+m} \, \overline{a_{n+k+\ell+m}} \Big)^{1/8}$$

Question: Is it possible to interpret the "double correlation sequence" $\underline{d}$:

$$d_{k,\ell} = \sum_{n\in\mathbb{Z}} a_n\,\overline{a_{n+k}}\,\overline{a_{n+\ell}}\,a_{n+k+\ell}\ .$$

in terms of double nilsequences?